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A study of n-subracks

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Abstract. In this paper, we introduce the notion of n-subracks ($n > 2$) and provide a characterization that enables us to obtain several results on n-racks. We also define a cohomology theory on n-racks.

1. Introduction

The category of n-racks [2] has been introduced as a generalization of the category of left distributive left quasigroups [9], or simply racks [6], and was shown to be associated to the category of Leibniz n-algebras [5]. In the pursue of studying the structure of this new category, we study in this paper the notion of n-subracks and explore several classical examples such as the normalizer, the center of a n-rack, and the components of a decomposable n-rack. In section 4, we provide several properties of decomposable n-racks.

In [8], Fenn, Rourke and Sanderson introduced a cohomology theory for racks which was modified in [4] by Carter, Jelsovsky, Kamada, Landford and Saito to obtain quandle cohomology, and several results have been recently established. In section 5, we use these cohomology theories to define cohomology theories on n-racks and n-quandles.

Let us recall a few definitions.

A *pointed rack* $(R, \circ, 1)$ is a set R with a binary operation $\circ$ and a specific element $1 \in R$ such that the following conditions are satisfied:

(R1) $x \circ (y \circ z) = (x \circ y) \circ (x \circ z)$.

(R2) For each $x, y \in R$, there exits a unique $a \in R$ such that $x \circ a = y$.

(R3) $1 \circ x = x$ and $x \circ 1 = 1$ for all $x \in R$.

A rack R is *decomposable* [1] if there are disjoints subracks X and Y of R such that $R = X \cup Y$. R is *indecomposable* if otherwise.

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2. n-racks

For the remaining of this paper, we assume $n \geqslant 2$, integer.

Definition 2.1. [2] A *n-rack* $(R, [\ldots])$ is a set R endowed with an n-ary operation $[\ldots] : R^n \longrightarrow R$ such that

(NR1) $\big[x_1, \ldots, x_{n-1}, [y_1, \ldots, y_{n-1}]\big] = \big[[x_1, \ldots, x_{n-1}, y_1], \ldots, [x_1, \ldots, x_{n-1}, y_n]\big]$
(This is the *left distributive property* of n-racks.)

(NR2) For $a_1, \ldots, a_{n-1}, b \in R$, there exists a unique $x \in R$ such that $[a_1, \ldots, a_{n-1}, x] = b$.

If in addition there is a distinguish element $1 \in R$, such that

(NR3) $[1, \ldots, 1, y] = y$ and $[x_1, \ldots, x_{n-1}, 1] = 1$ for all $x_1, \ldots, x_{n-1} \in R$,

then $(R, [\ldots], 1)$ is said to be a *pointed n-rack.*

An n-rack in which $[x_1, \ldots, x_{n-1}, y] = y$ if $x_i = y$ for some $i \in \{1, \ldots, n-1\}$, is an *n-quandle.*

Definition 2.2. A n-rack R is *involutive* if

$$\big[x_1, \ldots, x_{n-1}, [x_1, \ldots, x_{n-1}, y]\big] = y \quad \text{for all } x_1, \ldots, x_{n-1}, y \in R.$$

Note that an involutive n-quandle is an n-kei [2].

A n-rack R is *trivial* if it satisfies $[x_1, x_2, \ldots, x_{n-1}, y] = y$ for all $x_i, y \in R$.

For $n = 2$, one recovers involutive racks [1] and trivial racks [3].

Definition 2.3. Let K be a ring and M a K-module. Then M endowed with the n-ary operation $[\ldots]$ defined by

$$[x_1, \ldots, x_n] = q_1x_1 + q_2x_2 + \ldots + q_nx_n \quad \text{with} \quad \sum_{i=1}^{n} q_i = 1$$

is a n-rack called an *affine n-rack* associated to the K-module M.

Example 2.4. A $\mathbb{Z}_4$-module M endowed with the operation $[\ldots]_M$ defined by

$$[x_1, \ldots, x_n]_M = 2x_1 + 2x_2 + \ldots + 2x_{n-1} + x_n$$

is an affine n-rack if n is odd.

