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# Intra-regular, left quasi-regular and semisimple fuzzy ordered semigroups

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**Abstract.** We characterize the ordered semigroup which are both intra-regular and left quasi-regular also the ordered semigroups which are both intra-regular and semisimple in terms of fuzzy sets.

In this paper we prove that an ordered semigroup $S$ is intra-regular and left quasi-regular if and only if for every fuzzy subset $f$ of $S$ we have $f \preceq 1 \circ f^2 \circ 1 \circ f$. It is intra-regular and semisimple if and only if for every fuzzy subset $f$ of $S$ we have $f \preceq 1 \circ f^2 \circ 1 \circ f \circ 1$. Moreover, the property $f \preceq f \circ 1 \circ f^2 \circ 1$ characterizes the ordered semigroups which are intra-regular and right quasi-regular. An ordered semigroup $(S, \cdot, \leqslant)$ is called left (resp. right) quasi-regular if $a \in (SaSa]$ (resp. $a \in (aSaS]$) for every $a \in S$. In other words, $S$ is left (resp. right) quasi-regular if for every $a \in S$ there exist $x, y \in S$ such that $a \leqslant xaya$ (resp. $a \leqslant axay$). An ordered semigroup $S$ is called semisimple if $a \in (SaSaS]$ for every $a \in S$. That is, if for every $a \in S$ there exist $x, y, z \in S$ such that $a \leqslant xayaz$ [2]. Intra-regular ordered semigroups are well known. These are the ordered semigroups in which $a \in (Sa^2S]$ for each $a \in S$. We remind that for a subset $H$ of $S$, $(H]$ is the set $\{t \in S \mid t \leqslant h \text{ for some } h \in H\}$. As always, denote by 1 the fuzzy subset of $S$ defined by $1(x) = 1$ for every $x \in S$. Recall that if $S$ is an intra-regular ordered semigroup, then $1 \circ 1 = 1$. If $f, g$ are fuzzy subsets of $S$ such that $f \preceq g$, then for any fuzzy subset $h$ of $S$ we have $f \circ h \preceq g \circ h$ and $h \circ f \preceq h \circ g$. Denote $f^2 := f \circ f$, and by $f_a$ the characteristic function on the set $S$ defined by $f_a(x) = 1$ if $x = a$ and $f_a(x) = 0$ if $x \neq a$ ($a \in S$). Denote by $A_a$ the subset of $S \times S$ defined by $A_a := \{(x, y) \in S \times S \mid a \leqslant xy\}$ [3]. The paper in a continuation of our papers in [1,5], for information not given in the present paper we refer to those papers. Exactly as in [1,5], our aim is to present a proof which is drastically simplified than the usual one.

**Lemma 1.** *Let $(S, \cdot, \leqslant)$ be an ordered groupoid, $f$, $g$ fuzzy subsets of $S$ and $a \in S$. The following are equivalent:*

(1) $(f \circ g)(a) \neq 0$.

(2) *There exists $(x, y) \in A_a$ such that $f(x) \neq 0$ and $g(y) \neq 0$.*

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**Lemma 2.** *Let $(S, \cdot, \leqslant)$ be an ordered groupoid, $f$ a fuzzy subset of $S$ and $a \in S$. The following are equivalent:*

(1) $(f \circ 1)(a) \neq 0$.

(2) *There exists $(x, y) \in A_a$ such that $f(x) \neq 0$.*

**Lemma 3.** *Let $(S, \cdot, \leqslant)$ be an ordered groupoid, $g$ a fuzzy subset of $S$ and $a \in S$. The following are equivalent:*

(1) $(1 \circ g)(a) \neq 0$.

(2) *There exists $(x, y) \in A_a$ such that $g(y) \neq 0$.*

**Theorem 4.** *An ordered semigroup $S$ is intra-regular and left quasi-regular if and only if for every fuzzy subset $f$ of $S$ we have*

$$f \preceq 1 \circ f^2 \circ 1 \circ f.$$

*Proof.* $(\Rightarrow)$. Let $a \in S$. By hypothesis, there exist $x, y, z, t \in S$ such that $a \leqslant xa^2y$ and $a \leqslant zata$. Then we have $a \leqslant z(xa^2y)ta$. Since $(zxa^2yt, a) \in A_a$, we have $A_a \neq \emptyset$ and

$$\begin{aligned}(1 \circ f^2 \circ 1 \circ f)(a) &= \bigvee_{(u,v) \in A_a} \min\{(1 \circ f^2 \circ 1)(u), f(v)\} \\ &\geqslant \min\{(1 \circ f^2 \circ 1)(zxa^2yt), f(a)\}.\end{aligned}$$

