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Zariski-topology for co-ideals of commutative semirings

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Abstract. Let R be a semiring and *co-spec*(R) be the collection of all prime strong co-ideals of R. In this paper, we introduce and study a generalization of the Zariski topology of ideals in rings to co-ideals of semirings. We investigate the interplay between the algebraic-theoretic properties and the topological properties of *co-spec*(R). Semirings whose Zariski topology is respectively T_1, Hausdorff or cofinite are studied, and several characterizations of such semirings are given.

1. Introduction

As a generalization of rings, semirings have been found useful for solving problems in different areas of applied mathematics and information sciences, since the structure of a semiring provides an algebraic framework for modelling and studying the key factors in these applied areas. Let R be a commutative ring with identity. The prime spectrum $spec(R)$ and the topological space obtained by introducing Zariski topology on the set of prime ideals of R play an important role in the ideals of commutative algebra, algebraic geometry and lattice theory. Also, recently the notion of prime submodules and Zariski topology on $spec(M)$, the set of all prime submodules of a module M over R, are studied by many authors. In this paper, we concentrate on Zariski topology for co-ideals of semirings and generalize the some well known results of Zariski topology on the sets of prime ideals of a commutative ring to the sets of prime strong co-ideals of a commutative semiring and investigate the basic properties of this topology. For example, we prove that if R is a $*$-semiring, then *co-spec*(R) is a T_0-space; it is a compact space; the quasi-compact open subsets of its are closed under finite intersection and it is a sober space. Consequently, it is a spectral space. Equivalently, it is homeomorphic to $spec(S)$, with the Zariski topology, for some commutative ring S.

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2. Preliminaries

In order to make this paper easier to follow, we recall in this section various notions from topology theory and co-ideals theory of commutative semirings which will be used in the sequel. A *commutative semiring* R is defined as an algebraic system $(R, +, .)$ such that $(R, +)$ and $(R, cdot)$ are commutative semigroups, connected by $a(b+c) = ab+ac$ for all $a, b, c \in R$, and there exists $0, 1 \in R$ such that $r+0 = r$ and $r0 = 0r = 0$ and $r1 = 1r = r$ for each $r \in R$. In this paper all semirings considered will be assumed to be commutative with non-zero identity.

Let R be a semiring. A non-empty subset I of R is called *co-ideal*, if it is closed under multiplication and satisfies the condition $r + a \in I$ for all $a \in I$ and $r \in R$. A co-ideal I in R is called *strong* provided that $1 \in I$. (Clearly, $0 \in I$ if and only if $I = R$) [4, 7, 8, 10]. A strong co-ideal I of R is called *subtractive* if $x, xy \in I$, then $y \in I$ [7]. A proper strong co-ideal P of R is *prime* if $x + y \in P$, then $x \in P$ or $y \in P$. The notation *co-spec*(R) denotes the set of all prime strong co-ideals of R. A proper strong co-ideal I of R is said to be *maximal* if J is a strong co-ideal in R with $I \subseteq J$ and $I \neq J$, then $J = R$. If D is an arbitrary nonempty subset of R, then the set $F(D)$ consisting of all elements of R of the form $d_1 d_2 \cdots d_n + r$ (with $d_i \in D$ for all $1 \leqslant i \leqslant n$ and $r \in R$) is a co-ideal of R containing D [8, 10].

We need the following propositions, proved in [7].

Proposition 2.1. *Let R be a semiring. Then any proper co-ideal of R is contained in a maximal co-ideal of R. Moreover, any maximal co-ideal of R is a prime and subtractive strong co-ideal of R.* □

A topological space X is called *irreducible* if $X \neq \emptyset$ and every finite intersection of non-empty open sets of X is non-empty. A (non-empty) subset Y of a topology space X is an *irreducible set* if the subspace Y of X is irreducible. For this to be so, it is necessary and sufficient that, for every pair of sets Y_1, Y_2 which are closed in X and satisfy $Y \subseteq Y_1 \cup Y_2$, then $Y \subseteq Y_1$ or $Y \subseteq Y_2$.

