Quasigroups and Related Systems **21** (2013), 131 − 146

Hom-Bol algebras

Sylvain Attan and A. Nourou Issa

Abstract. Hom-Bol algebras are defined as a twisted generalization of (left) Bol algebras. Hom-Bol algebras generalize multiplicative Hom-Lie triple systems in the same way as Bol algebras generalize Lie triple systems. The notion of an nth derived (binary) Hom-algebra is extended to the one of an nth derived binary-ternary Hom-algebra and it is shown that the category of Hom-Bol algebras is closed under the process of taking nth derived Hom-algebras. It is also closed by self-morphisms of binary-ternary Hom-algebras. Every Bol algebra is twisted into a Hom-Bol algebra. Some examples of low-dimensional Hom-Bol algebras are given.

1. Introduction

A *Bol algebra* is a triple $(A, \cdot, [,,])$, where A is a vector space, $\cdot : A^{\otimes 2} \to A$ a bilinear map (the binary operation on A) and $[,,] : A^{\otimes 3} \to A$ a trilinear map (the ternary operation on A) such that

(B1) $x \cdot y = -y \cdot x$,
(B2) $[x, y, z] = -[y, x, z]$,
(B3) $\circlearrowleft_{x,y,z}[x, y, z] = 0$,
(B4) $[x, y, u \cdot v] = [x, y, u] \cdot v + u \cdot [x, y, v] + [u, v, x \cdot y] - uv \cdot xy$,
(B5) $[x, y, [u, v, w]] = [[x, y, u], v, w] + [u, [x, y, v], w] + [u, v, [x, y, w]]$

for all u, v, w, x, y, z in A, where $\circlearrowleft_{x,y,z}$ denotes the sum over cyclic permutation of x, y, z and juxtaposition will be used (here and in the sequel) in order to reduce the number of braces (so, in (B4), $uv \cdot xy$ means $(u \cdot v) \cdot (x \cdot y)$).

Observe that when $x \cdot y = 0$ in a Bol algebra $(A, \cdot, [,,])$, then it reduces to a *Lie triple system* $(A, [,,])$ so one could think of a Bol algebra $(A, \cdot, [,,])$ as a Lie triple system $(A, [,,])$ with an additional anticommutative binary operation "$\cdot$" such that (B4) holds.

The definition of a Bol algebra given above is the one of a *left* Bol algebra (see, e.g., [21], [22], [24]) and the reader is advised not to confuse it with the one of the *right* Bol algebra $(A, \diamond, (;,))$ (see, e.g., [23]). However, a left Bol algebra $(A, \cdot, [,,])$ is obtained from a right one $(A, \diamond, (;,))$ if set $x \cdot y = -x \diamond y$, and $[x, y, z] = -(z; x, y)$. In this paper, by a Bol algebra we always mean a left Bol algebra.

2010 Mathematics Subject Classification: 17A30, 17D99
Keywords: Lie triple system, Bol algebra, Hom-Lie algebra, Hom-Lie triple system, Hom-Akivis algebra, Hom-Bol algebra.

Bol algebras are introduced (see [21], [24], [25]) in a study of the differential geometry of smooth Bol loops. Such a study could be seen as a generalization of the differential geometry of Lie groups, where the left-invariant vector fields on a given Lie group constitute a Lie algebra. The tangent structure to a smooth Bol loop turns out to be a Bol algebra and, locally, there is a correspondence between Bol algebras and (local) smooth Bol loops ([21], [24]). For local smooth loops in general, the correspondence is established in terms of a vector space equipped with a family of multilinear operations, called *hyperalgebra* [26] (now called *Sabinin algebra*). Bol algebras are further studied in [14], [22].

It is shown [21] that Bol algebras, as tangent algebras to local smooth Bol loops, are Akivis algebras with additional conditions. *Akivis algebras* are introduced ([1], [2]) in a study of the differential geometry of 3-webs (see references in [1], [2]; originally, M. A. Akivis called "W-algebras" such algebraic structures and the term "Akivis algebra" is introduced in [10]). Akivis algebras have also a close connection with the theory of smooth loops [1].

The aim of this paper is a study of a Hom-type generalization of Bol algebras. Roughly, a Hom-type generalization of a kind of algebra is obtained by a certain twisting of the defining identities by a linear self-map, called the twisting map, in such a way that when the twisting map is the identity map, then one recovers the original kind of algebra. In this scheme, e.g., associative algebras and Leibniz algebras are twisted into Hom-associative algebras and Hom-Leibniz algebras respectively [19] and, likewise, Hom-type analogues of Novikov algebras, alternative algebras, Jordan algebras or Malcev algebras are defined and discussed in [18], [30], [31]. The Hom-type generalization of some classes of ternary algebras are discussed in [3], [30], [32]. One could say that the theory of Hom-algebras originated in [9] (see also [15], [16]) in a study of deformations of the Witt and the Virasoro algebras (in fact, some q-deformations of the Witt and the Virasoro algebras have a structure of a Hom-Lie algebra [9]). Some algebraic abstractions of this study are given in [19], [29]. For more recent results regarding Hom-Lie algebras or Hom-Leibniz algebras, one may refer to [5], [6]. For further information on other Hom-type algebras, one may refer to, e.g., [3], [7], [18], [30], [31], [32].

The Hom-type generalization of binary algebras or ternary algebras is extended to the one of binary-ternary algebras in [8], [12]. Our present study of a Hom-type generalization of Bol algebras is included in this setting.

A description of the rest of this paper is as follows.

In section 2, we first recall some basics on Hom-algebras and then extend to binary-ternary Hom-algebras the notion of an nth-derived (binary) Hom-algebra introduced in [32]. Theorem 2.7 says that the category of Hom-Akivis algebras is closed under taking derived binary-ternary Hom-algebras (see Definition 2.6). Theorem 2.9, as well as Theorem 2.7, produce a sequence of Hom-Akivis algebras. However the construction of Theorem 2.9 is not based on derived binary-ternary Hom-algebras.

In section 3 we define Hom-Bol algebras and we point out that Bol algebras are particular instances of Hom-Bol algebras. Also Hom-Bol algebras generalize multiplicative Hom-Lie triple systems in the same way as Bol algebras generalize Lie triple systems. Next we prove some construction theorems (Theorems 3.2, 3.5 and 3.6, Corollary 3.3). The category of Hom-Bol algebras is closed under self-morphisms (Theorem 3.2) and, subsequently, every Bol algebra is twisted, along any self-morphism, into a Hom-Bol algebra (Corollary 3.3). Theorem 3.5 says that the category of Hom-Bol algebras is closed under taking derived binary-ternary Hom-algebras. Theorems 3.5 and 3.6 describe some constructions of sequences of Hom-Bol algebras. With the notion of derived Hom-algebras, one observes that each Hom-algebra gives rise to a sequence of Hom-algebras. Moreover, the closure of a given class of Hom-algebras under taking derived Hom-algebras (as well as twisting by self-morphisms) is a unique property to Hom-type algebras. In fact, the notion of derived Hom-algebras is useless for ordinary algebras and a given class of ordinary algebras may not be closed under twisting by morphisms (e.g., the class of Malcev algebras is not closed by self-morphisms; see [31], Example 2.13).