Since $(zxa^2, yt) \in A_{zxa^2yt}$, we have $A_{zxa^2yt} \neq \emptyset$ and

$$\begin{aligned}(1 \circ f^2 \circ 1)(zxa^2yt) &= \bigvee_{(u,v) \in A_{zxa^2yt}} \min\{(1 \circ f^2)(u), 1(v)\} \\ &\geqslant \min\{(1 \circ f^2)(zxa^2), 1(yt)\} \\ &= (1 \circ f^2)(zxa^2).\end{aligned}$$

Since $(zxa, a) \in A_{zxa^2}$, we have $A_{zxa^2} \neq \emptyset$ and

$$\begin{aligned}(1 \circ f^2)(zxa^2) &= \bigvee_{(u,v) \in A_{zxa^2}} \min\{(1 \circ f)(u), f(v)\} \\ &\geqslant \min\{(1 \circ f)(zxa), f(a)\}.\end{aligned}$$

Since $(zx, a) \in A_{zxa}$, we have $A_{zxa} \neq \emptyset$ and

$$\begin{aligned}(1 \circ f)(zxa) &= \bigvee_{(u,v) \in A_{zxa}} \min\{1(u), f(v)\} \\ &\geqslant \min\{1(zx), f(a)\} \\ &= f(a).\end{aligned}$$

Thus we have

$$\begin{aligned}(1 \circ f^2 \circ 1 \circ f)(a) &\geqslant \min\{(1 \circ f^2 \circ 1)(zxa^2yt), f(a)\}\\ &\geqslant \min\{(1 \circ f^2)(zxa^2), f(a)\}\\ &\geqslant \min\{\min\{(1 \circ f)(zxa), f(a)\}, f(a)\}\\ &\geqslant \min\{f(a), f(a)\}\\ &= f(a).\end{aligned}$$

($\Leftarrow$). Let $a \in S$. Since $f_a$ is a fuzzy set in $S$, by hypothesis, we have $1 = f_a(a) \leqslant (1 \circ f_a^2 \circ 1 \circ f_a)(a)$. Since $1 \circ f_a^2 \circ 1 \circ f_a$ is a fuzzy set in $S$, we have $(1 \circ f_a^2 \circ 1 \circ f_a)(a) \leqslant 1$. Thus we have $(1 \circ f_a^2 \circ 1 \circ f_a)(a) = 1$. By Lemma 1, there exists $(x, y) \in A_a$ such that $(1 \circ f_a^2)(x) \neq 0$ and $(1 \circ f_a)(y) \neq 0$. Since $(1 \circ f_a^2)(x) \neq 0$, by Lemma 3, there exists $(z, t) \in A_x$ such that $f_a^2(t) \neq 0$. Since $(1 \circ f_a)(y) \neq 0$, by Lemma 3, there exists $(u, v) \in A_y$ such that $f_a(v) \neq 0$. Since $f_a^2(t) \neq 0$, by Lemma 1, there exists $(w, h) \in A_t$ such that $f_a(w) \neq 0$ and $f_a(h) \neq 0$. Since $f_a(v) \neq 0$, we have $f_a(v) = 1$. Similarly $f_a(w) = 1$, $f_a(h) = 1$. Hence we obtain

$$a \leqslant xy \leqslant (zt)(uv) \leqslant z(wh)uv \text{ and } v = w = h = a.$$

Then $a \leqslant za^2ua \in Sa^2S \cap SaSa$. Then $a \in (Sa^2S]$ and $a \in (SaSa]$, that is, $S$ is intra-regular and left quasi-regular. □

In an analogous way we prove the next theorem.

