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Bipolar fuzzy soft Lie algebras

Muhammad Akram

Abstract. We introduce the notion of bipolar fuzzy soft Lie subalgebras and investigate some of their properties. We also introduce the concept of an $(\in, \in \vee q)$-bipolar fuzzy (soft) Lie subalgebra and present some of its properties.

1. Introduction

In 1994, Zhang [13] initiated the concept of bipolar fuzzy sets as a generalization of fuzzy sets. Bipolar fuzzy sets are an extension of fuzzy sets [12] whose membership degree range is $[-1, 1]$. In a bipolar fuzzy set, the membership degree 0 of an element means that the element is irrelevant to the corresponding property, the membership degree $(0, 1]$ of an element indicates that the element somewhat satisfies the property, and the membership degree $[-1, 0)$ of an element indicates that the element somewhat satisfies the implicit counter-property. Although bipolar fuzzy sets and intuitionistic fuzzy sets look similar to each other, they are essentially different sets. In many domains, it is important to be able to deal with bipolar information. It is noted that positive information represents what is granted to be possible, while negative information represents what is considered to be impossible. This domain has recently motivated new research in several directions. In particular, fuzzy and possibilistic formalisms for bipolar information have been proposed, because when we deal with spatial information in image processing or in spatial reasoning applications, this bipolarity also occurs. For instance, when we assess the position of an object in a space, we may have positive information expressed as a set of possible places and negative information expressed as a set of impossible places. As another example, let us consider the spatial relations. Human beings consider "left" and "right" as opposite directions. But this does not mean that one of them is the negation of the other. The semantics of "opposite" captures a notion of symmetry rather than a strict complementation. In particular, there may be positions which are considered neither to the right nor to the left of some reference object, thus leaving some room for indetermination. This corresponds to the idea that the union of positive and negative information does not cover the whole space.

In 1999, Molodtsov [8] initiated the novel concept of soft set theory to deal with uncertainties which can not be handled by traditional mathematical tools. He

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successfully applied the soft set theory into several disciplines, such as game theory, Riemann integration, Perron integration, measure theory etc. Applications of soft set theory in real life problems are now catching momentum due to the general nature parametrization expressed by a soft set. Yang and Li [10] introduced the notion of bipolar fuzzy soft sets. Recently, Akram and Feng introduced the notion of soft Lie subalgebras of Lie algebras in [11] and studied some of their results.

In this paper, we introduce the notion of bipolar fuzzy soft Lie subalgebras and investigate some of their properties. We introduce the concept of an $(\in, \in \vee q)$-bipolar fuzzy Lie subalgebra and present some of its properties. We also introduce the notion of an $(\in, \in \vee q)$-bipolar fuzzy soft Lie subalgebra and discuss some of its related properties.

2. Preliminaries

A *Lie algebra* is a vector space L over a field F (equal to $\mathbf{R}$ or $\mathbf{C}$) on which $L \times L \to L$ denoted by $(x, y) \to [x, y]$ is defined satisfying the following axioms:

(L1) $[x, y]$ is bilinear,

(L2) $[x, x] = 0$ for all $x \in L$,

(L3) $[[x, y], z] + [[y, z], x] + [[z, x], y] = 0$ for all $x, y, z \in L$ (Jacobi identity).

Throughout this paper, L is a Lie algebra and F is a field. We note that the operation $[.,.]$ is not associative, but it is *anticommutative*, i.e., $[x, y] = -[y, x]$. A subspace H of L closed under $[\cdot, \cdot]$ will be called a *Lie subalgebra*.

Let X be a nonempty set. A fuzzy subset μ of X is defined as a mapping from X into $[0, 1]$, where $[0, 1]$ is the usual interval of real numbers. We denote by $\mathbb{F}(X)$ the set of all fuzzy subsets of X.

A fuzzy set μ in a set X of the form

$$\mu(y) = \begin{cases} t \in (0, 1], & \text{if } y = x, \\ 0, & \text{if } y \neq x, \end{cases}$$

is said to be a *fuzzy point* with support x and value t and is denoted by x_t. For a fuzzy point x_t and a fuzzy set μ in a set X, Pu and Liu [9] gave meaning to the symbol $x_t \alpha \mu$, where $\alpha \in \{\in, q, \in \vee q, \in \wedge q\}$. A fuzzy point x_t is called *belong to* a fuzzy set μ, written as $x_t \in \mu$, if $\mu(x) \geqslant t$. A fuzzy point x_t is said to be *quasicoincident with* a fuzzy set μ, written as $x_t q \mu$, if $\mu(x) + t > 1$. To say that $x_t \in \vee q \mu$ (resp. $x_t \in \wedge q \mu$) means that $x_t \in \mu$ or $x_t q \mu$ (resp. $x_t \in \mu$ and $x_t q \mu$). $x_t \overline{\alpha} \mu$ means that $x_t \alpha \mu$ does not hold, where $\alpha \in \{\in, q, \in \vee q, \in \wedge q\}$.