In section 4, relying on a classification of real 2-dimensional Bol algebras given in [14], we classify all the algebra morphisms on all the real 2-dimensional Bol algebras and then construct (for the case of nontrivial Bol algebras) their associated Hom-Bol algebras (applying thusly Corollary 3.3). We observe that these real 2-dimensional Hom-Bol algebras are actually Bol algebras. Applying Proposition 3.4, we construct a nontrivial 4-dimensional Hom-Bol algebra.

Throughout this paper we will work over a ground field of characteristic 0.

2. Derived binary-ternary Hom-algebras

The main purpose of this section is the extension to binary-ternary Hom-algebras of the notion of an nth derived (binary) Hom-algebra that is introduced in [32]. We show (Theorem 2.7) that the category of Hom-Akivis algebras is closed under taking derived binary-ternary Hom-algebras. When deriving Hom-algebras, one constructs a sequence of Hom-algebras and thus Theorem 2.7 describes a sequence of Hom-Akivis algebras. Another sequence is described in Theorem 2.9, relying on a result in [12].

First we begin with some basic definitions and facts that could be found in [3], [9], [12], [19], [29], [32].

Definition 2.1. Let $n \geqslant 2$ be an integer.

(i) An *n-ary Hom-algebra* $(A, [\ldots], \alpha = (\alpha_1, \ldots, \alpha_{n-1}))$ consists of a vector space A, an n-linear map $[\ldots] : A^{\otimes n} \to A$ (the n-ary operation) and linear maps $\alpha_i : A \to A$ (the twisting maps), $i = 1, \ldots, n-1$.

(ii) An n-ary Hom-algebra $(A, [\ldots], \alpha)$ is said to be *multiplicative* when the twisting maps α_i are all equal, $\alpha_1 = \ldots = \alpha_{n-1} := \alpha$, and $\alpha([x_1, \ldots, x_n]) =$

$[\alpha(x_1), \dots, \alpha(x_n)]$ for all $x_1, \dots, x_n$ in A.

(iii) A linear map $\theta : A \to B$ of n-ary Hom-algebras is called a *weak morphism* if $\theta([x_1, \dots, x_n]_A) = [\theta(x_1), \dots, \theta(x_n)]_B$ for all $x_1, \dots, x_n$ in A. The weak morphism θ is called a *morphism* of the n-ary Hom-algebras A and B if $\theta \circ (\alpha_i)_A = (\alpha_i)_B \circ \theta$ for $i = 1, \dots, n-1$.

Remark. If all $n-1$ twisting maps are the identity map Id in an n-ary Hom-algebra $(A, [\dots], \alpha)$, then it reduces to a usual n-ary algebra $(A, [\dots])$ so that the category of n-ary Hom-algebras contains the one of n-ary algebras. In this case, the weak morphism coincides with the morphism.

For $n = 2$ (resp. $n = 3$), an n-ary Hom-algebra is called a *binary* (resp. *ternary*) Hom-algebra. In the sequel, for our purpose and convenience, we shall consider only multiplicative Hom-algebras.

Hom-Lie algebras [9] constitute the first introduced class of (binary) Hom-algebras.

Definition 2.2. Let $(A, [\ \ ], \alpha)$ be a multiplicative binary Hom-algebra.

(i) The (binary) *Hom-Jacobian* of A is the trilinear map $J_\alpha : A^{\otimes 3} \to A$ defined as

$$J_\alpha(x, y, z) := \circlearrowleft_{x,y,z} [[x, y], \alpha(z)]$$

for all x, y, z in A.

(ii) The Hom-algebra $(A, [,], \alpha)$ is called a *Hom-Lie algebra* if

$$[x, y] = -[y, x] \qquad \text{and}$$

$$J_\alpha(x, y, z) = 0 \quad (\text{the } \textit{Hom-Jacobi identity})$$

for all $x, y, z \in A$.

If $\alpha = Id$, a Hom-Lie algebra reduces to a usual Lie algebra. The n-ary Hom-Jacobian of an n-ary Hom-algebra is defined in [3]. Other binary Hom-type algebras are introduced and discussed in [3], [18], [19], [30], [31].

The classes of ternary Hom-algebras that are of interest in our setting are the ones of Hom-triple systems and Hom-Lie triple systems defined in [32].

Definition 2.3. (i) A multiplicative *Hom-triple system* is a multiplicative ternary Hom-algebra $(A, [\ \ ], \alpha)$.

(ii) A multiplicative *Hom-Lie triple system* is a multiplicative Hom-triple system $(A, [\ \ ], \alpha)$ that satisfies

- $[u, v, w] = -[v, u, w]$ (left skew-symmetry),
- $\circlearrowleft_{u,v,w} [u, v, w] = 0$ (ternary Jacobi identity),
- $$[\alpha(x), \alpha(y), [u, v, w]] = [[x, y, u], \alpha(v), \alpha(w)] + [\alpha(u), [x, y, v], \alpha(w)] + [\alpha(u), \alpha(v), [x, y, w]] \tag{2.1}$$

for all $u, v, w, x, y \in A$.

When $\alpha = Id$, we recover the usual notions of triple systems and Lie triple systems ([13], [17]). The identity (2.1) is known as the ternary *Hom-Nambu identity* (see [3] for the definition of an *n-ary Hom-Nambu algebra*). Other ternary Hom-algebras such as ternary Hom-Nambu algebras [3], ternary Hom-Nambu-Lie algebras [3], ternary Hom-Lie algebras [3], Hom-Jordan triple systems [32] are also considered. In [8], the Hom-triple system is defined to be a ternary Hom-algebra satisfying the left skew-symmetry and the ternary Jacobi identity (this allowed to establish a natural connection between Hom-triple systems and the Hom-version of nonassociative algebras called non-Hom-associative algebras [12], or Hom-nonassociative algebras [18], or either nonassociative Hom-algebras [29]).

Hom-Lie-Yamaguti algebras introduced in [8] constitute some generalization of multiplicative Hom-Lie triple systems in the same way as Lie-Yamaguti algebras generalize Lie triple systems. The basic object of this paper (see section 3) may also be viewed as some generalization of multiplicative Hom-Lie triple systems.

Moving forward in the general theory of Hom-algebras, a study of "binary-ternary" Hom-algebras is initiated in [12] by defining the class of Hom-Akivis algebras as a Hom-analogue of the class of Akivis algebras ([1], [2], [10]) which are a typical example of binary-ternary algebras.

Definition 2.4. A *Hom-Akivis algebra* is a quadruple $(A, [,], [,,], \alpha)$ consisting of a vector space A, a skew-symmetric bilinear map $[,] : A^{\otimes 2} \to A$ (the binary operation), a trilinear map $[,,] : A^{\otimes 3} \to A$ (the ternary operation) and a linear self-map α of A such that

$$\circlearrowleft_{x,y,z}[[x, y], \alpha(z)] = \circlearrowleft_{x,y,z}[x, y, z] - \circlearrowleft_{x,y,z}[y, x, z] \tag{2.2}$$

for all x, y, z in A. A Hom-Akivis algebra is said to be *multiplicative* if α is a weak morphism with respect to the binary operation $[,]$ and the ternary operation $[,,]$.