**Theorem 5.** *An ordered semigroup $S$ is intra-regular and right quasi-regular if and only if for every fuzzy subset $f$ of $S$ we have*

$$f \preceq f \circ 1 \circ f^2 \circ 1.$$

**Theorem 6.** *An ordered semigroup $S$ is intra-regular and semisimple if and only if for every fuzzy subset $f$ of $S$ we have*

$$f \preceq 1 \circ f^2 \circ 1 \circ f \circ 1.$$

*Proof.* ($\Rightarrow$). Let $a \in S$. By hypothesis, there exist $x, y, z, t, h \in S$ such that $a \leqslant xa^2y$ and $a \leqslant zatah$, then $a \leqslant z(xa^2y)tah$. Since $(zxa^2yta, h) \in A_a$, we have $A_a \neq \emptyset$ and

$$\begin{aligned}(1 \circ f^2 \circ 1 \circ f \circ 1)(a) &= \bigvee_{(u,v) \in A_a} \min\{(1 \circ f^2 \circ 1 \circ f)(u), 1(v)\}\\ &\geqslant \min\{(1 \circ f^2 \circ 1 \circ f)(zxa^2yta), 1(h)\}\\ &= (1 \circ f^2 \circ 1 \circ f)(zxa^2yta).\end{aligned}$$

Since $(zxa^2yt, a) \in A_{zxa^2yta}$, we have $A_{zxa^2yta} \neq \emptyset$ and

$$\begin{aligned}(1 \circ f^2 \circ 1 \circ f)(zxa^2yta) &= \bigvee_{(u,v) \in A_{zxa^2yta}} \min\{(1 \circ f^2 \circ 1)(u), f(v)\}\\ &\geqslant \min\{(1 \circ f^2 \circ 1)(zxa^2yt), f(a)\}.\end{aligned}$$

Since $(zxa, ayt) \in A_{zxa^2yt}$, we have $A_{zxa^2yt} \neq \emptyset$ and

$$\begin{aligned}(1 \circ f^2 \circ 1)(zxa^2yta) &= \bigvee_{(u,v)\in A_{zxa^2yt}} \min\{(1 \circ f^2)(u), 1(v)\} \\ &\geqslant \min\{(1 \circ f^2)(zxa), 1(ayt)\} \\ &= (1 \circ f^2)(zxa).\end{aligned}$$

Since $(zxa, a) \in A_{zxa^2}$, we have $A_{zxa^2} \neq \emptyset$ and

$$\begin{aligned}(1 \circ f^2)(zxa) &= \bigvee_{(u,v)\in A_{zxa^2}} \min\{(1 \circ f)(u), f(v)\} \\ &\geqslant \min\{(1 \circ f)(zxa), f(a)\}.\end{aligned}$$

Since $(zx, a) \in A_{zxa}$, we have $A_{zxa} \neq \emptyset$ and

$$\begin{aligned}(1 \circ f)(zxa) &= \bigvee_{(u,v)\in A_{zxa}} \min\{1(u), f(v)\} \\ &\geqslant \min\{1(zx), f(a)\} \\ &= f(a).\end{aligned}$$

Thus we have

$$\begin{aligned}(1 \circ f^2 \circ 1 \circ f \circ 1)(a) &\geqslant (1 \circ f^2 \circ 1 \circ f)(zxa^2yta) \\ &\geqslant \min\{(1 \circ f^2 \circ 1)(zxa^2yt), f(a)\} \\ &\geqslant \min\{(1 \circ f^2)(zxa), f(a)\} \\ &\geqslant \min\{\min\{(1 \circ f)(zxa), f(a)\}, f(a)\} \\ &= \min\{f(a), f(a)\} \\ &= f(a).\end{aligned}$$

($\Leftarrow$). Let $a \in S$. Since $f_a$ and $1 \circ f_a^2 \circ 1 \circ f_a \circ 1$ are fuzzy sets in $S$, by hypothesis, we have $1 = f_a(a) \leqslant (1 \circ f_a^2 \circ 1 \circ f_a \circ 1)(a) \leqslant 1$, then $(1 \circ f_a^2 \circ 1 \circ f_a \circ 1)(a) = 1$. By Lemma 1, there exists $(x, y) \in A_a$ such that $(1 \circ f_a^2)(x) \neq 0$ and $(1 \circ f_a \circ 1)(y) \neq 0$. Since $(1 \circ f_a^2)(x) \neq 0$, by Lemma 3, there exists $(z, t) \in A_x$ such that $f_a^2(t) \neq 0$. Since $(1 \circ f_a \circ 1)(y) \neq 0$, by Lemma 3, there exists $(u, v) \in A_y$ such that $(f_a \circ 1)(v) \neq 0$. Since $f_a^2(t) \neq 0$, by Lemma 1, there exists $(h, k) \in A_t$ such that $f_a(h) \neq 0$, $f_a(k) \neq 0$. Since $(f_a \circ 1)(v) \neq 0$, by Lemma 2, there exists $(g, w) \in A_v$ such that $f_a(g) \neq 0$. We have

$$a \leqslant xy \leqslant (zt)(uv) \leqslant z(hk)uv \leqslant z(hk)u(gw) \text{ and } h = k = g = a.$$