Let Y be a closed subset of a topological space. An element $y \in Y$ is called a *generic point* of Y if $Y = \overline{\{y\}}$. Note that a generic point of the irreducible closed subset Y of a topological space is unique if the topological space is a T_0-space.

The *cofinite topology* (sometimes called the *finite complement topology*) is a topology which can be defined on every set X. It has precisely the empty set and all cofinite subsets of X as open sets. As a consequence, in the cofinite topology, the only closed subsets are finite sets, or the whole of X. Then X is automatically compact in this topology, since every open set only omits finitely many points of X. Also, the cofinite topology is the smallest topology satisfying the T_1 axiom; i.e., it is the smallest topology for which every singleton set is closed. If X is not finite, then this topology is not Hausdorff.

Following Hochster [9], we say that a topological space X is a *spectral space* in case X is homeomorphic to *spec*(S), with the Zariski topology, for some commutative ring S. Spectral spaces have been characterized by Hochster [9] as the topological spaces X which is a quasi-compact T_0-space such that the quasi-compact open subsets of X are closed under finite intersection and each its irreducible closed subset has a generic point, i.e., X is a *sober space*.

3. Strong co-ideals and Zariski topology

Let R be a semiring with non-zero identity. For any subset E of R by $V(E)$ we mean the set of all prime strong co-ideals of R containing E.

Lemma 3.1. *Let R be a semiring. Then $V(R) = \emptyset$ and $V(F(\{1\})) = co - spec(R)$.*

Proof. This follows directly from definitions. □

Lemma 3.2. *Let P be a prime strong co-ideal of a semiring R. If I and J are co-ideals of R such that $I + J \subseteq P$, then $I \subseteq P$ or $J \subseteq P$.*

Proof. It suffices to show that if $I + J \subseteq P$ and $I \not\subseteq P$, then $J \subseteq P$. Let $b \in J$. By assumption, there exists $a \in I$ such that $a \notin P$. As $a + b \in P$, P prime gives $b \in P$, as needed. □

Proposition 3.3. *Let R be a semiring.*

(1) *If E is a subset of R, then $V(E) = V(F(E))$.*
(2) *If I and J are co-ideals of R with $I \subseteq J$, then $V(J) \subseteq V(I)$.*
(3) *If I and J are co-ideals of R, then $V(I + J) = V(J) \cup V(I)$.*
(4) *If $\{I_i\}_{i\in\Gamma}$ is a family of co-ideals of R, then $V(F(\bigcup_{i\in\Gamma} I_i)) = \bigcap_{i\in\Gamma} V(I_i)$.*

Proof. (1). Assume that $P \in V(E)$ (so $E \subseteq P$) and let $r + s_1 \cdots s_n \in F(E)$ where $s_1, \ldots, s_n \in E$ and $r \in R$. Since $s_1, \ldots, s_n \in E \subseteq P$, we must have $s_1 \cdots s_n \in P$; hence $r + s_1 \cdots s_n \in P$ since P is a co-ideal. Therefore $F(E) \subseteq P$, and so $P \in V(F(E))$. Thus $V(E) \subseteq V(F(E))$. For the reverse inclusion, assume that $P \in V(F(E))$. Since $E \subseteq F(E) \subseteq P$, we get $P \in V(E)$, and so we have equality.

(2). is clear.

(3). Let $P \in V(I + J)$. By Lemma 3.2, either $I \subseteq P$ or $J \subseteq P$. This implies that $P \in V(I) \cup V(J)$; hence $V(I+J) \subseteq V(J) \cup V(I)$. Since I and J are co-ideals, we have $I + J \subseteq I$ and $I + J \subseteq J$; thus $V(J) \cup V(I) \subseteq V(I + J)$ by (2). Therefore, $V(I + J) = V(J) \cup V(I)$.