The identity (2.2) is called the *Hom-Akivis identity* (if $\alpha = Id$ in (2.2), one gets the *Akivis identity* which defines Akivis algebras). From the definition above it clearly follows that the category of Hom-Akivis algebras contains the ones of Akivis algebras and Hom-Lie algebras. Some construction theorems for Hom-Akivis algebras are proved in [12]; in particular, it is shown that every Akivis algebra can be twisted along a linear self-morphism into a (multiplicative) Hom-Akivis algebra.

Another class of binary-ternary Hom-algebras is the one of Hom-Lie-Yamaguti algebras [8].

For binary Hom-algebras the notion of an nth derived Hom-algebra is introduced and studied in [32]. For completeness, we remind it in the following

Definition 2.5. Let (A, μ, α) be a Hom-algebra and $n \geqslant 0$ an integer (μ is the binary operation on A). The Hom-algebra A^n defined by

$$A^n := (A, \mu^{(n)}, \alpha^{2^n}), \quad \text{where } \mu^{(n)}(x,y) := \alpha^{2^n-1}(\mu(x,y)), \ \forall x, y \in A,$$

is called the *nth derived Hom-algebra* of A.

For simplicity of exposition, $\mu^{(n)}$ is written as $\mu^{(n)} = \alpha^{2^n-1} \circ \mu$. Then one notes that $A^0 = (A, \mu, \alpha)$, $A^1 = (A, \mu^{(1)} = \alpha \circ \mu, \alpha^2)$, and $A^{n+1} = (A^n)^1$.

Now we extend this notion of nth derived (binary) Hom-algebra to the case of binary-ternary Hom-algebras in the following

Definition 2.6. Let $\mathcal{A} := (A, [\,,\,], [\,,\,,\,], \alpha)$ be a binary-ternary Hom-algebra and $n \geqslant 0$ an integer. Define on A the nth derived binary operation $[\,,\,]^{(n)}$ and the nth derived ternary operation $[\,,\,,\,]^{(n)}$ by

$$[x,y]^{(n)} := \alpha^{2^n-1}([x,y]), \tag{2.3}$$

$$[x,y,z]^{(n)} := \alpha^{2^{n+1}-2}([x,y,z]), \tag{2.4}$$

for all x, y, z in A. Then $\mathcal{A}^{(n)} := (A, [\,,\,]^{(n)}, [\,,\,,\,]^{(n)}, \alpha^{2^n})$ will be called the *nth derived* (*binary-ternary*) *Hom-algebra* of $\mathcal{A}$.

Denote $[\,,\,]^{(n)} = \alpha^{2^n-1} \circ [\,,\,]$ and $[\,,\,,\,]^{(n)} = \alpha^{2^{n+1}-2} \circ [\,,\,,\,]$. Then we note that $\mathcal{A}^{(0)} = \mathcal{A}$, $\mathcal{A}^{(1)} = (A, [\,,\,]^{(1)} = \alpha \circ [\,,\,],\ [\,,\,,\,]^{(1)} = \alpha^2 \circ [\,,\,,\,],\ \alpha^2)$ and $\mathcal{A}^{(n+1)} = (\mathcal{A}^{(n)})^{(1)}$.

One observes that, from Definition 2.6, if set $[x,y,z] = 0$, $\forall x, y, z \in A$, we recover the nth derived (binary) Hom-algebra of Definition 2.5.

The category of Hom-Akivis algebras is closed under taking derived binary-ternary Hom-algebras as it could be seen from the following result.

Theorem 2.7. *Let $\mathcal{A} := (A, [\,,\,],\ [\,,\,,\,], \alpha)$ be a multiplicative Hom-Akivis algebra. Then, for each $n \geqslant 0$, the nth derived Hom-algebra $\mathcal{A}^{(n)}$ is a multiplicative Hom-Akivis algebra.*

Proof. For $n = 1$, $\mathcal{A}^{(1)}$ is a multiplicative Hom-Akivis algebra by Theorem 4.4 in [12] (when $\beta = \alpha$ and $n = 1$).

Now suppose that, up to n, the $\mathcal{A}^{(n)}$ are multiplicative Hom-Akivis algebras. To conclude by the induction argument, we must prove that $\mathcal{A}^{(n+1)}$ is also a multiplicative Hom-Akivis algebra.

The skew-symmetry of $[\,,\,]^{(n+1)}$ is quite obvious. In the transformations below we shall use the identity

$$J_{\mathcal{A}^{(n)}}(x,y,z) = \alpha^{2(2^n-1)}(J_{\mathcal{A}}(x,y,z)) \tag{2.5}$$

that holds for all $n \geqslant 0$ in derived (binary) Hom-algebras (see [32], Lemma 2.9).

We have

$$\begin{aligned}\circlearrowleft_{x,y,z}[[x,y]^{(n+1)},\alpha^{2^{n+1}}(z)]^{(n+1)} &= J_{\alpha^{2^{n+1}}}(x,y,z)\\ &= \alpha^{2(2^{n+1}-1)}(J_\alpha(x,y,z))\\ &= \alpha^{2(2^{n+1}-1)}(\circlearrowleft_{x,y,z}[x,y,z]-\circlearrowleft_{x,y,z}[y,x,z])\\ &= \circlearrowleft_{x,y,z}\alpha^{2^{n+2}-2}([x,y,z])-\circlearrowleft_{x,y,z}\alpha^{2^{n+2}-2}([y,x,z])\\ &= \circlearrowleft_{x,y,z}[x,y,z]^{(n+1)}-\circlearrowleft_{x,y,z}[y,x,z]^{(n+1)}\end{aligned}$$

and so the Hom-Akivis identity (2.2) holds in $\mathcal{A}^{(n+1)}$. Thus we conclude that $\mathcal{A}^{(n+1)}$ is a (multiplicative) Hom-Akivis algebra. □

As for Akivis algebras, Hom-flexibility and Hom-alternativity are defined for Hom-Akivis algebras [12]: a Hom-Akivis algebra $\mathcal{A} := (A, [\,,\,], [\,,\,,\,], \alpha)$ is said to be *Hom-flexible* if $[x,y,x]=0$, for all $x,y \in A$; it is said to be *Hom-alternative* if $[x,y,z]=0$ whenever any two of variables x,y,z are equal (i.e., $[\,,\,,\,]$ is alternating). The following result constitutes the analogue of Theorem 2.7.