Then $a \leqslant zhkugw = za^2uaw \in Sa^2S \cap SaSaS$, so $a \in (Sa^2S]$ and $a \in (SaSaS]$ which means that $S$ is intra-regular and semisimple. □

For a second proof of Theorems 4 and 6 we need the following lemmas.

**Lemma 7.** [4] *An ordered semigroup $S$ is intra-regular if and only if for any fuzzy subset $f$ of $S$ we have $f \preceq 1 \circ f^2 \circ 1$.*

**Lemma 8.** [2] *An ordered semigroup $S$ is left (resp. right) quasi-regular if and only if for any fuzzy subset $f$ of $S$ we have $f \preceq 1 \circ f \circ 1 \circ f$ (resp. $f \preceq f \circ 1 \circ f \circ 1$).*

**Lemma 9.** [2] *An ordered semigroup $S$ is semisimple if and only if for any fuzzy subset $f$ of $S$ we have $f \preceq 1 \circ f \circ 1 \circ f \circ 1$.*

*Proof of Theorem* 4. $(\Rightarrow)$. Let $f$ be a fuzzy subset of $S$. Since $S$ is intra-regular, by Lemma 7, we have $f \preceq 1 \circ f^2 \circ 1$. Since $S$ is left quasi-regular, by Lemma 8, we have $f \preceq 1 \circ f \circ 1 \circ f$. Thus we have

$$f \preceq 1 \circ f \circ 1 \circ f \preceq 1 \circ (1 \circ f^2 \circ 1) \circ 1 \circ f = 1 \circ f^2 \circ 1 \circ f.$$

$(\Leftarrow)$. By hypothesis, for any fuzzy subset $f$ of $S$, we have

$$f \preceq 1 \circ f^2 \circ 1 \circ f \preceq 1 \circ f^2 \circ 1, \ \ 1 \circ f \circ 1 \circ f.$$

By Lemmas 7 and 8, $S$ is intra-regular and left quasi-regular. □

*Proof of Theorem* 6. $(\Rightarrow)$. By Lemmas 7 and 9, for any fuzzy subset $f$ of $S$, we have $f \preceq 1 \circ f^2 \circ 1$ and $f \preceq 1 \circ f \circ 1 \circ f \circ 1$, then

$$f \preceq 1 \circ (1 \circ f^2 \circ 1) \circ 1 \circ f \circ 1 = 1 \circ f^2 \circ 1 \circ f \circ 1.$$

$(\Leftarrow)$. For any fuzzy subset $f$ of $S$, by hypothesis, we have

$$f \preceq 1 \circ f^2 \circ 1 \circ f \circ 1 \preceq 1 \circ f^2 \circ 1, \ \ 1 \circ f \circ 1 \circ f \circ 1.$$

By Lemmas 7 and 9, $S$ is intra-regular and semisimple. □

# References


[1] **N. Kehayopulu,** *Left quasi-regular and intra-regular ordered semigroups using fuzzy ideals*, Quasigroups and Related Systems **20** (2012), 239 – 249.

[2] **N. Kehayopulu,** *Characterization of left quasi-regular and semisimple ordered semigroups in terms of fuzzy sets*, Int. J. Algebra **6** (2012), 747 – 755.

[3] **N. Kehayopulu,** *On fuzzy ordered semigroups*, Quasigroups and Related Systems **20** (2012), 251 – 256.

[4] **N. Kehayopulu and M. Tsingelis,** *Characterization of some types of ordered semigroups in terms of fuzzy sets*, Lobachevskii J. Math. **29** (2008), 14 – 20.

[5] **N. Kehayopulu and M. Tsingelis,** *On fuzzy ordered semigroups*, Quasigroups and Related Systems **20** (2012), 61 – 70.

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