Molodtsov [8] defined the notion of soft set in the following way: Let U be an initial universe and E be a set of parameters. Let $P(U)$ denotes the power set of U and let A be a nonempty subset of E. Then a pair (F, A) is called a *soft set* over U, where F is a mapping given by $F : A \to P(U)$.

In other words, a soft set over U is a parameterized family of subsets of the universe U. For $\varepsilon \in A$, $F(\varepsilon)$ may be considered as the set of ε-approximate elements of the soft set (F, A). Clearly, a soft set is not just a subset of U.

Definition 2.1. [13] Let X be a nonempty set. A *bipolar fuzzy set* B in X is an object having the form

$$B = \{(x,\, \mu^P(x),\, \mu^N(x)) \,|\, x \in X\},$$

where $\mu^P : X \to [0,\, 1]$ and $\mu^N : X \to [-1,\, 0]$ are mappings.

We use the positive membership degree $\mu^P(x)$ to denote the satisfaction degree of an element x to the property corresponding to a bipolar fuzzy set B, and the negative membership degree $\mu^N(x)$ to denote the satisfaction degree of an element x to some implicit counter-property corresponding to a bipolar fuzzy set B. If $\mu^P(x) \neq 0$ and $\mu^N(x) = 0$, it is the situation that x is regarded as having only positive satisfaction for B. If $\mu^P(x) = 0$ and $\mu^N(x) \neq 0$, it is the situation that x does not satisfy the property of B but somewhat satisfies the counter property of B . It is possible for an element x to be such that $\mu^P(x) \neq 0$ and $\mu^N(x) \neq 0$ when the membership function of the property overlaps that of its counter property over some portion of X.

For the sake of simplicity, we shall use the symbol $B = (\mu^P,\, \mu^N)$ for the bipolar fuzzy set $B = \{(x,\, \mu^P(x),\, \mu^N(x)) \,|\, x \in X\}$.

Definition 2.2. Let $A = (\mu_A^P, \mu_A^N)$ be a bipolar fuzzy set on X and let $\alpha \in [0, 1]$. α-cut A_α of A can be defined as

$$A_\alpha = A_\alpha^P \cup A_\alpha^N, \quad A_\alpha^P = \{x \mid \mu_\alpha^P(x) \geq \alpha\}, \quad A_\alpha^N = \{x \mid \mu_\alpha^N(x) \leq -\alpha\}.$$

We call A_α^P as positive α-cut and A_α^N as negative α-cut.

Definition 2.3. [13] For every two bipolar fuzzy sets $A = (\mu_A^P, \mu_A^N)$ and $B = (\mu_B^P, \mu_B^N)$ in X, we define

- $(A \bigcap B)(x) = (\min(\mu_A^P(x), \mu_B^P(x)), \max(\mu_A^N(x), \mu_B^N(x)))$,
- $(A \bigcup B)(x) = (\max(\mu_A^P(x), \mu_B^P(x)), \min(\mu_A^N(x), \mu_B^N(x)))$.

The concept of bipolar fuzzy soft set was originally proposed in [10]. Let $BF(U)$ denote the family of all bipolar fuzzy sets in U.

Definition 2.4. [10] Let U be an initial universe and $A \subseteq E$ be a set of parameters. A pair (f, A) is called an ***bipolar fuzzy soft set*** over U, where f is a mapping given by $f : A \to BF(U)$. A bipolar fuzzy soft set is a parameterized family of bipolar fuzzy subsets of U. For any $\varepsilon \in A$, f_ε is referred to as the set of a-approximate elements of the bipolar fuzzy soft set (f, A), which is actually a bipolar fuzzy set on U and can be written as

$$f_\varepsilon = \{(\mu_{f_\varepsilon}^P(x), \mu_{f_\varepsilon}^N(x)) \mid x \in U\},$$

where $\mu^P_{f_\varepsilon}(x)$ denotes the degree of x keeping the parameter ε, $\mu^N_{f_\varepsilon}(x)$ denotes the degree of x keeping the non-parameter ε.