Proposition 2.5. [2] *Any pointed rack $(R, \circ, 1)$ has a pointed n-rack structure under the n-ary operation defined by*

$$[x_1, x_2, \ldots, x_n] = x_1 \circ (x_2 \circ (\ldots (x_{n-1} \circ x_n) \ldots)).$$

This process determines a functor $\mathfrak{G} : prack \longrightarrow {}_nprack$, which has as left adjoint, the functor $\mathfrak{G}' : {}_nprack \longrightarrow prack$ defined as follows:

Given a pointed n-rack $(R, [\ldots], 1)$, then R^{n-1} endowed with the binary operation

$$(x_1, \ldots, x_{n-1}) \circ (y_1, \ldots, y_{n-1}) = \big([x_1, \ldots, x_{n-1}, y_1], \ldots, [x_1, \ldots, x_{n-1}, y_{n-1}]\big) \quad (2.1)$$

is a rack pointed at $(1, 1, \ldots, 1)$.

Proposition 2.6. *Let m, n be nonnegative integers with $m = 2n - 1$. Then any pointed n-rack $(R, [\ldots], 1)$ has a pointed m-rack structure under the operation $\langle \ldots \rangle$ defined by*

$$\langle x_1, \ldots, x_m \rangle = \big[x_1, \ldots, x_{n-1}, [x_n, \ldots, x_m]\big].$$

Proof. To show (NR1), let $\{x_i\}_{i=1,\ldots,m-1}, \{y_i\}_{i=1,\ldots,m} \subseteq R$. We have by definition

$$\begin{aligned}\langle x_1, \ldots, x_{m-1}, \langle y_1, \ldots, y_m \rangle\rangle &= \langle x_1, \ldots, x_{m-1}, [y_1, \ldots, y_{n-1}, [y_n, \ldots, y_m]]\rangle \\ &= \big[x_1, \ldots, x_{n-1}, [x_n, \ldots, x_{m-1}, [y_1, \ldots, y_{n-1}, [y_n, \ldots, y_m]]]\big],\end{aligned}$$

then use consecutively (NR1) on $(R, [\ldots], 1)$ from inside out to obtain

$$\begin{aligned}&= \langle [x_1, \ldots, x_{n-1}, [x_n, \ldots, x_{m-1}, y_1]], \ldots, [x_1, \ldots, x_{n-1}, [x_n, \ldots, x_{m-1}, y_m]]\rangle \\ &= \langle \langle x_1, \ldots, x_{m-1}, y_1 \rangle \ldots, \langle x_1, \ldots, x_{m-1}, y_m \rangle\rangle.\end{aligned}$$

To show (NR2), let $\{x_i\}_{i=1,\ldots,m-1} \subseteq R$ and $y \in R$. Then by (NR2) on $(R, [\ldots], 1)$, there are unique $t, z \in R$ such that $y = [x_1, \ldots, x_{n-1}, t]$ and $t = [x_n, \ldots, x_{m-1}, z]$, i.e.,

$$y = \big[x_1, \ldots, x_{n-1}, [x_n, \ldots, x_{m-1}, z]\big] = \langle x_1, \ldots, x_{m-1}, z \rangle.$$

To show (NR3), we have by (NR3) on $(R, [\ldots], 1)$,

$$\langle 1, \ldots, 1, y \rangle = \big[1, \ldots, 1, [1, \ldots, 1, y]\big] = [1, \ldots, 1, y] = y \quad \text{for all } y \in R,$$

and for all $\{x_i\}_{i=1,\ldots,x_{m-1}} \subseteq R$,

$$\langle x_1, \ldots, x_{m-1}, 1 \rangle = \big[x_1, \ldots, x_{n-1}, [x_n, \ldots, x_{m-1}, 1]\big] = [x_1, \ldots, x_{n-1}, 1] = 1,$$

which completes the proof. □

3. n-subracks

Let $(R, [\ldots])$ be a n-rack (resp. pointed n-rack). A nonempty subset $S \subseteq R$ is called a n-*semisubrack* of R if S is closed under the n-rack operation. $(S, [\ldots])$ is called a n-*subrack* of R if it has a n-rack structure (resp. pointed n-rack structure).