(4). By (1), it suffices to show that $V(\bigcup_{i\in\Gamma} I_i) = \bigcap_{i\in\Gamma} V(I_i)$. Consider an arbitrary $P \in \bigcap_{i\in\Gamma} V(I_i)$. Then for each $i \in \Gamma$, $I_i \subseteq P$. Thus $\bigcup_{i\in\Gamma} I_i \subseteq P$. Therefore $P \in V(\bigcup_{i\in\Gamma} I_i)$. For the reverse inclusion, let $P \in V(\bigcup_{i\in\Gamma} I_i)$. From $I_i \subseteq \bigcup_{i\in\Gamma} I_i$ and $P \in V(\bigcup_{i\in\Gamma} I_i)$, we have $P \in V(I_i)$ for each $i \in \Gamma$. Therefore $V(\bigcup_{i\in\Gamma} I_i) \subseteq \bigcap_{i\in\Gamma} V(I_i)$. Hence $V(\bigcup_{i\in\Gamma} I_i) = \bigcap_{i\in\Gamma} V(I_i)$. □

Let R be a semiring. If $\xi(R)$ denotes the collection of all subsets $V(I)$ of *co-spec*(R), then $\xi(R)$ contains the empty set and *co-spec*$(R) = X$ and is closed under arbitrary intersection and finite union by Proposition 3.3. Thus $\xi(R)$ satisfies the axioms of closed subsets of a topological spaces, which is called the *Zariski-topology* for co-ideals of commutative semirings.

Let I be a co-ideal of R. Put

$$\textit{co-rad}(I) = \{x \in R \mid nx \in I \ for \ some \ n \in \mathbb{N}\}$$

and

$$\textit{co-rad}(R) = \{x \in R \mid nx \in F(\{1\}) \ for \ some \ n \in \mathbb{N}\}.$$

We will denote the closure of Y in *co-spec*(R) by $\overline{Y}$, and intersections of elements of Y by $\mathcal{T}(Y)$.

Proposition 3.4. *Let R be a semiring.*

(1) *If I is a co-ideal of R, then $V(I) = V(\textit{co-rad}(I))$.*

(2) *If I is a co-ideal of R, then $V(I) = V(\mathcal{T}(V(I)))$.*

(3) *If I and J are co-ideals of R with $V(I) \subseteq V(J)$, then $J \subseteq \mathcal{T}(V(I))$.*

(4) *$V(I) = V(J)$ if and only if $\mathcal{T}(V(I)) = \mathcal{T}(V(J))$ for each co-ideals I and J of R.*

Proof. (1). Since $I \subseteq$*co-rad*(I), $V(\textit{co-rad}(I)) \subseteq V(I)$ by Proposition 3.3. For the reverse inclusion, assume that $P \in V(I)$. If $x \in$*co-rad*(I), then $nx \in I$ for some $n \in \mathbb{N}$. Since $I \subseteq P$, $nx \in P$, consequently $x \in P$. Thus *co-rad*$(I) \subseteq P$ and so $V(I) \subseteq V(\textit{co-rad}(I))$. Hence $V(\textit{co-rad}(I)) = V(I)$.

(2). As $I \subseteq \mathcal{T}(V(I))$, we have $V(\mathcal{T}(V(I))) \subseteq V(I)$. Conversely, let $P \in V(I)$, hence $\mathcal{T}(V(I)) = \bigcap_{Q \in V(I)} Q \subseteq P$. Therefore we have $V(I) \subseteq V(\mathcal{T}(V(I)))$, and so $V(\mathcal{T}(V(I))) = V(I)$.

(3). Let I and J be co-ideals of R and $V(I) \subseteq V(J)$. Therefore we obtain $\mathcal{T}(V(J)) \subseteq \mathcal{T}(V(I))$. Since $J \subseteq \mathcal{T}(V(J))$, $J \subseteq \mathcal{T}(V(I))$.

(4). Let $V(I) = V(J)$. By (2), we have $V(I) = V(\mathcal{T}(V(J)))$; hence we get $\mathcal{T}(V(J)) \subseteq \mathcal{T}(V(I))$. Similarly, the reverse inclusion is hold. The converse implication is clear. □

Let $X =$*co-spec*(R). For each subset E of R, by $D(E)$ we mean $X - V(E) = \{P \in X \mid E \not\subseteq P\}$. If $E = \{f\}$, then by X_f we denote the set $\{P \in X \mid f \notin P\}$.

