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On classes of regularity in an ordered semigroup

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Abstract. Let m, n be nonnegative integers. An element a in an ordered semigroup $(S, \cdot, \leqslant)$ is said to be (m, n)*-regular* if there exists $x \in S$ such that $a \leqslant a^m x a^n$. This paper gives necessary and sufficient conditions for the set of all (m, n)-regular elements of S to be a subsemigroup of S. The results obtained extend the results on semigroups without order.

1. Introduction

Let S be a semigroup without order and m, n nonnegative integers. An element $a \in S$ is said to be (m, n)*-regular* [3] if there exists $x \in S$ such that $a = a^m x a^n$. Here, we let $a^0 x = x$ and $x a^0 = x$. In [4], the author investigated some sufficient conditions for classes of (m, n)-regularity to be subsemigroups of S. The purpose of this paper is to extend the results on semigroups without order to ordered semigroups.

The rest of this section we recall some definitions and results used throughout the paper.

A semigroup $(S, \cdot)$ together with a partial order $\leqslant$ (on S) that is *compatible* with the semigroup operation, meaning that for $x, y, z \in S$,

$$x \leqslant y \Rightarrow zx \leqslant zy, \quad xz \leqslant yz,$$

is called an *ordered semigroup* ([1], [5]). If A, B are nonempty subsets of S, we let

$$\begin{aligned} AB &= \{xy \in S \mid x \in A, y \in B\}, \\ (A] &= \{x \in S \mid x \leqslant a \text{ for some } a \in A\}. \end{aligned}$$

If $a \in S$, then we write Sa and aS instead of $S\{a\}$ and $\{a\}S$, respectively. It is well-known that the following conditions hold:

(1) $(S] = S$,

(2) $A \subseteq B$ implies $(A] \subseteq (B]$, and

(3) $((A]] = (A]$.

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Let $(S, \cdot, \leqslant)$ be an ordered semigroup and m, n nonnegative integers. An element $a \in S$ is said to be (m,n)-*regular* [10] if there exists $x \in S$ such that

$$a \leqslant a^m x a^n.$$

Here, we let $a^0 x = x$ and $x a^0 = x$. The set of all (m,n)-regular elements of S will be denoted by $R_S(m,n)$. The following conditions hold for nonnegative integers m, m_1, m_2, n, n_1, n_2:

(1) $R_S(0,0) = S$.

(2) If $m_1 \geqslant m_2$ and $n_1 \geqslant n_2$, then $R_S(m_1,n_1) \subseteq R_S(m_2,n_2)$.

(3) If $m_1 \geqslant m_2 \geqslant 2$, then $R_S(m_1,n) = R_S(m_2,n)$.

(4) If $n_1 \geqslant n_2 \geqslant 2$, then $R_S(m,n_1) = R_S(m,n_2)$.

(5) $R_S(1,2) = R_S(1,1) \cap R_S(0,2)$.

(6) $R_S(2,1) = R_S(1,1) \cap R_S(2,0)$.

A nonempty subset A of an ordered semigroup $(S, \cdot, \leqslant)$ is called a *left* (respectively, *right*) *ideal* [7] of S if

(i) $SA \subseteq A$ (respectively, $AS \subseteq A$);

(ii) for $x \in A$ and $y \in S$, $y \leqslant x$ implies $y \in A$.

If A is both a left and a right ideal of S, then A is called a (two-sided) *ideal* of S. The principal left (respectively, right) ideal of S containing $a \in S$, denoted by $L(a)$, is of the form $(a \cup Sa] := (\{a\} \cup Sa]$. Similarly, the principal right ideal of S containing $a \in S$ is of the form $R(a) := (a \cup aS]$.

It is easy to see for an ordered semigroup $(S, \cdot, \leqslant)$ that the following hold:

(1) If $R_S(1,0) \neq \emptyset$, then $R_S(1,0)$ is a left ideal of S.

(2) If $R_S(0,1) \neq \emptyset$, then $R_S(0,1)$ is a right ideal of S.

A nonempty subset A of an ordered semigroup $(S, \cdot, \leqslant)$ is called a *subsemigroup* of S if $AA \subseteq A$. It is clear that every left (respectively, right, two-sided) ideals of S is a subsemigroup of S.

2. Main Results

In [2], a left (respectively, right, two-sided) ideal A of an ordered semigroup $(S, \cdot, \leqslant)$ is said to be *complete* if $(SA] = A$ (respectively, $(AS] = A$, $(SAS] = A$). Since if A is a left ideal of S then $(SA] \subseteq A$, it follows that A is complete if $A \subseteq (SA]$. For complete right ideals and complete two-sided ideals of S can be considered similarly.