In particular, $\{1\}$ and R are n-subracks of R.

Example 3.1. Let S be a $\mathbb{Z}_4$-submodule of M (the n-rack of Example 2.4) annihilated by 2. Then S has a trivial n-rack structure when endowed with the operation $[\ldots]$ of M. Therefore S is a n-subrack of M when n is odd.

The following theorem provides a characterization of n-subracks in a pointed n-rack.

Theorem 3.2. *A n-semisubrack S of a pointed n-rack $(R, [\ldots], 1)$ is a n-subrack if and only if for all $b \in R$, $[a_1, a_2, \ldots, a_{n-1}, b] \in S$ and $\{a_i\}_{i=1,\ldots,n-1} \subseteq S$ implies $b \in S$.*

Proof. Assume that S is a n-subrack and let $\{a_i\}_{i=1,\ldots,n-1} \subseteq S$ and $b \in R$ with $[a_1, \ldots, a_{n-1}, b] \in S$. Then by (NR2), there is a unique $u \in S$ with $[a_1, \ldots, a_{n-1}, b]$ $= [a_1, a_2, \ldots, a_{n-1}, u]$. Thus $b = u \in S$ by uniqueness. For the converse, it is enough to establish (NR2) for the n-semisubrack S. Let $a_1, a_2, \ldots, a_{n-1}, x \in S \subseteq R$. Then there is a unique $b \in R$ with $x = [a_1, a_2, \ldots, a_{n-1}, b]$, and thus $b \in S$ by hypothesis. □

Proposition 3.3. *Let R, R' be pointed n-racks and $\phi : R \longrightarrow R'$ be a homomorphism. Let $K = \{x \in R : \phi(x) = 1_{R'}\}$ be the kernel of ϕ. Then K and $I = \phi(R)$ are n-subracks of R and R' respectively.*

Proof. $\phi(1_R) = 1_{R'}$. So $1_R \in K$ and $1_{R'} \in I$. Let $\{a_i\}_{i=1,\ldots,n} \subseteq K$. Then $[a_1, \ldots, a_n]_R \in K$ since $\phi([a_1, \ldots, a_n]_R) = [\phi(a_1), \ldots, \phi(a_n)]_{R'} = [1_{R'}, \ldots, 1_{R'}]_{R'}$ $= 1_{R'}$. Now let $b \in R$ and $\{a_i\}_{i=1,\ldots,n-1} \subseteq K$ with $[a_1, \ldots, a_{n-1}, b]_R \in K$. Then

$$\begin{aligned}\phi(b) &= [1_{R'}, \ldots, 1_{R'}, \phi(b)]_{R'} = [\phi(a_1), \ldots, \phi(a_{n-1}), \phi(b)]_{R'} \\ &= \phi([a_1, \ldots, a_{n-1}, b]_R) = 1_{R'}.\end{aligned}$$

Thus $b \in K$. Hence K is a n-subrack of R by Theorem 3.2. To show that I is an n-subrack, notice that $[\phi(x_1), \ldots, \phi(x_n))]_{R'} = \phi([x_1, \ldots, x_n]_R)$ for all $\{x_i\}_{i=1,\ldots,n} \subseteq R$. Now let $y \in R'$ such that $[\phi(x_1), \ldots, \phi(x_{n-1}), y]_{R'} = \phi(d)$ for some $d \in R$. We have by (NR2) on R that $[x_1, \ldots, x_{n-1}, c]_R = d$ for some unique $c \in R$. So $[\phi(x_1), \ldots, \phi(x_{n-1}), \phi(c)]_{R'} = \phi(d)$, and thus $y = \phi(c)$ by uniqueness. Hence I is a n-subrack of R' by Theorem 3.2. □