Theorem 3.5. *Let R be a semiring. Then $\mathcal{A} = \{X_f \mid f \in R\}$ forms a base for Zariski topology for co-ideals of R.*

Proof. Let U be an open set. Then $U = X - V(I)$ for some co-ideal I of R. Let $P \in U$. Then $I \not\subseteq P$, so there exists $f \in I$ such that $f \notin P$; hence $P \in X_f$. We claim that $X_f \subseteq U$. Let $Q \in X_f$. Then $f \notin Q$, so $I \not\subseteq Q$; thus $Q \in U$. Hence $X_f \subseteq U$. Therefore $\mathcal{A}$ is a base for Zariski topology on X. □

Proposition 3.6. *Let R be a semiring and $X = \bigcup_{i \in \Gamma} X_{a_i}$. If $I = F(\{a_i\}_{i \in \Gamma})$, then $I = R$.*

Proof. Suppose that $I \neq R$. Then there exists a maximal co-ideal P of R such that $I \subseteq P$ by Proposition 2.1. Since $P \in X$, there exists $i \in \Gamma$ such that $a_i \notin P$, a contradiction with $I \subseteq P$. Hence $I = R$. □

Theorem 3.7. *Let R be a semiring. Then the following statements are hold.*

(1) $X_f \cap X_g = X_{f+g}$ *for each* $f, g \in R$.

(2) $X_f = X$ *if and only if* f^n *has additive inverse for some* $n \in \mathbb{N}$.

(3) $X_f = \emptyset$ *if and only if* $f \in P$ *for each* $P \in$*co-spec*(R) (*or equivalently,* $f \in \mathcal{T}(V(\{1\}))$).

Proof. (1). If $P \in X_f \cap X_g$, then $f \notin P$ and $g \notin P$; hence $f + g \notin P$. Thus $X_f \cap X_g \subseteq X_{f+g}$. For the reverse inclusion, let $P \in X_{f+g}$. Then $f + g \notin P$, so $f \notin P$ and $g \notin P$. Therefore $P \in X_f \cap X_g$, and we have equality.

(2). Let $X_f = X$. By Proposition 3.6, $R = F(\{f\})$. Therefore $f^n + r = 0$ for some $n \in \mathbb{N}$ and $r \in R$. Conversely, assume that f^n has inverse for some $n \in \mathbb{N}$. We show that $X_f = X$. If $P \in X$ and $P \notin X_f$, then $f \in P$. It follows that $0 \in P$; hence $P = R$, which is a contradiction. Thus $X = X_f$.

(3). It is clear that $X_f = \emptyset$ if and only if $f \in P$ for each $P \in$*co-spec*(R). □

Proposition 3.8. *Let I be a strong co-ideal of semiring R. Then* $D(I) = \bigcup_{a \in I} X_a$. *In particular, if* $I = F(\{a_1, \ldots, a_n\})$, *then* $D(I) = \bigcup_{i=1}^{n} X_{a_i}$.

Proof. Let $P \in D(I)$. So $I \nsubseteq P$. Thus there exists $a \in I$ such that $a \notin P$; hence $P \in X_a$. Therefore, $P \in \bigcup_{a \in I} X_a$, and so $D(I) \subseteq \bigcup_{a \in I} X_a$. Conversely, assume that $P \in \bigcup_{a \in I} X_a$. Then $P \in X_a$ for some $a \in I$. Since $a \notin P$, $I \nsubseteq P$. Hence $P \in D(I)$ and so the equality is hold. The "in particular" statement is clear. □

Theorem 3.9. *Let R be a semiring. Then $X =$co-spec(R) is a compact space.*

Proof. Let $X = \bigcup_{i \in \Gamma} X_{a_i}$. By Proposition 3.6, $F(\{a_i\}_{i \in \Gamma}) = R$; hence $0 = r + a_1 \cdots a_n$ for some $a_1, \ldots, a_n \in \{a_i\}_{i \in \Gamma}$. We claim that $X \subseteq \bigcup_{i=1}^{n} X_{a_i}$. Let $P \in X$. If for each $1 \leqslant i \leqslant n$, $a_i \in P$, then $a_1 \cdots a_n \in P$, and so $0 = r + a_1 \cdots a_n \in P$ which is a contradiction. Therefore there exists $1 \leqslant i \leqslant n$ such that $a_i \notin P$. Hence $P \in X_{a_i}$, as desired. □

Definition 3.10. A semiring R is called $*$*-semiring* if *co-rad*$(I) = \mathcal{T}(V(I))$ for each proper strong co-ideal I of R.