Proposition 2.8. *Let $\mathcal{A}$ be a Hom-Akivis algebra.*

(*i*) *If $\mathcal{A}$ is Hom-flexible, then the derived Hom-algebra $\mathcal{A}^{(n)}$ is also Hom-flexible for each $n \geqslant 0$.*

(*ii*) *If $\mathcal{A}$ is Hom-alternative, then so is the derived Hom-algebra $\mathcal{A}^{(n)}$ for each $n \geqslant 0$.*

Proof. We get (i) (resp. (ii)) from the definition of $[\,,\,,\,]^{(n)}$ and the Hom-flexibility (resp. the Hom-alternativity) of $\mathcal{A}$. □

Theorem 2.7 describes a sequence of Hom-Akivis algebras (the derived Hom-Akivis algebras). Starting from a given Akivis algebra and its linear self-morphism, a sequence of Hom-Akivis algebras is constructed in [12] (Theorem 4.8 which is a variant of Theorem 4.4). From Theorem 4.4 in [12] when $\alpha=\beta$, we get the following sequence of Hom-Akivis algebras but starting from a given Hom-Akivis algebra.

Theorem 2.9. *Let $\mathcal{A}$ be a Hom-Akivis algebra. For each integer $n \geqslant 0$, define on A a binary operation $[\,,\,]_n$ and a ternary operation $[\,,\,,\,]_n$ by*

$$[x,y]_n := \beta^n([x,y]),$$

$$[x,y,z]_n := \beta^{2n}([x,y,z]).$$

Then $\mathcal{A}_n := (A, [,]_n, [\,,\,,\,]_n, \beta^{n+1})$ is a Hom-Akivis algebra.

Proof. The skew-symmetry of $[\,,\,]_n$ is obvious. Next, we have

$$\begin{aligned}&\circlearrowleft_{x,y,z}[[x,y]_n,\beta^{n+1}(z)]_n = \circlearrowleft_{x,y,z}\beta^n([\beta^n([x,y]),\beta^{n+1}(z)]) = \beta^{2n}(\circlearrowleft_{x,y,z}[[x,y],\beta(z)])\\ &\quad= \beta^{2n}(\circlearrowleft_{x,y,z}[x,y,z]-\circlearrowleft_{x,y,z}[y,x,z]) = \circlearrowleft_{x,y,z}\beta^{2n}([x,y,z])-\circlearrowleft_{x,y,z}\beta^{2n}([y,x,z])\\ &\quad= \circlearrowleft_{x,y,z}[x,y,z]_n-\circlearrowleft_{x,y,z}[y,x,z]_n\end{aligned}$$

which means that (2.2) holds in $\mathcal{A}_n$ and so $\mathcal{A}_n$ is a Hom-Akivis algebra (we used

(2.2) in $\mathcal{A}$). □

In Theorem 2.9 observe that $\mathcal{A}_0 = \mathcal{A}$. However, in strong contrast with derived Hom-Akivis algebras, $\mathcal{A}_{n+1}$ is not constructed from $\mathcal{A}_n$.

3. Definition and construction theorems

In this section we define a Hom-Bol algebra. It turns out that Hom-Bol algebras constitute a twisted generalization of Bol algebras. We prove some construction theorems for Hom-Bol algebras (Theorems 3.2, 3.5, Corollary 3.3). Theorem 3.2 shows that the category of Hom-Bol algebras is closed under self-morphisms while Corollary 3.3 says that every Bol algebra can be twisted, along any endomorphism, into a Hom-Bol algebra. Theorem 3.5 points out that Hom-Bol algebras are also closed under taking derived binary-ternary Hom-algebras.

We begin with the definition of the basic object of this paper.

Definition 3.1. A *Hom-Bol algebra* is a quadruple $(A, *, \{,,\}, \alpha)$ in which A is a vector space, "$*$" a binary operation and "$\{,,\}$" a ternary operation on A, and $\alpha : A \to A$ a linear map such that

(HB1) $\alpha(x * y) = \alpha(x) * \alpha(y)$,

(HB2) $\alpha(\{x, y, z\}) = \{\alpha(x), \alpha(y), \alpha(z)\}$,

(HB3) $x * y = -y * x$,

(HB4) $\{x, y, z\} = -\{y, x, z\}$,

(HB5) $\circlearrowleft_{x,y,z} \{x, y, z\} = 0$,

(HB6) $\{\alpha(x), \alpha(y), u*v\} = \{x, y, u\}*\alpha^2(v) + \alpha^2(u)*\{x, y, v\} + \{\alpha(u), \alpha(v), x*y\} - \alpha(u)\alpha(v) * \alpha(x)\alpha(y)$,

(HB7) $\{\alpha^2(x), \alpha^2(y), \{u, v, w\}\} = \{\{x, y, u\}, \alpha^2(v), \alpha^2(w)\} + \{\alpha^2(u), \{x, y, v\}, \alpha^2(w)\} + \{\alpha^2(u), \alpha^2(v), \{x, y, w\}\}$

for all $u, v, w, x, y, z \in A$.

The multiplicativity of $(A, *, \{,,\}, \alpha)$ (see (HB1) and (HB2)) is built into our definition for convenience.

Remark. (i) If $\alpha = Id$, then the Hom-Bol algebra $(A, *, \{,,\}, \alpha)$ reduces to a Bol algebra $(A, *, \{,,\})$ (see (B1)-(B5)). So Bol algebras may be seen as Hom-Bol algebras with the identity map Id as the twisting map.

(ii) If $x * y = 0$, for all $x, y \in A$, then $(A, *, \{,,\}, \alpha)$ becomes a (multiplicative) Hom-Lie triple system $(A, \{,,\}, \alpha^2)$ (see Definition 2.3). The identity (HB7) is in fact the ternary Hom-Nambu identity (2.1) but with α^2 as the twisting map.

The following result shows that the category of Hom-Bol algebras is closed under self-morphisms.

Theorem 3.2. *Let $\mathcal{A}_\alpha := (A, *, \{,,\}, \alpha)$ be a Hom-Bol algebra and β and endomorphism of $(A, *, \{,,\})$ such that $\beta\alpha = \alpha\beta$ (i.e., β is a self-morphism of $\mathcal{A}_\alpha$). Let $\beta^0 = Id$ and, for any $n \geqslant 1$, $\beta^n := \beta\beta^{n-1}$ $(= \beta \circ \beta^{n-1})$. Define on A the operations*

$$x *_\beta y := \beta^n(x * y) \quad and \quad \{x, y, z\}_\beta := \beta^{2n}(\{x, y, z\}).$$

Then $\mathcal{A}_{\beta^n} := (A, *_\beta, \{,,\}_\beta, \beta^n\alpha)$ is a Hom-Bol algebra with $n \geqslant 1$.

Proof. We observe first that the condition $\beta\alpha = \alpha\beta$ implies $\beta^n\alpha = \alpha\beta^n$. This last equality and the definitions of the operations "$*_\beta$" and "$\{,,\}_\beta$" lead to the validity of (HB1) and (HB2) for $\mathcal{A}_{\beta^n}$. Obviously the skew-symmetries (HB3) and (HB4) for $\mathcal{A}_{\beta^n}$ follow from the skew-symmetry of "$*$" and "$\{,,\}$" respectively. Next, we have

$$\circlearrowleft_{x,y,z}\{x, y, z\}_\beta = \circlearrowleft_{x,y,z}\beta^{2n}(\{x, y, z\}) = \beta^{2n}(\circlearrowleft_{x,y,z}(\{x, y, z\}) = \beta^{2n}(0) = 0$$

(by (HB5) for $\mathcal{A}_\alpha$), so we get (HB5) for $\mathcal{A}_{\beta^n}$.