Proposition 3.4. *Every pointed n-rack has a trivial n-subrack.*

Proof. Let R be a pointed n-rack and consider the subset

$$Z(R) = \big\{a \in R \,|\, [x_1, \ldots, x_{n-1}, a] = a,\ \forall \{x_i\}_{i=1,\ldots,n-1} \subseteq R\big\}.$$

Clearly, $1 \in Z(R)$ by (NR3). Let $\{x_i\}_{i=1,\ldots,n-1} \subseteq R$ and $\{a_i\}_{i=1,\ldots,n} \subseteq Z(R)$. Then by (NR1),

$$[x_1, \ldots, x_{n-1}, [a_1, \ldots, a_n]] = [[x_1, \ldots, x_{n-1}, a_1], \ldots, [x_1, \ldots, x_{n-1}, a_n]] = [a_1, \ldots, a_n].$$

Now, for $y \in R$ such that $[a_1, \ldots, a_{n-1}, y] \in Z(R)$, we have

$$\begin{aligned}[a_1, \ldots, a_{n-1}, y] &= \big[x_1, \ldots, x_{n-1}, [a_1, \ldots, a_{n-1}, y]\big] \\ &= \big[[x_1, \ldots, x_{n-1}, a_1], \ldots, [x_1, \ldots, x_{n-1}, a_{n-1}], [x_1, \ldots, x_{n-1}, y]\big] \\ &= \big[a_1, \ldots, a_{n-1}, [x_1, \ldots, x_{n-1}, y]\big].\end{aligned}$$

By uniqueness, $[x_1, \ldots, x_{n-1}, y] = y$ and thus $y \in Z(R)$. The result follows by Theorem 3.2. □

Definition 3.5. The n-subrack $Z(R)$ is called the *center* of R.

Proposition 3.6. *For every pointed n-rack, there is an involutive subrack of R^{n-1}.*

Proof. Recall by Proposition 2.5 that R^{n-1} has a pointed rack structure and denote the operation $\circ$ by $[-,-]$. Now consider the subset

$$\mathfrak{I}_R = \big\{(a_1, a_2, \ldots, a_{n-1}) \in R^{n-1} \,|\, [a_1, \ldots, a_{n-1}, [a_1, \ldots, a_{n-1}, y]] = y, \quad \forall y \in R\big\}.$$

Clearly, $(1, \ldots, 1) \in \mathfrak{I}_R$ by (NR3).

Now let $a = (a_1, \ldots, a_{n-1}), b = (b_1, \ldots, b_{n-1}) \in \mathfrak{I}_R$ and $x = (x_1, \ldots, x_{n-1}) \in R^{n-1}$. Then

$$\begin{aligned}\big[[a,b], [[a,b],x]\big] &= \big[[a,b], [[a,b], [a,[a,x]]]\big] \\ &= \big[[a,b], [a,[b,[a,x]]]\big] \\ &= \big[a, [b,[b,[a,x]]]\big] \\ &= \big[a,[a,x]\big] = x.\end{aligned}$$

So $\mathfrak{I}_R$ is closed under the rack operation. Moreover, this implies that for $a = (a_1, \ldots, a_{n-1}) \in \mathfrak{I}_R$ and $y = (y_1, \ldots, y_{n-1}) \in R^{n-1}$, we have

$$\begin{aligned}\big[a,[a,y]\big] &= \big[(a_1, \ldots, a_{n-1}), [(a_1, \ldots, a_{n-1}), (y_1, \ldots, y_{n-1})]\big] \\ &= \big[(a_1, \ldots, a_{n-1}), \big([a_1, \ldots, a_{n-1}, y_1], \ldots, [a_1, \ldots, a_{n-1}, y_{n-1}]\big)\big] \\ &= \big(\big[a_1, \ldots, a_{n-1}, [a_1, \ldots, a_{n-1}, y_1]\big], \ldots, \big[a_1, \ldots, a_{n-1}, [a_1, \ldots, a_{n-1}, y_{n-1}]\big]\big) \\ &= (y_1, y_2, \ldots, y_{n-1}) = y.\end{aligned}$$