Definition 2.5. [10] Let (f, A) and (g, B) be two bipolar fuzzy soft sets over U. We say that (f, A) is a *bipolar fuzzy soft subset* of (g, B) and write $(f, A) \Subset (g, B)$ if $A \subseteq B$ and $f(\varepsilon) \subseteq g(\varepsilon)$ for $\varepsilon \in A$. (f, A) and (g, B) are said to be *bipolar fuzzy soft equal sets* and write $(f, A) = (g, B)$ if $(f, A) \Subset (g, B)$ and $(g, B) \Subset (f, A)$.

According to [10] for any two bipolar fuzzy soft sets (f, A) and (g, B) over U we define

- the *extended intersection* $(h, C) = (f, A)\widetilde{\cap}(g, B)$, where $C = A \cup B$ and

$$h(\varepsilon) = \begin{cases} f_\varepsilon & \text{if } \varepsilon \in A - B, \\ g_\varepsilon & \text{if } \varepsilon \in B - A, \\ f_\varepsilon \cap g_\varepsilon & \text{if } \varepsilon \in A \cap B, \end{cases}$$

- the *extended union* $(h, C) = (f, A)\widetilde{\cup}(g, B)$, where $C = A \cup B$ and

$$h(\varepsilon) = \begin{cases} f_\varepsilon & \text{if } \varepsilon \in A - B, \\ g_\varepsilon & \text{if } \varepsilon \in B - A, \\ f_\varepsilon \cup g_\varepsilon & \text{if } \varepsilon \in A \cap B, \end{cases}$$

- the operation $(f,A) \wedge (g,B) = (h, A \times B)$, where $h(a, b) = h(a) \cap g(b)$ for all $(a,b) \in A \times B$.

3. Bipolar fuzzy soft Lie algebras

Definition 3.1. Let (f, A) be a bipolar fuzzy soft set over L. Then (f, A) is said to be a *bipolar fuzzy soft Lie subalgebra* over L if $f(x)$ is a bipolar fuzzy Lie subalgebra of L for all $x \in A$, that is, a bipolar fuzzy soft set (f, A) over L is called a *bipolar fuzzy soft Lie subalgebra* of L if the following conditions are satisfied:

(1) $\mu^P_{f_\varepsilon}(x + y) \geqslant \min\{\mu^P_{f_\varepsilon}(x), \mu^P_{f_\varepsilon}(y)\}$,

(2) $\mu^N_{f_\varepsilon}(x + y) \leqslant \max\{\mu^N_{f_\varepsilon}(x), \mu^N_{f_\varepsilon}(y)\}$,

(3) $\mu^P_{f_\varepsilon}(mx) \geqslant \mu^P_{f_\varepsilon}(x)$, $\quad \mu^N_{f_\varepsilon}(mx) \leqslant \mu^N_{f_\varepsilon}(x)$,

(4) $\mu^P_{f_\varepsilon}([x, y]) \geqslant \min\{\mu^P_{f_\varepsilon}(x), \mu^P_{f_\varepsilon}(y)\}$,

(5) $\mu^N_{f_\varepsilon}([x, y]) \leqslant \max\{\mu^N_{f_\varepsilon}(x), \mu^N_{f_\varepsilon}(y)\}$

for all $x, y \in L$ and $m \in K$.

Example 3.2. The real vector space $\Re^2$ with $[x,y] = x \times y$ is a real Lie algebra. Let $\mathbb{N}$ and $\mathbb{Z}$ denote the set of all natural numbers and the set of all integers, respectively. By routine computations, we can easily check that $(f,\mathbb{Z})$, where $f : \mathbb{Z} \to ([0,1] \times [-1,0])^{\Re^2}$ with $f(n) = (\mu^P_{f_n}, \mu^N_{f_n}) : \Re^2 \to [0,1] \times [-1,0]$ for all $n \in \mathbb{Z}$,

$$\mu^P_{f_n}(x) = \begin{cases} 0.6 & \text{if } x = (0,0) = \underline{0}, \\ 0.2 & \text{if } x = (0,a),\ a \neq 0, \\ 0 & \text{otherwise,} \end{cases} \qquad \mu^N_{f_n}(x) = \begin{cases} -0.3 & \text{if } x = (0,0) = \underline{0}, \\ -0.2 & \text{if } x = (0,a),\ a \neq 0, \\ -1 & \text{otherwise,} \end{cases}$$

is a bipolar fuzzy soft Lie subalgebra of $\Re^2$. □

We state the following propositions without their proofs.