Example 3.11. (1) Let $R = (\mathbb{Z}^+, +, \times)$. Then the only strong co-ideals of R is $I_1 = \{n \in \mathbb{Z}^+ \mid 1 \leqslant n\}$ and $\mathbb{Z}^+$. Also the only prime strong co-ideals of R is I_1. Therefore, R is a $*$-semiring.

(2) Let Y={a,b,c} and $S = (P(Y), \cup, \cap)$ a semiring, where $P(Y)$ is the family of all subsets of Y. An inspection will show that S is a $*$-semiring.

(3) Let $T = (\mathbb{Z}^+ \cup \{\infty\}, \max, \min)$. An inspection will show that the list of strong co-ideals of T are T, $I_n = \{k \mid k \geqslant n\}$. It is clear that each proper strong co-ideal of T is prime and T is a $*$-semiring. □

The following example shows that a semiring need not be a $*$-semiring.

Example 3.12. Let $R = \{0, 1, 2, 3, 4, 5\}$. Define

$$a + b = \begin{cases} 5 & \text{if } a \neq 0,\ b \neq 0,\ a \neq b, \\ a & \text{if } a = b, \\ b & \text{if } a = 0, \\ a & \text{if } b = 0, \end{cases}$$

and

$$a * b = \begin{cases} 0 & \text{if } a = 0 \text{ or } b = 0, \\ 2 & \text{if } a = b = 3, \\ b & \text{if } a = 1, \\ a & \text{if } b = 1, \\ 5 & \text{otherwise.} \end{cases}$$

Then $(R, +, *)$ is easily checked to be a commutative semiring. Suppose that $I = \{1, 4, 5\}$. It is clear that I is a strong co-ideal of R and $V(I) = \{P_1, P_2\}$, where

$$P_1 = \{1, 2, 4, 5\}, \quad P_2 = \{1, 2, 3, 4, 5\}.$$

Hence $\mathcal{T}(V(I)) = P_1$. It can be seen $\mathcal{T}(V(I)) \neq$*co-rad*(I) because $2 \in \mathcal{T}(V(I))$ and $2 \notin$ *co-rad*(I). Therefore R is not $*$-semiring. □

Theorem 3.13. *Let R be a $*$-semiring. For every $a \in R$, the set X_a is compact. Specifically, the whole space $X_0 = X$ is compact.*

Proof. Assume that $X_a \subseteq \bigcup_{i \in \Gamma} X_{b_i}$ and let $I = F(\{b_i\}_{i \in \Gamma})$. We claim that $V(I) \subseteq V(\{a\})$. Assume that $P \in V(I)$, so $I \subseteq P$; hence $P \notin \bigcup_{i \in \Gamma} X_{b_i}$. Since $X_a \subseteq \bigcup_{i \in \Gamma} X_{b_i}$, $P \notin X_a$. This implies that $a \in P$. Therefore $V(I) \subseteq V(\{a\})$. It follows that $a \in \mathcal{T}(V(I))$. As R is $*$-semiring, $a \in$*co-rad*(I). Therefore $na \in I$ for some $n \in \mathbb{N}$. Hence $na = b_{i_1} \cdots b_{i_n} + r$ for some $b_{i_j} \in \{b_i\}_{i \in \Gamma}$, $r \in R$. We show that $X_a \subseteq \bigcup_{j=1}^{n} X_{b_{i_j}}$. Let $P \in X_a$ (so $a \notin P$). If for each $1 \leqslant j \leqslant n$, $b_{i_j} \in P$, then $na = b_{i_1} \cdots b_{i_n} + r \in P$, consequently $a \in P$, a contradiction. Therefore there exists $1 \leqslant j \leqslant n$ such that $b_{i_j} \notin P$. Hence $P \in \bigcup_{j=1}^{n} X_{b_{i_j}}$. Thus $X_a \subseteq \bigcup_{j=1}^{n} X_{b_{i_j}}$. □