The result follows by Theorem 3.2. □

Proposition 3.7. *Let S be a n-semisubrack of a pointed n-rack R. Let*

$$\mathfrak{N}(S) = \big\{a \in R \,|\, [u_1, \ldots, u_{n-1}, a] \in S, \ \forall \ \{u_i\}_{i=1,\ldots,n-1} \subseteq S\big\}.$$

Then

(1) $1 \in S$ *iff* $1 \in \mathfrak{N}(S)$.

(2) $\mathfrak{N}(S) \subseteq J$ *for any n-subrack J of R containing S as a n-semisubrack.*

(3) $S \subseteq \mathfrak{N}(S)$. *The equality holds (thus* $\mathfrak{N}(R)$ *is a* n*-subrack of* R*) if* S *is a* n*-subrack of* R*.*

Proof. (1). By (NR3), $1 = [u_1, \ldots, u_{n-1}, 1]$ for all $\{u_i\}_{i=1,\ldots,n-1} \subseteq S$. Thus $1 \in S$ iff $1 \in \mathfrak{N}(S)$.

(2). Let J be a n-subrack of R containing S as a n-semisubrack, and let $a \in \mathfrak{N}(S)$. Then $[u_1, \ldots, u_{n-1}, a] \in S \subseteq J$, for all $\{u_i\}_{i=1,\ldots,n-1} \subseteq S \subseteq J$. This implies that $a \in J$ as J is a n-subrack. Hence $\mathfrak{N}(S) \subseteq J$.

(3). It is clear that $S \subseteq \mathfrak{N}(S)$ as S is closed under the n-rack operation. Now let $\{a_i\}_{i=1,\ldots,n} \subseteq \mathfrak{N}(S)$. Then by (NR1) on S,

$$\big[u_1, \ldots, u_{n-1}, [a_1, \ldots, a_n]\big] = \big[[u_1, \ldots, u_{n-1}, a_1], \ldots, [u_1, \ldots, u_{n-1}, a_n]\big] \in S$$

for all $\{u_i\}_{i=1,\ldots,n-1} \subseteq S$. So $[a_1, \ldots, a_n] \in \mathfrak{N}(S)$ and thus $\mathfrak{N}(S)$ is closed under the n-rack operation. In addition, for $y \in R$ such that $[a_1, \ldots, a_{n-1}, y] \in \mathfrak{N}(S)$, we have $[u_1, \ldots, u_{n-1}, [a_1, \ldots, a_{n-1}, y]] \in S$, i.e.,

$$\big[[u_1, \ldots, u_{n-1}, a_1], \ldots, [u_1, \ldots, u_{n-1}, a_{n-1}], [u_1, \ldots, u_{n-1}, y]\big] \in S.$$

So $[u_1, \ldots, u_{n-1}, y] \in S$ if S is a n-subrack, and thus $y \in \mathfrak{N}(S)$. Hence $\mathfrak{N}(S)$ is a n-subrack of R. □

$\mathfrak{N}(S)$ is called *normalizer* of S. The right normalizer of the n-semisubrack S is dually defined by

$$\mathfrak{N}_r(S) = \{a \in R \,|\, [a, u_1, \ldots, u_{n-1}] \subseteq S, \text{ for all } \{u_i\}_{i=1\ldots,n-1} \subseteq S\}$$

and does not appear to be of interest for left n-racks. However $\mathfrak{N}_r(S)$ satisfies the same properties above for right n-racks.

4. Decomposition of n-racks

In this section we assume that the n-rack R is not pointed.