Proposition 3.3. *Let (f,A) be a bipolar fuzzy soft Lie subalgebra on L, then*

(i) $\mu^P_{f_\varepsilon}(\underline{0}) \geqslant \mu^P_{f_\varepsilon}(x), \quad \mu^N_{f_\varepsilon}(\underline{0}) \leqslant \mu^N_{f_\varepsilon}(x)$,

(ii) $\mu^P_{f_\varepsilon}([x,y]) = \mu^P_{f_\varepsilon}(-[y,x]) = \mu^P_{f_\varepsilon}([y,x])$,

(iii) $\mu^N_{f_\varepsilon}([x,y]) = \mu^N_{f_\varepsilon}(-[y,x]) = \mu^N_{f_\varepsilon}([y,x])$

for all $x,y \in L$. □

Proposition 3.4. *Let (f,A) and (g,B) be bipolar fuzzy soft Lie subalgebras over L, then $(f,A)\widetilde{\cap}(g,B)$ and $(f,A) \wedge (g,B)$ are bipolar fuzzy soft Lie subalgebras over L. If $A \cap B = \emptyset$, then also $(f,A)\widetilde{\cup}(g,B)$ is a bipolar fuzzy soft Lie subalgebra .* □

Proposition 3.5. *Let (f,A) be a bipolar fuzzy soft Lie subalgebra over L and let $\{(h_i,B_i) \mid i \in I\}$ be a nonempty family of bipolar fuzzy soft Lie subalgebras of (f,A). Then*

(a) *$\widetilde{\cap}_{i\in I}(h_i,B_i)$ is a bipolar fuzzy soft Lie subalgebra of (f,A),*

(b) *$\bigwedge_{i\in I}(h_i,B_i)$ is a bipolar fuzzy soft Lie subalgebra of $\bigwedge_{i\in I}(f,A)$,*

(c) *If $B_i \cap B_j = \emptyset$ for all $i,j \in I$, $i \neq j$, then $\widetilde{\bigvee}_{i\in I}(H_i,B_i)$ is a bipolar fuzzy soft Lie subalgebra of $\widetilde{\bigvee}_{i\in I}(f,A)$.* □

Definition 3.6. Let (f,A) be a bipolar fuzzy soft set over U. For each $s \in [0,1]$, $t \in [-1,0]$, the set $(f,A)^{(s,t)} = (f^{(s,t)},A)$, where

$$(f,A)^{(s,t)}_\varepsilon = \{x \in U \mid \mu^P_{f_\varepsilon}(x) \geqslant s,\ \ \mu^N_{f_\varepsilon}(x) \leqslant t\} \ \text{ for all } \varepsilon \in A,$$

is called an (s,t)-*level soft set* of (f,A). Clearly, $(f,A)^{(s,t)}$ is a soft set over U.

Theorem 3.7. *Let (f,A) be a bipolar fuzzy soft set over L. (f,A) is a bipolar fuzzy soft Lie subalgebra if and only if $(f,A)^{(s,t)}$ is a soft Lie subalgebra over L for each $s \in [0,1]$, $t \in [-1,0]$.*

Proof. Suppose that (f, A) is a bipolar fuzzy soft Lie subalgebra. Then for each $s \in [0,1]$, $t \in [-1,0]$, $\varepsilon \in A$ and $x_1, x_2 \in (f, A)_\varepsilon^{(s,t)}$ we have $\mu_{f_\varepsilon}^P(x_1) \geqslant s$, $\mu_{f_\varepsilon}^P(x_2) \geqslant s$ and $\mu_{f_\varepsilon}^N(x_1) \leqslant t$, $\mu_{f_\varepsilon}^N(x_2) \leqslant t$. From Definition 3.1, it follows that $(f, A)_\varepsilon^{(s,t)}$ is a bipolar fuzzy Lie subalgebra over L. Thus $\mu_{f_\varepsilon}^P(x_1 + x_2) \geqslant \min(\mu_{f_\varepsilon}^P(x_1), \mu_{f_\varepsilon}^P(x_2))$, $\mu_{f_\varepsilon}^P(x_1 + x_2) \geqslant s$, $\mu_{f_\varepsilon}^N(x_1 + x_2) \leqslant \max(\mu_{f_\varepsilon}^N(x_1), \mu_{f_\varepsilon}^N(x_2))$, $\mu_{f_\varepsilon}^N(x_1 + x_2) \leqslant t$. This implies that $x_1 + x_2 \in f_\varepsilon^s$. The verification for other conditions is similar. Thus $(f, A)^{(s,t)}$ is a soft Lie subalgebra over L for each $s \in [0,1], t \in [-1,0]$.