Corollary 3.14. *Let R be a $*$-semiring. Then an open subset of $X =$ co-spec(R) is compact if and only if it is a finite union of basic open sets.*

Proof. Apply Theorem 3.5 and Theorem 3.13. □

Theorem 3.15. *Let R be a semiring. Then the toplologic space $X =$ co-spec(R) is a T_0-space.*

Proof. Let $P, Q \in X$ and $P \neq Q$. We note that the set X_a is a neighborhood of P if and only if $a \notin P$. Suppose that $Q \in X_b$ for all $b \notin P$. Then we conclude that $b \in Q$ implies that $b \in P$; hence $Q \subset P$. Now let $c \in P - Q$. Then $c \notin Q$ gives X_c is a neighborhood of Q, but $c \in P$, so $P \notin X_c$. This completes the proof. □

Definition 3.16. A semiring R is called *p-subtractive* if every prime strong co-ideal of R is subtractive.

Example 3.17. (1) Let Y={a,b,c} and $R = (P(Y), \cup, \cap)$ a semiring, where $P(Y)$= the set of all subsets of Y. An inspection will shows that *co-spec*$(R) = \{P_1, P_2, P_3\}$, where

$$P_1 = \{\{a\}, \{a, b\}, \{a, c\}, X\},$$

$$P_2 = \{\{b\}, \{a, b\}, \{b, c\}, X\},$$

$$P_3 = \{\{c\}, \{a, c\}, \{b, c\}, X\}.$$

Since P_1, P_2 and P_3 are maximal co-ideal, they are subtractive by Proposition 2.1. Hence R is a p-subtractive semiring.

(2) Let $S = (\mathbb{Z}^+, +, \times)$. Then $P = S - \{0\}$ is the only prime co-ideal of S which is subtractive. Hence S is a p-subtractive semiring. □

Theorem 3.18. *Let R be a p-subtractive semiring. If the only elements of R such that $a + b \in P$ and $ab \notin P$ for each $P \in$ co-spec(R) are $0, 1$, then $X =$ co-spec(R) is connected.*

Proof. Suppose that X is not connected. Let $X = X_a \cup X_b$ and $X_a \cap X_b = \emptyset$ for some $a, b \in R$. Since $X_a \cap X_b = \emptyset$, $X_{a+b} = \emptyset$ by Theorem 3.7. Thus $a + b \in P$ for all $P \in$ *co-spec*(R) by Theorem 3.7. We claim that $X_{ab} = X$. Let $P \in X$ and $ab \in P$. Since $X_{a+b} = \emptyset$, $a + b \in P$, therefore $a \in P$ or $b \in P$. As P is subtractive and $ab \in P$, $P \notin X_a \cup X_b$. This contradicts our hypothesis that $X = X_a \cup X_b$. Therefore $ab \notin P$ and $X_{ab} = X$. Hence $ab \notin P$ for all $P \in X$ by Theorem 3.7. Hence $\{a, b\} = \{0, 1\}$. Thus X is connected. □

Example 3.19. (1) Let Y={a,b,c} and $R = (P(Y), \cup, \cap)$ be a semiring, where $P(Y)$ is the collection of all subsets of Y. Then *co-spec*$(R) = X_{\{a\}} \bigcup X_{\{b,c\}}$ and $X_{\{a\}} \bigcap X_{\{b,c\}} = \emptyset$. Therefore *co-spec*$(R)$ is not connected.