Let ${}_nAut(R)$ be the set of all automorphisms of the n-rack R, i.e., bijective maps $\xi : R \longrightarrow R$ such that $\xi([x_1, \ldots, x_n]) = [\xi(x_1), \ldots, \xi(x_n)]$.

It is not difficult to see that for all $x_1, \ldots, x_{n-1} \in R$ the map

$$\phi(x_1, ..., x_{n-1})(y) = [x_1, ..., x_{n-1}, y]$$

is an automorphism of R. So, we can consider the map

$$\phi : R^{n-1} \longrightarrow {}_nAut(R) \quad \text{such that} \quad \phi : (x_1, \ldots, x_{n-1}) \mapsto \phi(x_1, \ldots, x_{n-1}).$$

If ϕ is injective, then R is called *faithful*.

Definition 4.1. A n-rack R is *decomposable* if there are two disjoint n-subracks of R such that $R = X_1 \cup X_2$.

Proposition 4.2. *If R is a decomposable n-rack, then the following statements are true:*

(1) $[X_1, \dots, X_1, X_2] \subseteq X_2,$ (4.1)

(2) $(X_1)^{n-1}$ *and* $(X_2)^{n-1}$ *are subracks of the rack* R^{n-1} *satisfying*
$[(X_1)^{n-1}, (X_2)^{n-1}]_{R^{n-1}} \subseteq (X_2)^{n-1}$ *and* $[(X_2)^{n-1}, (X_1)^{n-1}]_{R^{n-1}} \subseteq (X_1)^{n-1}$,

(3) $\phi((X_1)^{n-1}) \in {}_nAut(X_2)$ *and* $\phi((X_2)^{n-1}) \in {}_nAut(X_1)$.

Proof. (1). Let $\{x_i\}_{i=1,\dots,n-1} \subseteq X_1$ and $y \in X_2$ with $[x_1 \dots, x_{n-1}, y] \notin X_2$, i.e., $[x_1, \dots, x_{n-1}, y] \in X_1$. Then by Theorem 3.3, $y \in X_1$ as X_1 is a n-subrack, and thus $y \in X_1 \cap X_2$. A contradiction.

(2). Recall that the rack operation on R^{n-1} is given by the equality (2.1). So $(X_1)^{n-1}$ is closed under this operation and satisfies (R2) as X_1 is a n-subrack of R. Moreover, it is clear by (4.1) that each coordinate of the right hand side of the equality above is in X_2 for $\{x_i\}_{i=1,\dots,n-1} \subseteq X_1$ and $\{y_i\}_{i=1,\dots,n-1} \subseteq X_2$. Thus $[(X_1)^{n-1}, (X_2)^{n-1}]_{R^{n-1}} \subseteq (X_2)^{n-1}$. The other inclusion is obtained similarly.

(3). Let $\{x_i\}_{i=1,\dots,n-1} \subseteq X_1$. The restriction of the map $\phi(x_1, \dots, x_{n-1})$ to X_2 together with (4.1) completes the proof. The proof that $\phi((X_2)^{n-1}) \in {}_nAut(X_1)$ is similar. □

Proposition 4.3. *If R is a decomposable rack, then R is decomposable as a n-rack for all integer $n > 2$.*

Proof. Let $n > 2$ (integer), and $R = X_1 \cup X_2$ be a decomposition of the rack $(R, \circ)$. It is enough to show that X_1 and X_2 are n-subracks. Indeed, for $\{x_i\}_{i=1,\dots,n}$ from X_1, we have, by Proposition 2.5, $[x_1, x_2, \dots, x_n] = x_1(x_2(\dots(x_{n-1} \circ x_n)\dots)) \in X_1$ as X_1 is closed under $\circ$. Also for $y \in X_1$, there is by $(R2)$ a unique $t_1 \in X_1$ with $y = x_1 \circ t_1$. Repeating the process, there exists uniquely $t_2, t_3, \dots, t_{n-1}, z \in X_1$ with $t_i = x_{i+1} \circ t_{i+1}$ and $t_{n-2} = x_{n-1} \circ z$ such that

$$y = x_1 \circ t_1 = x_1 \circ (x_2 \circ t_2) = \dots = x_1 \circ (x_2(\dots(x_{n-1} \circ z)\dots)) = [x_1, x_2, \dots x_{n-1}, z].$$