Conversely, assume that $(f, A)^{(s,t)}$ is a soft Lie subalgebra over L for each $s \in [0,1], t \in [-1,0]$. For each $\varepsilon \in A$ and $x_1, x_2 \in G$, let $s = \min\{\mu_{f_\varepsilon}^P(x_1), \mu_{f_\varepsilon}^P(x_2)\}$ and let $t = \max\{\mu_{f_\varepsilon}^N(x_1), \mu_{f_\varepsilon}^N(x_2)\}$, then $x_1,\ x_2 \in (f, A)_\varepsilon^{(s,t)}$. Since $(f, A)_\varepsilon^{(s,t)}$ is a Lie subalgebra over L, then $x_1 + x_2 \in (f, A)_\varepsilon^{(s,t)}$. This means that $\mu_{f_\varepsilon}^P(x_1 + x_2) \geqslant \min(\mu_{f_\varepsilon}^P(x_1), \mu_{f_\varepsilon}^P(x_2))$ and $\mu_{f_\varepsilon}^N(x_1 + x_2) \leqslant \max(\mu_{f_\varepsilon}^N(x_1), \mu_{f_\varepsilon}^N(x_2))$. The verification for other conditions is similar. Thus according to Definition 3.1, (f, A) is a bipolar fuzzy soft Lie subalgebra over L. This completes the proof. □

Definition 3.8. Let $\phi : L_1 \to L_2$ and $\psi : A \to B$ be two functions, A and B are parametric sets from the crisp sets L_1 and L_2, respectively. Then the pair (ϕ, ψ) is called a *bipolar fuzzy soft function* from L_1 to L_2.

Definition 3.9. Let (f, A) and (g, B) be two bipolar fuzzy soft sets over L_1 and L_2, respectively and let (ϕ, ψ) be a bipolar fuzzy soft function from L_1 to L_2.

The *image of* (f, A) under the bipolar fuzzy soft function (ϕ, ψ), denoted by $(\phi, \psi)(f, A)$, is the bipolar fuzzy soft set on $\mathcal{K}_2$ defined by $(\phi, \psi)(f, A) = (\phi(f), \psi(A))$, where for all $k \in \psi(A), y \in L_2$

$$\mu_{\phi(f)_k}^P(y) = \begin{cases} \bigvee_{\phi(x)=y} \bigvee_{\psi(a)=k} f_a(x) & \text{if } x \in \psi^{-1}(y), \\ 1 & \text{otherwise,} \end{cases}$$

$$\mu_{\phi(f)_k}^N(y) = \begin{cases} \bigwedge_{\phi(x)=y} \bigwedge_{\psi(a)=k} f_a(x) & \text{if } x \in \psi^{-1}(y), \\ -1 & \text{otherwise.} \end{cases}$$

The *preimage of* (g, B) under the bipolar fuzzy soft function (ϕ, ψ), denoted by $(\phi, \psi)^{-1}(g, B)$, is the bipolar fuzzy soft set over $\mathcal{K}_1$ defined by $(\phi, \psi)^{-1}(g, B) = (\phi^{-1}(g), \psi^{-1}(B))$, where for all $a \in \psi^{-1}(A)$ for all $x \in L_1$,

$$\mu_{\phi^{-1}(g)_a}^P(x) = \mu_{g_{\psi(a)}}^P(\phi(x)), \quad \mu_{\phi^{-1}(g)_a}^N(x) = \mu_{g_{\psi(a)}}^N(\phi(x)).$$

Definition 3.10. Let (ϕ, ψ) be a bipolar fuzzy soft function from L_1 to L_2. If ϕ is a homomorphism from L_1 to L_2 then (ϕ, ψ) is said to be a *bipolar fuzzy soft homomorphism*. If ϕ is a isomorphism from L_1 to L_2 and ψ is one-to-one mapping from A onto B then (ϕ, ψ) is said to be a *bipolar fuzzy soft isomorphism*.

Theorem 3.11. *Let (g, B) be a bipolar fuzzy soft Lie subalgebra over L_2 and let (ϕ, ψ) be a bipolar fuzzy soft homomorphism from L_1 to L_2. Then $(\phi, \psi)^{-1}(g, B)$ is a bipolar fuzzy soft Lie subalgebra over L_1.*