Theorem 2.1. *Let $(S, \cdot, \leqslant)$ be an ordered semigroup. Then $R_S(1,0)$ (respectively, $R_S(0,1)$) is nonempty if and only if at least one of the principal right (left) ideal of S is complete.*

Proof. Assume that $R_S(1,0)$ is nonempty. Then there exists $a \in R_S(1,0)$, that is $a \in (aS]$. We have

$$(a \cup aS] \subseteq (aS] \subseteq ((a \cup aS]S],$$

hence $(a \cup aS]$ is complete.

Conversely, assume that there exists $a \in S$ such that $(a \cup aS]$ is complete. Then

$$a \in (a \cup aS] = ((a \cup aS]S] \subseteq ((aS]] = (aS].$$

This proves that $a \in R_S(1,0)$, and so $R_S(1,0)$ is nonempty.

The second statement can be proved similarly. □

A left (respectively, right, two-sided) ideal A of an ordered semigroup $(S, \cdot, \leqslant)$ is said to be *semiprime* [6] if for $a \in A$ and any positive integer k, $a^k \in A$ implies $a \in A$.

Theorem 2.2. *Let $(S, \cdot, \leqslant)$ be an ordered semigroup. If at least one principal right (left) ideal of S generated by a^2 for some $a \in S$ is semiprime, then $R_S(2,0)$ (respectively, $R_S(0,2)$) is nonempty.*

Proof. Let $a \in S$ be such that $(a^2 \cup a^2S]$ is semiprime. Since $a^2 \in (a^2 \cup a^2S]$, we obtain $a \in (a^2 \cup a^2S]$, and so $a \leqslant a^2$ or $a \in (a^2S]$. Each of the cases implies that $a \in R_S(2,0)$.

The second statement can be proved analogously. □

Theorem 2.3. *Let $(S, \cdot, \leqslant)$ be an ordered semigroup. The class of regularity $R_S(1,1)$ (also $R_S(2,1), R_S(1,2), R_S(2,2)$) is nonempty if and only if S contains an element a such that $a \leqslant a^2$.*

Proof. Assume that $R_S(1,1)$ is nonempty. Then there exists $a \in (aSa]$. If $x \in S$ such that $a \leqslant axa$, then $ax \leqslant axax = (ax)^2$.

The opposite direction is clear. □

An ordered semigroup $(S, \cdot, \leqslant)$ is said to be *left* (respectively, *right*) *simple* [8] if S has no left (respectively, right) proper ideal. It is easy to see that S is left (respectively, right) simple if and only if $S = (Sa]$ for all $a \in S$ (respectively, $S = (aS]$ for all $a \in S$). Note that if S is left simple then $S = R_S(0,1)$.

Theorem 2.4. *Let $(S, \cdot, \leqslant)$ be an ordered semigroup such that $R_S(1,1)$ is nonempty. If (1), (2) or (3) holds, then $R_S(1,1)$ is a subsemigroup of S.*

(1) *If $a, b \in R_S(1,1)$, then $ab \leqslant (ab)^2$.*

(2) *$R_S(1,1) = R_S(1,0) \cap R_S(0,1)$.*

(3) *For* $a, b \in S$, $a \leqslant a^2$ *and* $b \leqslant b^2$ *imply* $ab = ba$.

Proof. Clearly, if (1) holds then $R_S(1,1)$ is a subsemigroup of S. Since $R_S(1,0)$ is a left ideal of S and $R_S(0,1)$ is a right ideal of S, it follows that $R_S(1,0) \cap R_S(0,1)$ is an ideal of S, and so this is a subsemigroup of S. Hence (2) holds.

Assume that (3) holds. Let $a, b \in R_S(1,1)$. Then there exist $x, y \in S$ such that $a \leqslant axa$ and $b \leqslant byb$. Since $xa \leqslant (xa)^2$ and $by \leqslant (by)^2$, we have $(xa)(by) = (by)(xa)$, and so

$$ab \leqslant a(xa)(by)b = a(by)(xa)b = (ab)(yx)(ab).$$

Thus $ab \in R_S(1,1)$. □

If $(S, \cdot, \leqslant)$ is an ordered semigroup, then the *center* of S is defined by

$$Z = \{a \in S \mid ax = xa \text{ for all } x \in S\}.$$

Theorem 2.5. *Let* $(S, \cdot, \leqslant)$ *be an ordered semigroup such that* $R_S(2,0)$ *is nonempty. If* $(1), (2)$ *or* (3) *holds, then* $R_S(2,0)$ *is a subsemigroup of* S.

(1) *If* $a, b \in R_S(2,0)$, *then* $ab \leqslant (ab)^2$.

(2) *For* $a, b \in R_S(2,0)$, *if* $a \leqslant a^2x$ *and* $b \leqslant b^2y$ *for some* $x, y \in S$, *then* $ab \leqslant (ab)(ax)(by)$.