Hence X_1 is a n-subrack. The proof that X_2 is a n-subrack is similar. □

Proposition 4.4. *If R is a decomposable n-rack, then R is decomposable as a $(2n-1)$-rack.*

Proof. The proof is similar to the proof of Proposition 4.3 and follows by Proposition 2.6. □

5. A homology theory on n-racks

Recall that for a rack $(X, \circ)$, one defines (see [4] for the right rack version) the rack homology $H_*^R(X)$ of X as the homology of the chain complex $\{C_k^R(X), \partial_k\}$ where

$C_k^R(X)$ is the free abelian group generated by k-uples $(x_1, x_2, \dots, x_k)$ of elements of X and the boundary maps $\partial_k : C_k^R(X) \longrightarrow C_{k-1}^R(X)$ are defined by

$\partial_k(x_1, x_2, \dots, x_k) =$

$\sum_{i=2}^k (-1)^i [(x_1, \dots, x_{i-1}, \widehat{x_i}, x_{i+1}, \dots, x_k) - (x_i \circ x_1, \dots, x_i \circ x_{i-1}, \widehat{x_i}, x_{i+1}, \dots, x_k)]$

for $k \geqslant 2$ and $\partial_k = 0$ for $k \leqslant 1$, where $\widehat{x_i}$ means that x_i is deleted. If X is a quandle, the subgroups $C_k^D(X)$ of $C_k^R(X)$ generated by k-tuples $(x_1, x_2, \dots, x_k)$ with $x_i = x_{i+1}$ for some i, $1 \leqslant i < k$ form a subcomplex $C_*^D(X)$ of $C_*^R(X)$ whose homology $H_*^D(X)$ is called the *degeneration homology* of X. The homology $H_*^Q(X)$ of the quotient complex $\{C_k^Q(X) = C_k^R(X)/C_k^D(X), \partial_k\}$ is called the *quandle homology* of X.

Lemma 5.1. *Let $\mathcal{X}$ be a n-rack. Then $\mathcal{X}^{n-1}$ has a rack structure. $\mathcal{X}^{n-1}$ is a quandle if $\mathcal{X}$ is a n-quandle.*

Proof. Endow $\mathcal{X}^{n-1}$ with the binary operation

$$(x_1, \dots, x_{n-1}) \circ (y_1, \dots y_{n-1}) = ([x_1, \dots, x_{n-1}, y_1], \dots, [x_1, \dots, x_{n-1}, y_{n-1}]). \quad \square$$

We define the chain complexes ${}_nC_*^R(\mathcal{X}) := C_*^R(\mathcal{X}^{n-1})$ if $\mathcal{X}$ is an n-rack, ${}_nC_*^D(\mathcal{X}) := C_*^D(\mathcal{X}^{n-1})$ and ${}_nC_*^Q(\mathcal{X}) := C_*^Q(\mathcal{X}^{n-1})$ if $\mathcal{X}$ is a n-quandle.

Definition 5.2. Let $\mathcal{X}$ be an n-rack. The *kth n-rack homology group* of $\mathcal{X}$ with trivial coefficients is defined by

$${}_nH_k^R(\mathcal{X}) = H_k({}_nC_*^R(\mathcal{X})).$$

Definition 5.3. Let $\mathcal{X}$ be a n-quandle.