Proof. Let $x_1, x_2 \in L_1$, then

$$\begin{aligned}\phi^{-1}(\mu^P_{g_\varepsilon})(x_1+x_2) &= \mu^P_{g_{\psi(\varepsilon)}}(\phi(x_1+x_2)) = \mu^P_{g_{\psi(\varepsilon)}}(\phi(x_1)+\phi(x_2))\\ &\geqslant \min\{\mu^P_{g_{\psi(\varepsilon)}}(\phi(x_1)), \mu^P_{g_{\psi(\varepsilon)}}(\phi(x_2))\}\\ &= \min\{\phi^{-1}(\mu^P_{g_\varepsilon})(x_1), \phi^{-1}(\mu^P_{g_\varepsilon})(x_2)\},\end{aligned}$$

$$\begin{aligned}\phi^{-1}(\mu^N_{g_\varepsilon})(x_1+x_2) &= \mu^N_{g_{\psi(\varepsilon)}}(\phi(x_1+x_2)) = \mu^N_{g_{\psi(\varepsilon)}}(\phi(x_1)+\phi(x_2))\\ &\leqslant \max\{\mu^N_{g_{\psi(\varepsilon)}}(\phi(x_1)), \mu^N_{g_{\psi(\varepsilon)}}(\phi(x_2))\}\\ &= \max\{\phi^{-1}(\mu^N_{g_\varepsilon})(x_1), \phi^{-1}(\mu^N_{g_\varepsilon})(x_2)\}.\end{aligned}$$

The verification for other conditions is similar and hence we omit the detail. Hence $(\phi,\psi)^{-1}(g,B)$ is a bipolar fuzzy soft Lie subalgebra over L_1. □

Note that $(\phi,\psi)(f,A)$ may not be a bipolar fuzzy soft Lie subalgebra over L_2.

4. $(\in, \in \vee q)$- bipolar fuzzy soft Lie algebras

Let $c \in G$ be fixed. If $\gamma \in (0,1]$ and $\delta \in [-1,0)$ are two real numbers, then $c(\gamma,\delta) = \langle x, c_\gamma, c_\delta\rangle$ is called a ***bipolar fuzzy point*** in G, where γ(resp, δ) is the positive degree of membership (resp, negative degree of membership) of $c(\gamma,\delta)$ and $c \in G$ is the support of $c(\gamma,\delta)$. Let $c(\gamma,\delta)$ be a bipolar fuzzy in G and let $A = \langle x, \mu^P_A, \mu^N_A\rangle$ be a bipolar fuzzy in G. Then $c(\gamma,\delta)$ is said to belong to A, written $c(\gamma,\delta) \in A$ if $\mu^P_A(c) \geqslant \gamma$ and $\mu^N_A(c) \leqslant \delta$. We say that $c(\gamma,\delta)$ is quasicoincident with A, written $c(\gamma,\delta)qA$, if $\mu^P_A(c)+\gamma > 1$ and $\mu^N_A(c)+\delta < -1$. To say that $c(\gamma,\delta) \in \vee qA$ (resp, $c(\gamma,\delta) \in \wedge qA$) means that $c(\gamma,\delta) \in A$ or $c(\gamma,\delta)qA$ (resp, $c(\gamma,\delta) \in A$ and $c(\gamma,\delta)qA$) and $c(\gamma,\delta)\overline{\in \vee q}A$ means that $c(\gamma,\delta) \in \vee qA$ does not hold.

Definition 4.1. A bipolar fuzzy set $A = (\mu^P_A, \mu^N_A)$ in L is called an $(\in, \in \vee q)$-*bipolar fuzzy Lie subalgebra* of L if it satisfies the following conditions:

(a) $x(s_1,t_1), y(s_2,t_2) \in A \Rightarrow (x+y)(\min(s_1,s_2), \max(t_1,t_2)) \in \vee qA$,

(b) $x(s,t) \in A \Rightarrow (mx)(s,t) \in \vee qA$,

(c) $x(s_1,t_1), y(s_2,t_2) \in A \Rightarrow ([x,y])(\min(s_1,s_2), \max(t_1,t_2)) \in \vee qA$

for all $x, y \in L$, $m \in K$, $s, s_1, s_2 \in (0,1]$, $t, t_1, t_2 \in [-1,0)$.

Example 4.2. Let $\Re^2$ be as in Example 3.2. We define a bipolar fuzzy set $A : G \to [0,1] \times [-1,0]$ by

$$\mu^P_A(x) = \begin{cases} 1 & \text{if } x = e, \\ 0.4 & \text{otherwise,} \end{cases} \qquad \mu^N_A(x) = \begin{cases} 0 & \text{if } x = e, \\ -0.2 & \text{otherwise.} \end{cases}$$

By routine computations, it is easy to see that A is not an $(\in, \in \vee q)$-bipolar fuzzy Lie subalgebra of L. □

Theorem 4.3. *A bipolar fuzzy set A in a Lie algebra L is an $(\in, \in \vee q)$-bipolar fuzzy Lie subalgebra of L if and only if*