Consider now $\{(\beta^n\alpha)(x), (\beta^n\alpha)(y), u \ *_\beta \ v\}_\beta$ in $\mathcal{A}_{\beta^n}$. We have (using the condition $\beta\alpha = \alpha\beta$ and (HB6) for $(A, *, \{,,\}, \alpha)$)

$$\begin{aligned}
&\{(\beta^n\alpha)(x), (\beta^n\alpha)(y), u \ *_\beta \ v\}_\beta = \beta^{3n}(\{\alpha(x), \alpha(y), u * v\}) \\
&\beta^n(\beta^{2n}(\{x, y, u\}) * (\beta^{2n}\alpha^2)(v) + \beta^n((\beta^{2n}\alpha^2)(u) * \beta^{2n}(\{x, y, v\})) \\
&+\beta^{2n}(\{(\beta^n\alpha)(u), (\beta^n\alpha)(v), \beta^n(x * y)\}) - \beta^n(\beta^{2n}(\alpha(u)\alpha(v)) * \beta^{2n}(\alpha(x)\alpha(y))) \\
&= \{x, y, u\}_\beta *_\beta (\beta^n\alpha)^2(v) + (\beta^n\alpha)^2(u) *_\beta \{x, y, v\}_\beta \\
&+\{(\beta^n\alpha)(u), (\beta^n\alpha)(v), x *_\beta y\}_\beta - ((\beta^n\alpha)(u) *_\beta (\beta^n\alpha)(v)) *_\beta ((\beta^n\alpha)(x) *_\beta (\beta^n\alpha)(y))
\end{aligned}$$

and thus we get (HB6) for $\mathcal{A}_{\beta^n}$. Finally, using repeatedly the condition $\beta\alpha = \alpha\beta$ and (HB7) for $\mathcal{A}_\alpha$, we compute:

$$\begin{aligned}
&\{(\beta^n\alpha)^2(x), (\beta^n\alpha)^2(y), \{u, v, w\}_\beta\}_\beta = \{(\beta^{2n}\alpha^2)(x), (\beta^{2n}\alpha^2)(y), \{u, v, w\}_\beta\}_\beta \\
&= \beta^{2n}(\{(\beta^{2n}\alpha^2)(x), (\beta^{2n}\alpha^2)(y), \beta^{2n}(\{u, v, w\})\}) = \beta^{4n}(\{\alpha^2(x), \alpha^2(y), \{u, v, w\}\}) \\
&= \beta^{2n}(\{(\beta^{2n}\alpha^2)(u), (\beta^{2n}\alpha^2)(v), \beta^{2n}(\{x, y, w\})\}) \\
&+\beta^{2n}(\{\beta^{2n}(\{x, y, u\}), (\beta^{2n}\alpha^2)(v), (\beta^{2n}\alpha^2)(w)\}) \\
&+\beta^{2n}(\{(\beta^{2n}\alpha^2)(u), \beta^{2n}(\{x, y, v\}), (\beta^{2n}\alpha^2)(w)\}) \\
&= \{(\beta^n\alpha)^2(u), (\beta^n\alpha)^2(v), \{x, y, w\}_\beta\}_\beta + \{\{x, y, u\}_\beta, (\beta^n\alpha)^2(v), (\beta^n\alpha)^2(w)\}_\beta \\
&+\{(\beta^n\alpha)^2(u), \{x, y, v\}_\beta, (\beta^n\alpha)^2(w)\}_\beta.
\end{aligned}$$

Thus (HB7) holds for $\mathcal{A}_{\beta^n}$.

Therefore, we proved the validity for $\mathcal{A}_{\beta^n}$ of the set of identities of type (HB1)-(HB7) and so we get that $\mathcal{A}_{\beta^n}$ is a Hom-Bol algebra. This finishes the proof. $\square$

From Theorem 3.2 we have the following

Corollary 3.3. *Let $\mathcal{A} := (A, *, [,,])$ be a Bol algebra and β an endomorphism of $\mathcal{A}$. If define on A a binary operation "$\widetilde{*}$" and a ternary operation "$\{,,\}$" by*

$$x \widetilde{*} y := \beta(x * y) \quad and \quad \{x, y, z\} := \beta^2([x, y, z]),$$

then $(A, \widetilde{*}, \{,,\}, \beta)$ is a Hom-Bol algebra.

Proof. The proof follows from the one of Theorem 3.2 for $\alpha = Id$ and $n = 1$. □

Observe that Corollary 3.3 gives a method for constructing Hom-Bol algebras from Bol algebras. This is an extension of a result due to D. Yau [29] giving a general construction method of Hom-algebras from their corresponding untwisted version (see also [8], [12] for a use of such an extension).

A *Malcev algebra* ([20], [27]) is a binary algebra $(A, *)$ such that the operation $*$ is anticommutative and that the *Malcev identity*

$$J(x, y, x * z) = J(x, y, z) * z$$

is satisfied for all $x, y, z \in A$, where $J(x, y, z)$ $(= J_{Id}(x, y, z))$ is the usual Jacobian.

From Corollary 3.3, we get the following

Proposition 3.4. *Let $(A, *)$ be a Malcev algebra and β any endomorphism of $(A, *)$. Define on A the operations*

$$x \widetilde{*} y := \beta(x * y), \quad \{x, y, z\} := (1/3)\beta^2(2(x * y) * z - (y * z) * x - (z * x) * y).$$

Then $(A, \widetilde{*}, \{,,\}, \beta)$ is a Hom-Bol algebra.

Proof. If consider on A the ternary operation

$$[x, y, z] := (1/3)(2(x * y) * z - (y * z) * x - (z * x) * y),$$

$\forall x, y, z \in A$, then $(A, *, [,,])$ is a Bol algebra [21]. Moreover, since β is an endomorphism of $(A, *)$, we have $\beta([x, y, z]) = [\beta(x), \beta(y), \beta(z)]$ so that β is also an endomorphism of $(A, *, [,,])$. Then Corollary 3.3 implies that $(A, \widetilde{*}, \{,,\}, \beta)$ is a Hom-Bol algebra. □

The following construction result for Hom-Bol algebras is the analogue of Theorem 2.7. It shows that Hom-Bol algebras are closed under taking derived Hom-algebras.

Theorem 3.5. *Let $\mathcal{A} := (A, *, \{,,\}, \alpha)$ be a Hom-Bol algebra. Then, for each $n \geqslant 0$, the nth derived Hom-algebra*

$$\mathcal{A}^{(n)} := (A, *^{(n)} = \alpha^{2^n - 1} \circ *, \{,,\}^{(n)} = \alpha^{2^{n+1}-2} \circ \{,,\}, \alpha^{2^n})$$

is a Hom-Bol algebra.