1. The *kth n-degeneration homology group* of $\mathcal{X}$ with trivial coefficients is defined by

$${}_nH_k^D(\mathcal{X}) = H_k({}_nC_*^D(\mathcal{X})).$$

2. The *kth n-quandle homology group* of $\mathcal{X}$ with trivial coefficients is defined by

$${}_nH_k^Q(\mathcal{X}) = H_k({}_nC_*^Q(\mathcal{X})).$$

Definition 5.4. Let A be a abelian group, we define the chain complexes

$${}_nC_*^W(\mathcal{X}; A) = {}_nC_*^W(\mathcal{X}) \otimes A, \quad \partial = \partial \otimes id \text{ with } W = D, R, Q.$$

1. The *kth n-rack homology group* of $\mathcal{X}$ with coefficients in A is defined by

$${}_nH_k^R(\mathcal{X}; A) = H_k({}_nC_*^R(\mathcal{X}; A)).$$

2. The *kth n-degenerate homology group* of $\mathcal{X}$ with coefficients in A is defined by

$${}_nH_k^D(\mathcal{X}; A) = H_k({}_nC_*^D(\mathcal{X}; A)).$$

3. The *kth n-quandle homology group* of $\mathcal{X}$ with coefficients in A is defined by

$${}_nH_k^Q(\mathcal{X}; A) = H_k({}_nC_*^Q(\mathcal{X}; A)).$$

One defines the cohomology theory of n-racks and n-quandles by duality. Note that for $n = 2$, one recovers the homology and cohomology theories defined by Carter, Jelsovsky, Kamada, Landford and Saito [4].

Proposition 5.5. *Let $\mathcal{X}$ be a n-quandle and $S \subset \mathcal{X}$ a n-subquandle. The following diagram of long exact sequences commutes:*

$$\begin{array}{ccccccc}
{}_nH_k^D(S) & \longrightarrow & {}_nH_k^R(S) & \longrightarrow & {}_nH_k^Q(S) & \longrightarrow & {}_nH_{k+1}^D(S) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
{}_nH_k^D(\mathcal{X}) & \longrightarrow & {}_nH_k^R(\mathcal{X}) & \longrightarrow & {}_nH_k^Q(\mathcal{X}) & \longrightarrow & {}_nH_{k+1}^D(\mathcal{X}) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
{}_nH_k^D(\mathcal{X}_S) & \longrightarrow & {}_nH_k^R(\mathcal{X}_S) & \longrightarrow & {}_nH_k^Q(\mathcal{X}_S) & \longrightarrow & {}_nH_{k+1}^D(\mathcal{X}_S)
\end{array}$$

where ${}_nH_k^W(\mathcal{X}_S)$ stands for the homology of the complex

$$\{{}_nC_k^W(X_S) =_n C_k^W(\mathcal{X})/{}_nC_k^W(S), \partial_k\}, \quad W = R, D, Q.$$

Proof. The diagram above is induced by the following commutative diagram of short exact sequences:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & {}_nC_*^D(S) & \longrightarrow & {}_nC_*^R(S) & \longrightarrow & {}_nC_*^Q(S) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & {}_nC_*^D(\mathcal{X}) & \longrightarrow & {}_nC_*^R(\mathcal{X}) & \longrightarrow & {}_nC_*^Q(\mathcal{X}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & {}_nC_*^D(\mathcal{X}_S) & \longrightarrow & {}_nC_*^R(\mathcal{X}_S) & \longrightarrow & {}_nC_*^Q(\mathcal{X}_S) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

□

Remark. Since $\mathcal{X}^{n-1}$ carries most of the properties of $\mathcal{X}$, several results established on racks are valid on n-racks. For instance; if $\mathcal{X}$ is finite, then $\mathcal{X}^{n-1}$ is also finite. Cohomology of finite racks were studied by Etingof and Graña in [7].

Proposition 5.6. *Let $\mathcal{X}$ be a trivial n-rack. Then we have the following isomorphisms:*

$${}_nH_*^R(\mathcal{X}) \cong \left(\mathbb{Z}\mathcal{X}^{n-1}\right)^*$$

Proof. It is easy to check with Lemma 2.1 that $\mathcal{X}^{n-1}$ is a trivial rack. That all chains are cycles follows by definition. □

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