(2) Let $T = (\mathbb{Z}^+ \cup \{\infty\}, \max, \min)$ and $I_i = \{n \in T \,|\, n \geqslant i\}$. It is clear that I_i is a prime strong co-ideal of T for each $i \in \mathbb{N}$. Then for each $n \in T$, $X_n = \{I_i \,|\, i \geqslant n + 1\}$. Therefore $X_0 \supseteq X_1 \supseteq \cdots \supseteq X_\infty$. This implies that *co-spec*(T) is connected. □

Theorem 3.20. *Let R be a semiring. Then co-spec(R) is irreducible if and only if $\mathcal{T}(V(\{1\}))$ is a prime strong co-ideal.*

Proof. Let *co-spec*(R) be irreducible, and $a + b \in \mathcal{T}(V(\{1\}))$ for some $a, b \in R$. Then $X_{a+b} = X_a \cap X_b = \emptyset$ by Theorem 3.7. Since *co-spec*(R) is irreducible, $X_a = \emptyset$ or $X_b = \emptyset$. Thus $a \in \mathcal{T}(V(\{1\}))$ or $b \in \mathcal{T}(V(\{1\}))$. Therefore $\mathcal{T}(V(\{1\}))$ is prime.

Conversely, let $\mathcal{T}(V(\{1\}))$ be prime; we show that *co-spec*(R) is irreducible. If $X_a \cap X_b = \emptyset$, then by Theorem 3.7, $X_{a+b} = \emptyset$. Hence $a + b \in \mathcal{T}(V(\{1\}))$. As $\mathcal{T}(V(\{1\}))$ is prime, $a \in \mathcal{T}(V(\{1\}))$ or $b \in \mathcal{T}(V(\{1\}))$. Thus $X_a = \emptyset$ or $X_b = \emptyset$. Therefore, *co-spec*(R) is irreducible. □

Proposition 3.21. *Let R be a semiring and $P, Q \in X =$ co-spec(R). Then:*

(1) $\overline{\{P\}} = V(P)$ *for each* $P \in$ *co-spec*(R),

(2) $Q \in \overline{\{P\}}$ *if and only if* $P \subseteq Q$,

(3) $\{P\}$ *is closed in* X *if and only if* P *is a maximal co-ideal of* R.

Proof. (1). As $\overline{\{P\}} = \bigcap_{P \in V(I)} V(I)$, and $P \in V(P)$, we have $\overline{\{P\}} \subseteq V(P)$. On the other hand, if $Q \in V(P)$, then $P \subseteq Q$. Thus $Q \in V(I)$ for each $I \subseteq P$. Hence $Q \in \overline{\{P\}}$. Therefore $\overline{\{P\}} = V(P)$.

(2) is a consequence of (1), (3) is a consequence of (2). □

Theorem 3.22. *Let R be a semiring. Then X is a T_1-space if and only if each prime strong co-ideal is maximal.*

Proof. Let X be a T_1-space, then for each $P \in X$, $\{P\}$ is closed in X. Hence P is maximal strong co-ideal by Proposition 3.21. Conversely, assume that each prime strong co-ideal of R is maximal, then using Proposition 3.21 we see that each singleton $\{P\}$ is closed in X, for each $P \in X$. Hence X is a T_1-space. □

Let R be a semiring with $|$*co-spec*$(R)| \leqslant 1$. Then *co-spec*(R) is the trivial space and so it is a Hausdorff space. The following theorem gives a relation between Hausdorff axiom and T_1 axiom for Zariski-topology for co-ideals of semirings.

Theorem 3.23. *Let R be a semiring. If $X =$ co-spec(R) is a Hausdorff space, then it is a T_1-space.*

Proof. Let $P_1, P_2 \in X$. Since X is a Hausdorff space, there exist $a, b \in R$ such that $P_1 \in X_a$ and $P_2 \in X_b$ and $X_a \cap X_b = \emptyset$. Hence $X_{a+b} = \emptyset$. Therefore, $a + b \in P_1$ and $a + b \in P_2$. This implies that $a \in P_2$ and $b \in P_2$. Consequently, $P_1 \not\subseteq P_2$ and $P_2 \not\subseteq P_1$. Hence each prime strong co-ideal is maximal. Therefore, X is a T_1-space. □

It is well-known that if X is a finite space, then X is a T_1-space if and only if X is the discrete space. Thus we have the following Proposition.