Proof. The identities (HB1)-(HB5) for $\mathcal{A}^{(n)}$ are obvious. The checking of (HB6) for $\mathcal{A}^{(n)}$ is as follows.

$$\begin{aligned}
&\{\alpha^{2^n}(x), \alpha^{2^n}(y), u *^{(n)} v\}^{(n)} = \alpha^{2^{n+1}-2}(\{\alpha^{2^n}(x), \alpha^{2^n}(y), \alpha^{2^n-1}(u * v)\}) \\
&= \alpha^{2^{n+1}-2}(\{\alpha^{2^n-1}\alpha(x), \alpha^{2^n-1}\alpha(y), \alpha^{2^n-1}(u * v)\}) \\
&= \alpha^{2^{n+1}-2}\alpha^{2^n-1}(\{\alpha(x), \alpha(y), u * v\}) \\
&= \alpha^{2^{n+1}-2}(\{x, y, u\} *^{(n)} \alpha^2(v)) + \alpha^{2^{n+1}-2}(\alpha^2(u) *^{(n)} \{x, y, v\}) \\
&\qquad + \alpha^{2^{n+1}-2}(\{\alpha^{2^n}(u), \alpha^{2^n}(v), x *^{(n)} y\}) - \alpha^{2^{n+1}-2}(\alpha(u)\alpha(v) *^{(n)} \alpha(x)\alpha(y)) \\
&= \{x, y, u\}^n *^{(n)} \alpha^{2^{n+1}}(v) + \alpha^{2^{n+1}}(u) *^{(n)} \{x, y, v\}^n \\
&\qquad + \{\alpha^{2^n}(u), \alpha^{2^n}(v), x *^{(n)} y\}^{(n)} - (\alpha^{2^{n-1}})^2(\alpha(u)\alpha(v) *^{(n)} \alpha(x)\alpha(y))
\end{aligned}$$

$$= \{x,y,u\}^n *^{(n)} (\alpha^{2^n})^2(v) + (\alpha^{2^n})^2(u) *^{(n)} \{x,y,v\}^n$$
$$+\{\alpha^{2^n}(u), \alpha^{2^n}(v), x *^{(n)} y\}^{(n)} - (\alpha^{2^n}(u) *^{(n)} \alpha^{2^n}(v)) *^{(n)} (\alpha^{2^n}(x) *^{(n)} \alpha^{2^n}(y))$$

and thus (HB6) holds for $\mathcal{A}^{(n)}$ (we used (HB6) for $\mathcal{A}$). Finally, we compute

$$\{(\alpha^{2^n})^2(x), (\alpha^{2^n})^2(y), \{u,v,w\}^{(n)}\}^{(n)}$$
$$= \alpha^{2^{n+1}-2}(\{(\alpha^{2^n})^2(x), (\alpha^{2^n})^2(y), \alpha^{2^{n+1}-2}(\{u,v,w\})\})$$
$$= \alpha^{2^{n+1}-2}(\{\alpha^{2^{n+1}}(x), \alpha^{2^{n+1}}(y), \alpha^{2^{n+1}-2}(\{u,v,w\})\})$$
$$= (\alpha^{2^{n+1}-2})^2(\{\alpha^2(x), \alpha^2(y), \{u,v,w\}\})$$
$$= \alpha^{2^{n+1}-2}(\{\alpha^{2^{n+1}-2}(\{x,y,u\}), \alpha^{2^{n+1}}(v), \alpha^{2^{n+1}}(w)\})$$
$$+\alpha^{2^{n+1}-2}(\{\alpha^{2^{n+1}}(u), \alpha^{2^{n+1}-2}(\{x,y,v\}), \alpha^{2^{n+1}}(w)\})$$
$$+\alpha^{2^{n+1}-2}(\{\alpha^{2^{n+1}}(u), \alpha^{2^{n+1}}(v), \alpha^{2^{n+1}-2}(\{x,y,v\})\})$$
$$= \{\{x,y,u\}^{(n)}, (\alpha^{2^n})^2(v), (\alpha^{2^n})^2(w)\}^{(n)} + \{(\alpha^{2^n})^2(u), \{x,y,v\}^{(n)}, (\alpha^{2^n})^2(w)\}^{(n)}$$
$$+\{(\alpha^{2^n})^2(u), (\alpha^{2^n})^2(v), \{x,y,w\}^{(n)}\}^{(n)}$$

and so we see that (HB7) holds for $\mathcal{A}^{(n)}$ (we used (HB7) for $\mathcal{A}$). Thus we get that $\mathcal{A}^{(n)}$ verifies the system (HB1)-(HB7), which means that $\mathcal{A}^{(n)}$ is a Hom-Bol algebra. □

The construction described in Theorem 2.9 is reported to Hom-Bol algebras as follows.

Theorem 3.6. *Let $\mathcal{A} := (A, *, \{,,\}, \beta)$ be a Hom-Bol algebra. For each $n \geqslant 0$, the Hom-algebra $\mathcal{A}_n := (A, *_n = \beta^n \circ *, \{,,\}_n = \beta^{2n} \circ \{,,\}, \beta^{n+1})$ is a Hom-Bol algebra.*

Proof. We get the proof from the one of Theorem 3.2 if set $\alpha = \beta$. □

4. Examples

In this section, we give some few examples of Hom-Bol algebras. We found that the Hom-Bol algebras obtained from the classification of real 2-dimensional Bol algebras are again (isomorphic to) Bol algebras (subsection 4.1). However, we constructed a 4-dimensional Hom-Bol algebra that is not Bol (subsection 4.2).

4.1. We classify all algebra morphisms on all the real 2-dimensional Bol algebras, using the classification of real 2-dimensional Bol algebras given in [14] (another classification, inferred from the one of 2-dimensional hyporeductive triple algebras, is independently given in [11], but not as distinct isomorphic classes). Then, from Corollary 3.3 we obtain their corresponding 2-dimensional Hom-Bol algebras. We will work over the ground field of real numbers.

Note that we do *not* classify *all* 2-dimensional Hom-Bol algebras (such a classification remains an open problem). In contrast of 2-dimensional Hom-Lie algebras where the Hom-Jacobi identity is independent of the twisting map (see [5]), they are defining identities of 2-dimensional Hom-Bol algebras which depends on the twisting map. This feature could be observed also for low-dimensional Hom-Novikov algebras [30].

We consider only nontrivial cases, i.e. we will omit mentioning the Lie triple systems (as particular cases of Bol algebras) and the 0-map as twisting map (since it is always an algebra morphism and it gives rise to the zero Hom-algebra).