- $\mu_A^P(x+y) \geqslant \min(\mu_A^P(x), \mu_A^P(y), 0.5)$, $\mu_A^N(x+y) \leqslant \max(\mu_A^N(x), \mu_A^N(y), -0.5)$,
- $\mu_A^P(mx) \geqslant \min(\mu_A^P(x), 0.5)$, $\mu_A^N(mx) \leqslant \max(\mu_A^N(x), -0.5)$,
- $\mu_A^P([x,y]) \geqslant \min(\mu_A^P(x), \mu_A^P(y), 0.5)$, $\mu_A^N([x,y]) \leqslant \max(\mu_A^N(x), \mu_A^N(y), -0.5)$

hold for all $x, y \in L$, $m \in K$. □

Theorem 4.4. *A bipolar fuzzy set A of a Lie algebra of L is an $(\in, \in \vee q)$-bipolar fuzzy Lie subalgebra of L if and only if for all $s \in (0.5, 1]$, $t \in [-1, -0.5)$ each nonempty $A_{(s,t)}$ is a Lie subalgebra of L.*

Proof. Assume that A is an $(\in, \in \vee q)$-bipolar fuzzy Lie subalgebra of L and let $s \in (0.5, 1]$, $t \in [-1, -0.5)$. If $x, y \in A_{(s,t)}$, then $\mu_A^P(x) \geq s$ and $\mu_A^P(y) \geq s$, $\mu_A^N(x) \leq t$ and $\mu_A^N(y) \leq t$. Thus, $\mu_A^P(x+y) \geqslant \min(\mu_A^P(x), \mu_A^P(y), 0.5) \geqslant \min(s, 0.5) = s$ and $\mu_A^N(x+y) \leqslant \max(\mu_A^N(x), \mu_A^N(y), -0.5) \leqslant \max(t, -0.5) = t$, so $x + y \in A_{(s,t)}$. The verification for other conditions is similar. The proof of converse part is obvious. □

Theorem 4.5. *If A is a bipolar fuzzy set in a Lie algebra L, then $A_{(s,t)}$ is a Lie subalgebra of L if and only if*

- $\max(\mu_A^P(x+y), 0.5) \geqslant \min(\mu_A^P(x), \mu_A^P(y))$,
  $\min(\mu_A^N(x+y), -0.5) \leqslant \max(\mu_A^N(x), \mu_A^N(y))$,
- $\max(\mu_A^P(mx), 0.5) \geqslant \min(\mu_A^P(x))$,
  $\min(\mu_A^N(mx), -0.5) \leqslant \max(\mu_A^N(x))$,
- $\max(\mu_A^P([x,y]), 0.5) \geqslant \min(\mu_A^P(x), \mu_A^P(y))$,
  $\min(\mu_A^N([x,y]), -0.5) \leqslant \max(\mu_A^N(x), \mu_A^N(y))$

for all $x, y \in L$, $m \in K$. □

Definition 4.6. Let (f, A) be a bipolar fuzzy soft set over a Lie algebra L. Then (f, A) is called an $(\in, \in\vee q)$-*bipolar fuzzy soft Lie subalgebra* if $f(\alpha)$ is an $(\in, \in\vee q)$-bipolar fuzzy Lie subalgebra of L for all $\alpha \in A$.

Theorem 4.7. *Let (f, A) and (g, B) be two $(\in, \in\vee q)$-bipolar fuzzy soft Lie subalgebras over a Lie algebra L. Then $(f, A) \wedge (g, B)$ is an $(\in, \in\vee q)$-bipolar fuzzy soft Lie subalgebra over L.*

Proof. By the definition, we can write $(f, A) \wedge (g, B) = (h, C)$, where $C = A \times B$ and $h(\alpha, \beta) = f(\alpha) \cap g(\beta)$ for all $(\alpha, \beta) \in C$. Now for any $(\alpha, \beta) \in C$, since (f, A) and (g, B) are $(\in, \in\vee q)$-bipolar fuzzy soft Lie subalgebras over L, we have both $f(\alpha)$ and $g(\beta)$ are $(\in, \in\vee q)$-bipolar fuzzy Lie subalgebras of L. Thus $h(\alpha, \beta) = f(\alpha) \cap g(\beta)$ is an $(\in, \in\vee q)$-bipolar fuzzy Lie subalgebra of L. Hence, $(f, A) \wedge (g, B)$ is an $(\in, \in\vee q)$-bipolar fuzzy soft Lie subalgebra over L. □

Theorem 4.8. *Let (f,A) and (g,B) be two $(\in,\in\vee q)$-bipolar fuzzy soft Lie subalgebras over a Lie algebra L. Then $(f,A)\widetilde{\cap}(g,B)$ is an $(\in,\in\vee q)$-bipolar fuzzy soft Lie subalgebra over L.*

Proof. We have $(f,A)\widetilde{\cap}(g,B)=(h,C)$, where $C=A\cup B$ and

$$h(\varepsilon)=\begin{cases} f(\varepsilon) & \text{if } \varepsilon\in A-B,\\ g(\varepsilon) & \text{if } \varepsilon\in B-C,\\ f(\varepsilon)\cap g(\varepsilon) & \text{if } \varepsilon\in A\cap B\end{cases}$$

for all $\alpha\in C$.