(3) *For* $a \in R_S(2,0)$, *if* $a \leqslant a^2x$ *for some* $x \in S$, *then* $ax \in Z$.

Proof. Let (1) hold. If $a, b \in R_S(2,0)$, then $ab \leqslant (ab)^2$, and so $ab \leqslant (ab)^3$. Thus $ab \in R_S(2,0)$.

Assume that (2) holds. Let $a, b \in R_S(2,0)$. Then $a \leqslant a^2x$ and $b \leqslant b^2y$ for some $x, y \in S$. We have $ab \leqslant ab(ax)(by)$ and $ba \leqslant ba(by)(ax)$. Since

$$ab \leqslant ab(ax)(by) = a(ba)(xby) \leqslant a(ba(by)(ax))(xby) = (ab)^2(yax)(xby),$$

we get $ab \in R_S(2,0)$.

Finally, we assume that (3) holds. Let $a, b \in R_S(2,0)$ be such that $a \leqslant a^2x$ and $b \leqslant b^2y$ for some $x, y \in S$. Then $ax \in Z$, and so

$$ab \leqslant (a^2x)(b^2y) = a(ax)b(by) = ab(ax)(by).$$

This shows that the condition (2) holds, hence $R_S(2,0)$ is a subsemigroup of S. □

Analogous to Theorem 2.5, we have:

Theorem 2.6. *Let* $(S, \cdot, \leqslant)$ *be an ordered semigroup such that* $R_S(0,2)$ *is nonempty. If* $(1), (2)$ *or* (3) *holds, then* $R_S(0,2)$ *is a subsemigroup of* S.

(1) *If* $a, b \in R_S(0,2)$, *then* $ab \leqslant (ab)^2$.

(2) *For* $a, b \in R_S(0,2)$, *if* $a \leqslant xa^2$ *and* $b \leqslant yb^2$ *for some* $x, y \in S$, *then* $ab \leqslant (xa)(yb)(ab)$.

(3) *For* $a \in R_S(0,2)$, *if* $a \leqslant xa^2$ *for some* $x \in S,$, *then* $xa \in Z$.

Lemma 2.7. *The following holds for an ordered semigroup* $(S, \cdot, \leqslant)$:

$$R_S(2,2) = R_S(2,1) \cap R_S(1,2).$$

Proof. It is clear that $R_S(2,2) \subseteq R_S(2,1) \cap R_S(1,2)$. For the reverse inclusion, let $a \in R_S(2,1) \cap R_S(1,2)$. Then there exist $x, y \in S$ such that $a \leqslant a^2xa$ and $a \leqslant aya^2$. Since $a \leqslant a^2xaya^2$, $a \in R_S(2,2)$. □

Theorem 2.8. *Let* $(S, \cdot, \leqslant)$ *be an ordered semigroup such that for* $a \in S$, *if* $a \leqslant a^2$ *then* $a \in Z$. *If* $R_S(1,1)$ *(respectively,* $R_S(2,1), R_S(1,2), R_S(2,2)$*) is nonempty, then* $R_S(1,1)$ *(respectively,* $R_S(2,1), R_S(1,2), R_S(2,2)$*) is a subsemigroup of* S.

Proof. If $R_S(1,1)$ is nonempty, then by Theorem 2.4 we have $R_S(1,1)$ is a subsemigroup of S.

Assume that $R_S(2,1)$ is nonempty. Let $a, b \in R_S(2,1)$. Then there exist $x, y \in S$ such that $a \leqslant a^2xa$ and $b \leqslant b^2yb$. Since $a^2x, b^2y \in Z$, we have

$$\begin{aligned}
ab &\leqslant a^2xab^2yb = (a^2x)a(b^2y)b = (a^2x)(b^2y)(ab) = a(ax)b(by)(ab) \\
&\leqslant (a^2xa)(ax)(b^2yb)(by)(ab) = a(a^2x)(axb)(b^2y)(by)(ab) \\
&= a(a^2x)(ax)(b^2y)(b^2y)(ab) = a(b^2y)(a^2x)(ax)(b^2y)(ab) \\
&= (ab)(by)(a^2x)(ax)(b^2y)(ab) = (ab)(a^2x)(by)(ax)(b^2y)(ab) \\
&= (ab)(a^2x)(b^2y)(by)(ax)(ab) = (ab)a(ax)(b^2y)(by)(ax)(ab) \\
&= (ab)a(b^2y)(ax)(by)(ax)(ab) = (ab)^2(by)(ax)(by)(ax)(ab).
\end{aligned}$$

Therefore, $ab \in R_S(2,1)$. Similarly, if $R_S(1,2)$ is nonempty, then $R_S(1,2)$ is a subsemigroup of S.

By Lemma 2.7, if $R_S(2,2)$ is nonempty then $R_S(2,2)$ is a subsemigroup of S. □

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