If $(A, *, [,,])$ is a 2-dimensional Bol algebra with basis $\{e_1, e_2\}$, then a linear map $\beta : A \to A$ defined by $\beta(e_1) = a_1 e_1 + a_2 e_2$, $\beta(e_2) = b_1 e_1 + b_2 e_2$ is a self-morphism of $(A, *, [,,])$ if and only if

$$\beta(e_1 * e_2) = \beta(e_1) * \beta(e_2), \quad \beta([e_1, e_2, e_i]) = [\beta(e_1), \beta(e_2), \beta(e_i)],\ i = 1, 2. \quad (4.1)$$

It is proved ([14], Theorem 2) that any nontrivial 2-dimensional (left) Bol algebra over the field of real numbers is isomorphic to only one of the algebras described below:

(A1) $e_1 * e_2 = -e_2$, $[e_1, e_2, e_1] = e_1$, $[e_1, e_2, e_2] = -e_2$;

(A2) $e_1 * e_2 = -e_2$, $[e_1, e_2, e_1] = \lambda e_2$, $[e_1, e_2, e_2] = 0$;

(A3) $e_1 * e_2 = -e_2$, $[e_1, e_2, e_1] = \lambda e_2$, $[e_1, e_2, e_2] = \pm e_1$.

We recall that we consider here left Bol algebras, instead of the right-sided case studied in [14]. Also, in [14] the types (A2) and (A3) are gathered in a single one type (thus λ takes the same value in (A2) and (A3); see [14]).

Now, with the linear map β given as above, we shall discuss the conditions (4.1) for each of the types (A1), (A2) and (A3).

• Case of type (A1):

For this type, the conditions (4.1) lead to the following simultaneous constraints on the coefficients a_i, b_j of β:

$$b_1 = b_2(a_1 - 1) = a_1(1 - a_1 b_2) = a_2(1 + a_1 b_2) = b_2(1 - a_1 b_2) = 0.$$

From these constraints, we see that the identity map $\beta = Id$ is the only nonzero self-morphism of (A1). So, by Corollary 3.3, any Bol algebra of type (A1), regarded as a Hom-Bol algebra with Id as the twisting map, is the only one nonzero Hom-Bol algebra correspondig to (A1).

• Case of type (A2):

The conditions (4.1) then imply that

$$b_1 = b_2(a_1 - 1) = \lambda b_2(1 - {a_1}^2) = 0.$$

If $b_2 = 0$, from the equations above, we see that the nonzero morphism β is defined as

$$\beta(e_1) = a_1 e_1 + a_2 e_2, \quad (\text{with } (a_1, a_2) \neq (0, 0)), \quad \beta(e_2) = 0 \quad (4.2)$$

and, applying Corollary 3.3, we get that (A2) is twisted into the zero Hom-Bol algebra.

If $b_2 \neq 0$, then the equation $b_2(a_1 - 1) = 0$ (see above) implies $a_1 = 1$ and so the nonzero morphism β is defined as

$$\beta(e_1) = e_1 + a_2 e_2, \quad \beta(e_2) = b_2 e_2 \quad (b_2 \neq 0). \tag{4.3}$$

The application of Corollary 3.3 then gives rise to the Hom-Bol algebra $(A, \widetilde{*}, \{,,\}, \beta)$ defined by

$$e_1 \widetilde{*} e_2 = -b_2 e_2, \quad \{e_1, e_2, e_1\} = \lambda {b_2}^2 e_2, \quad \{e_1, e_2, e_2\} = 0 \tag{4.4}$$

with $b_2 \neq 0$. Observe that this Hom-Bol algebra is isomorphic to (A2) (by an isomorphism $\phi : A \to A$ given by $\phi(e_1) = (1/b_2)e_1 + ke_2$, $\phi(e_2) = le_2$ with $l \neq 0$).

• Case of type (A3):

In this case, the conditions (4.1) are expressed as the following simultaneous equations:

$$b_1 = b_2(a_1 - 1) = a_1 a_2 b_2 = \lambda b_2(1 - {a_1}^2) = a_1(1 - {b_2}^2) = a_2 = 0.$$

So β must be found in the form

$$\beta(e_1) = a_1 e_1, \quad \beta(e_2) = b_2 e_2.$$

If $b_2 = 0$, then the equation $a_1(1 - {b_2}^2) = 0$ implies $a_1 = 0$ and then β is the 0-map, which is the case that is already precluded.

Let $b_2 \neq 0$. Then the equation $b_2(a_1 - 1) = 0$ implies $a_1 = 1$ and next the equation $a_1(1 - {b_2}^2) = 0$ implies $b_2 = \pm 1$. So β is defined as

$$\beta(e_1) = e_1, \quad \beta(e_2) = \pm e_2 \tag{4.5}$$

that is, either β is the identity map ($\beta = Id$) or β is defined as $\beta(e_1) = e_1$, $\beta(e_2) = -e_2$.

Therefore Corollary 3.3 says that (A3) is twisted into itself (observe that, by the self-morphism $\beta(e_1) = e_1$, $\beta(e_2) = -e_2$, (A3) is twisted into the Hom-Bol algebra $(A, \widetilde{*}, \{,,\}, \beta)$ defined by

$$e_1 \widetilde{*} e_2 = e_2, \quad \{e_1, e_2, e_1\} = \lambda e_2, \quad \{e_1, e_2, e_2\} = \pm e_1,$$

which is isomorphic to (A3) by an isomorphism $\phi : A \to A$ given by $\phi(e_1) = -e_1$, $\phi(e_2) = e_2$).

The considerations above are summarized in the following table (for each of the Bol algebras (A1), (A2) or (A3), all of its nonzero algebra morphisms are listed and the corresponding Hom-Bol algebra is given up to isomorphisms):

Bol algebra	Algebra morphisms β	Hom-Bol algebra
(A1)	$\beta = Id$	(A1)
(A2)	$\beta(e_1) = a_1e_1 + a_2e_2,\ \ (a_1, a_2) \neq (0, 0)$ $\beta(e_2) = 0$	0
	$\beta(e_1) = e_1 + a_2e_2$ $\beta(e_2) = b_2e_2,\ (b_2 \neq 0)$	(A2)
(A3)	$\beta(e_1) = e_1,\ \ \beta(e_2) = \pm e_2$	(A3)

4.2. Let A be a 4-dimensional vector space with basis $\{e_1, e_2, e_3, e_4\}$. If define

$$e_1 * e_2 = -e_2 = -e_2 * e_1, \quad e_1 * e_3 = -e_3 = -e_3 * e_1,$$
$$e_1 * e_4 = e_4 = -e_4 * e_1, \quad e_2 * e_3 = 2e_4 = -e_3 * e_2$$

(the unspecified products are 0), then $(A, *)$ is a Malcev algebra ([27], Example 3.1). Now, let $\beta : A \to A$ a linear map given by

$$\beta(e_1) = e_1 + e_4, \quad \beta(e_2) = e_2 + e_3, \quad \beta(e_3) = e_3, \quad \beta(e_4) = e_4.$$

Then β is an algebra morphism of $(A, *)$ (see [31], Example 2.13). The application of Proposition 3.4 to $(A, *)$ gives rise to a 4-dimensional Hom-Bol algebra $(A, \widetilde{*}, \{,,\}, \beta)$, where "$\widetilde{*}$" and "$\{,,\}$" are defined by

$$e_1 \widetilde{*} e_2 = -e_2 - e_3 = -e_2 \widetilde{*} e_1, \quad e_1 \widetilde{*} e_3 = -e_3 = -e_3 \widetilde{*} e_1,$$
$$e_1 \widetilde{*} e_4 = e_4 = -e_4 \widetilde{*} e_1, \qquad e_2 \widetilde{*} e_3 = 2e_4 = -e_3 \widetilde{*} e_2,$$
$$\{e_1, e_2, e_1\} = -e_2 - 2e_3 = -\{e_2, e_1, e_1\},$$
$$\{e_1, e_3, e_1\} = -e_3 = -\{e_3, e_1, e_1\},$$
$$\{e_1, e_4, e_1\} = e_4 = -\{e_4, e_1, e_1\}$$

(again, the unspecified products are 0). We note that this Hom-Bol algebra is not Bol since, e.g., $\{e_1, e_2, e_1 \widetilde{*} e_2\} = 0$ while $\{e_1, e_2, e_1\} \widetilde{*} e_2 + e_1 \widetilde{*} \{e_1, e_2, e_2\} + \{e_1, e_2, e_1 \widetilde{*} e_2\} - e_1 e_2 \widetilde{*} e_1 e_2 = 4e_4$ and so (B4) fails for $(A, \widetilde{*}, \{,,\})$.