Now for any $\alpha\in C$, we consider the following cases.

1. $\alpha\in A-B$. Then $h(\alpha)=f(\alpha)$ is an $(\in,\in\vee q)$-bipolar fuzzy Lie subalgebra of L since (f,A) is an $(\in,\in\vee q)$-bipolar fuzzy soft Lie subalgebra over L.

2. $\alpha\in B-A$. Then $h(\alpha)=g(\alpha)$ is an $(\in,\in\vee q)$-bipolar fuzzy Lie subalgebra of L since (g,B) is an $(\in,\in\vee q)$-bipolar fuzzy soft Lie subalgebra over L.

3. $\alpha\in A\cap B$. Then $h(\alpha)=f(\alpha)\cap g(\alpha)$ is an $(\in,\in\vee q)$-bipolar fuzzy Lie subalgebra of L by the assumption. Thus, in any case, $h(\alpha)$ is an $(\in,\in\vee q)$-bipolar fuzzy Lie subalgebra of L. Therefore, $(f,A)\widetilde{\cap}(g,B)$ is an $(\in,\in\vee q)$-bipolar fuzzy soft Lie subalgebra over L. □

Theorem 4.9. *Let (f,A) and (g,B) be two $(\in,\in\vee q)$-bipolar fuzzy soft Lie subalgebras over a Lie algebra L. If $A\cap B\neq\emptyset$, then $(f,A)\widetilde{\cap}(g,B)$ is an $(\in,\in\vee q)$-bipolar fuzzy soft Lie subalgebra over L.*

Proof. $(f,A)\widetilde{\cap}(g,B)=(h,C)$, where $C=A\cap B$ and $h(\alpha)=f(\alpha)\cap g(\alpha)$ for all $\alpha\in C$. Now for any $\alpha\in C$, since (f,A) and (g,B) are $(\in,\in\vee q)$-bipolar fuzzy soft Lie subalgebras over L, we have both $f(\alpha)$ and $g(\alpha)$ are $(\in,\in\vee q)$-bipolar fuzzy Lie subalgebras of L. Thus $h(\alpha)=f(\alpha)\cap g(\alpha)$ is an $(\in,\in\vee q)$-bipolar fuzzy Lie subalgebra of L. Therefore, $(f,A)\widetilde{\cap}(g,B)$ is an $(\in,\in\vee q)$-bipolar fuzzy soft Lie subalgebra over L. □

Theorem 4.10. *Let (f,A) be an $(\in,\in\vee q)$- bipolar fuzzy soft Lie subalgebra over L and let $\{(h_i,B_i)\mid i\in I\}$ be a nonempty family of $(\in,\in\vee q)$-bipolar fuzzy soft Lie subalgebras of (f,A). Then*

(a) $\widetilde{\cap}_{i\in I}(h_i,B_i)$ *is an $(\in,\in\vee q)$- bipolar fuzzy soft Lie subalgebra of (f,A),*

(b) $\bigwedge_{i\in I}(h_i,B_i)$ *is an $(\in,\in\vee q)$- bipolar fuzzy soft Lie subalgebra of $\bigwedge_{i\in I}(f,A)$,*

(c) *If $B_i\cap B_j=\emptyset$ for all $i,\ j\in I$, then $\widetilde{\bigvee}_{i\in I}(H_i,B_i)$ is an $(\in,\in\vee q)$- bipolar fuzzy soft Lie subalgebra of $\widetilde{\bigvee}_{i\in I}f,A)$.* □

Theorem 4.11. *Let (f,A) and (g,B) be two $(\in,\in\vee q)$-bipolar fuzzy soft Lie subalgebras over a Lie algebra L. If A and B are disjoint, then $(f,A)\widetilde{\cup}(g,B)$ is an $(\in,\in\vee q)$-bipolar fuzzy soft Lie subalgebra over L.* □

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Punjab University College of Information Technology, University of the Punjab,
Old Campus, Lahore-54000, Pakistan
E-mail: m.akram@pucit.edu.pk, makrammath@yahoo.com