Proposition 3.24. *For a semiring R with a finite $X =$ co-spec(R) the following conditions are equivalent:*

(1) *X is a Hausdorff space,*

(2) *X is a T_1-space,*

(3) *X has a cofinite topology,*

(4) *X is discrete,*

(5) *every prime co-ideal is maximal.* □

Lemma 3.25. *Let R be a semiring. Then for each $P \in$ co-spec(R), $V(P)$ is irreducible.*

Proof. Let $V(P) \subseteq Y_1 \cup Y_2$, where Y_1 and Y_2 are closed sets; so $P \in V(P)$ gives, $P \in Y_1$ or $P \in Y_2$. Let $P \in Y_1$. As $V(P) = \overline{\{P\}}$ by Proposition 3.21, we have $V(P) = \cap\{Y \mid P \in Y, \ Y \textit{is closed set}\} \subseteq Y_1$. Similarly, if $P \in Y_2$, then $V(P) \subseteq Y_2$. Hence $V(P)$ is irreducible. □

Theorem 3.26. *Let R be a semiring. Then $Y \subseteq$ co-spec(R) is irreducible if and only if $\mathcal{T}(Y)$ is a prime strong co-ideal.*

Proof. Let Y be irreducible and $a+b \in \mathcal{T}(Y)$. We claim that $Y \subseteq V(\{a\}) \cup V(\{b\})$. Let $P \in Y$. Since $Y \subseteq V(\mathcal{T}(Y))$ and $a+b \in \mathcal{T}(Y)$, $a+b \in P$. Hence $a \in P$ or $b \in P$. Therefore $Y \subseteq V(\{a\}) \cup V(\{b\})$. As Y is irreducible, $Y \subseteq V(\{a\})$ or $Y \subseteq V(\{b\})$. If $Y \subseteq V(\{a\})$, then $a \in \mathcal{T}(Y)$. Similarly, If $Y \subseteq V(\{b\})$, then $b \in \mathcal{T}(Y)$. Hence $\mathcal{T}(Y)$ is prime. Conversely, assume that $\mathcal{T}(Y)$ is a prime strong co-ideal. We show that Y is irreducible. Let $Y \subseteq Y_1 \cup Y_2$ for some closed subset Y_1 and Y_2 of *co-spec*(R). Thus $Y_1 = V(I_1)$ and $Y_2 = V(I_2)$ for some strong co-ideals I_1 and I_2. As $Y \subseteq V(I_1) \cup V(I_2)$, for each $P \in Y$, $I_1 \subseteq P$ or $I_2 \subseteq P$. Hence $I_1 + I_2 \subseteq P$ for each $P \in Y$. Thus $I_1 + I_2 \subseteq \mathcal{T}(Y)$. Since $\mathcal{T}(Y)$ is prime $I_1 \subseteq \mathcal{T}(Y)$ or $I_2 \subseteq \mathcal{T}(Y)$ by Lemma 3.2. Therefore $Y \subseteq Y_1$ or $Y \subseteq Y_2$, as needed. □

Theorem 3.27. *For every $*$-semiring R, co-spec(R) is spectral.*

Proof. Let R be a $*$-semiring. We show that $X =$ *co-spec*(R) is spectral in four steps.

1. X is a T_0-space by Theorem 3.15.
2. X is quasi-compact by Theorem 3.9.
3. The quasi-compact open subsets of X are closed under finite intersection by Corollary 3.14.
4. Let Y be an irreducible closed subset of X. Then $Y = V(I)$ for some strong co-ideal I of R. By Theorem 3.26, $P = \mathcal{T}(Y)$ is a prime strong co-ideal of R. An inspection will show that $V(P) = Y$. Since $\overline{\{P\}} = V(P) = Y$, $\{P\}$ is a generic point of Y. Thus X is spectral. □

Corollary 3.28. *Let R be a $*$-semiring, then $X =$ co-spec(R) is a T_1-space if and only if it is a Hausdorff space.*

Proof. By Theorem 3.27, *co-spec*(R) is homeomorphic to *spec*(S), with the Zariski topology, for some commutative ring S. By [1], *spec*(S) is a Hausdorff space if and only if it is T_1. Therefore X is Hausdorff if and only if it is T_1. □

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