Using (2.3) and (2.4), one may compute the nth derived Hom-Bol algebras coming from $(A, \widetilde{*}, \{,,\}, \beta)$.

Acknowledgment. The authors thank the referees for valuable comments and suggestions.

References

[1] **M. A. Akivis**, *Local differentiable quasigroups and* 3*-webs of multidimensional surfaces*, (Russian), in: Studies in the theory of quasigroups and loops, Ştiinţa, Kishinev 1973, 3 – 12.

[2] **M. A. Akivis**, *Local algebras of a multidimensional* 3*-web*, Siberian Math. J. **17** (1976), 3 – 8.

[3] **H. Ataguema, A. Makhlouf and S. D. Silvestrov**, *Generalization of n-ary Nambu algebras and beyond*, J. Math. Phys. **50** (2009), 083501.

[4] **S. Benayadi and A. Makhlouf**, *Hom-Lie algebras with symmetric invariant non-degenerate bilinear forms*, ArXiv:1009.4226.

[5] **J. M. Casas, M. A. Insua and N. Pacheco Rego**, *On universal central extensions of Hom-Lie algebras*, ArXiv:1209.5887v1.

[6] **J. M. Casas, M. A. Insua and N. Pacheco Rego**, *On universal central extensions of Hom-Leibniz algebras*, ArXiv:1209.6266v1.

[7] **Y. Frégier, A. Gohr and S. D. Silvestrov**, *Unital algebras of Hom-associative type and surjective or injective twistings*, J. Gen. Lie Theory Appl. **3** (2009), 285 – 295.

[8] **D. Gaparayi and A. N. Issa**, *A twisted generalization of Lie-Yamaguti algebras*, Int. J. Algebra **6** (2012), 339 – 352.

[9] **J. T. Hartwig, D. Larsson and S. D. Silvestrov**, *Deformations of Lie algebras using σ-derivations*, J. Algebra **295** (2006), 314 – 361.

[10] **K. H. Hofmann and K. Strambach**, *Lie's fundamental theorems for local analytic loops*, Pacific J. Math. **123** (1986), 301 – 327.

[11] **A. N. Issa**, *Classifying two-dimensional hyporeductive triple algebras*, Int. J. Math. Math. Sci. **2006** (2006), 1 – 10.

[12] **A. N. Issa**, *Hom-Akivis algebras*, Comment. Math. Univ. Carolin. **52** (2011), 485 – 500.

[13] **N. Jacobson**, *Lie and Jordan triple systems*, Amer. J. Math. **71** (1949), 149 – 170.

[14] **E. N. Kuz'min and O. Zaĭdi**, *Solvable and semisimple Bol algebras*, Alg. Logic **32** (1993), 361 – 371.

[15] **D. Larsson and S. Silvestrov**, *Quasi-Hom-Lie algebras, central extensions and 2-cycle-like identities*, J. Algebra **288** (2005), 321 – 344.

[16] **D. Larsson and S. Silvestrov**, *Quasi-Lie algebras*, Contemp. Math. **391** (2005), 241 – 248.

[17] **W. G. Lister**, *A structure theory of Lie triple systems*, Trans. Amer. Math. Soc. **52** (1952), 217 – 242.

[18] **A. Makhlouf**, *Hom-alternative algebras and Hom-Jordan algebras*, Int. Electron. J. Alg. **8** (2010), 177-190.

[19] **A. Makhlouf and S.D. Silvestrov**, *Hom-algebra structures*, J. Gen. Lie Theory Appl. **2** (2008), 51 – 64.

[20] **A. I. Mal'tsev**, *Analytic loops*, (Russian), Mat. Sb. (N.S.) **36** (1955), 569 – 575.

[21] **P. O. Mikheev**, *Geometry of smooth Bol loops*, (Russian), PhD. Thesis, Friendship University, Moscow 1986.

[22] **J. M. Pérez-Izquierdo**, *An envelope for Bol algebras*, J. Algebra **284** (2005), 480 – 493.

[23] **L. V. Sabinin**, *Smooth quasigroups and loops*, Math. Appl. **492**, Kluwer, Dordrecht 1999.

[24] **L. V. Sabinin and P. O. Mikheev**, *Analytic Bol loops*, (Russian), in: Webs and Quasigroups, Kalinin. Gos. Univ., Kalinin 1982, 102 – 109.

[25] **L. V. Sabinin and P. O. Mikheev**, *On the geometry of smooth Bol loops*, (Russian), in: Webs and Quasigroups, Kalinin. Gos. Univ., Kalinin 1984, 144 – 154.

[26] **L. V. Sabinin and P. O. Mikheev**, *On the infinitesimal theory of local analytic loops*, Soviet Math. Dokl. **36** (1988), 45 – 548.

[27] **A. A. Sagle**, *Malcev algebras*, Trans. Amer. Math. Soc. **101** (1961), 426 – 458.

[28] **Y. Sheng**, *Representations of hom-Lie algebras*, Algebr. Represent. Theory **15** (2012), 1081 – 1098.

[29] **D. Yau**, *Hom-algebras and Homology*, J. Lie Theory **19** (2009), 409 – 421.

[30] **D. Yau**, *Hom-Novikov algebras*, J. Phys. A **44** (2011), 085202.

[31] **D. Yau**, *Hom-Maltsev, Hom-alternative, and Hom-Jordan algebras*, Int. Electron. J. Algebra **11** (2012), 177 – 217.

[32] **D. Yau**, *On n-ary Hom-Nambu and Hom-Nambu-Lie algebras*, J. Geom. Phys. **62** (2012), 506 – 522.

Received February 07, 2013
Revised June 22, 2013

S. Attan
Institut de Mathématiques et de Sciences Physiques, Université d'Abomey-Calavi,
01 BP 613-Oganla, Porto-Novo, Bénin
e-mail: sylvain.attan@imsp-uac.org

A. Nourou Issa
Département de Mathématiques, Université d'Abomey-Calavi, 01 BP 4521, Cotonou 01, Bénin
e-mail: woraniss@yahoo.fr