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On finite loops whose inner mapping groups are direct products of dihedral groups and abelian groups

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Abstract. We show that a finite loop, whose inner mapping group is a direct product of a dihedral group and an abelian group, is solvable provided that the components in the direct product have coprime orders.

1. Introduction

Let Q be a groupoid with a neutral element e. If each of the two equations $ax = b$ and $ya = b$ has a unique solution for any $a, b \in Q$, then we say that Q is a loop. The two mappings $L_a(x) = ax$ and $R_a(x) = xa$ are permutations on Q for every $a \in Q$. The permutation group $M(Q) = \langle L_a, R_a : a \in Q\rangle$ is called the *multiplication group of the loop* Q. Clearly, $M(Q)$ is transitive on Q. The stabilizer of the neutral element e is denoted by $I(Q)$ and is called the *inner mapping group* of Q.

A subloop H of Q is *normal* in Q if $x(yH) = (xy)H$, $(Hx)y = H(xy)$ and $xH = Hx$ for every $x, y \in Q$. A loop Q is *solvable* if it has a series $1 = Q_0 \subseteq \cdots \subseteq Q_n = Q$, where Q_{i-1} is a normal subloop of Q_i and Q_i/Q_{i-1} is an abelian group for each i. In 1996 Vesanen [8] managed to show that the solvability of $M(Q)$ (in the group theoretical sense) implies the solvability of Q (in the loop theoretical sense) if Q is a finite loop. After this we were naturally interested in those properties of $I(Q)$ which imply the solvability of $M(Q)$.

In 2000 Csörgő and Niemenmaa [1] considered the case where $I(Q)$ is a nonabelian group of order $2p$ (here p is an odd prime number) and they showed that $M(Q)$ is then a solvable group. In 2002, Drápal [2] investigated the case where $I(Q)$ is a nonabelian group of order pq (p and q are two different prime numbers) and again the solvability of $M(Q)$ followed. Finally, in 2004 Niemenmaa [5] showed that finite loops with dihedral inner mapping groups are solvable. Now we are able to prove the following: *If Q is a finite loop and $I(Q) = S \times L$, where S is dihedral, L is abelian and* $\gcd(|S|, |L|) = 1$*, then $M(Q)$ is solvable.* By the

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result of Vesanen, Q is solvable, too. The result also holds in the case that S is a nonabelian group of order pq, where p and q are two different prime numbers.

Many properties of loops and their multiplication groups can be reduced to the properties of connected transversals in groups. Thus in section two we shall give the needed background material about connected transversals and their connections to loop theory. Section three contains our main results about the solvability of finite loops with given inner mapping groups.

2. Connected transversals

Let G be a group, $H \leq G$ and let A and B be two left transversals to H in G. We say that the two transversals A and B are *H-connected* if $a^{-1}b^{-1}ab \in H$ for every $a \in A$ and $b \in B$. We denote by H_G the core of H in G (the largest normal subgroup of G contained in H). If Q is a loop, then $A = \{L_a : a \in Q\}$ and $B = \{R_a : a \in Q\}$ are $I(Q)$-connected transversals in $M(Q)$ and the core of $I(Q)$ in $M(Q)$ is trivial. Niemenmaa and Kepka proved in 1990 the following [6, Theorem 4.1]

Theorem 2.1. *A group G is isomorphic to the multiplication group of a loop if and only if there exist a subgroup H and H-connected transversals A and B such that $H_G = 1$ and $G = \langle A, B\rangle$.* □

In the following results, which are needed later, we assume that A and B are H-connected transversals in G.

Lemma 2.2. *If $C \subseteq A \cup B$ and $K = \langle H, C\rangle$, then $C \subseteq K_G$.* □

Lemma 2.3. *If $G = \langle A, B\rangle$ and H is cyclic, then $G' \leq H$.* □

Theorem 2.4. *If G is finite and H is abelian or dihedral, then G is solvable.* □

For the proofs, see [6, Lemma 2.5 and Theorem 3.5], [7, Theorem 4.1] and [5, Theorem 3.1].

Next we wish to show that the solvability of G also follows in the case that H is a nonabelian subgroup of order pq (here $p \neq q$ are prime numbers). For the proof we need the following loop theoretical result by Drápal [2, Corollary 4.7].

Theorem 2.5. *If Q is a loop and $I(Q)$ is a nonabelian group of order pq, where $p \neq q$ are prime numbers, then $M(Q)$ is solvable.* □

We also need

Lemma 2.6. *Let $G = AH$ be a finite group, where A is an abelian subgroup, H is a subgroup of order pq and $p \neq q$ are prime numbers. Then G is solvable.* □

For the proof, see [4, Lemma 2.5].

Theorem 2.7. *Let G be a finite group, $H \leq G$ and $|H| = pq$, where $p \neq q$ are prime numbers. If there exist H-connected transversals A and B in G, then G is solvable.*

Proof. If $H_G > 1$, then we consider the group G/H_G and the subgroup H/H_G. Since H/H_G is cyclic, the claim follows from Theorem 2.4. Thus we may assume that $H_G = 1$.

If $G = \langle A, B\rangle$, then we apply Theorems 2.1 and 2.5, and the solvability of G follows. Thus we may assume that $E = \langle A, B\rangle < G$. If we write $K = E \cap H$, then $K < H$ and we have K-connected transversals A and B in E. Then $E' \leq K$ by Lemma 2.3 and K is normal in E. As $G = EH$, we may conclude that $K^G = \langle K^g : g \in G\rangle \leq H$. If $K \neq 1$, then we get a contradiction, as $H_G = 1$. Thus $K = 1$ and it follows that $E = A = B$ is an abelian group. Now $G = EH$ and we can apply Lemma 2.6. □

3. Main results

The following classical result of Wielandt is needed in the proof of our main theorem.

Theorem 3.1. *Let G be a finite group and let G contain a nilpotent Hall π-subgroup H. Then every π-subgroup of G is contained in a conjugate of H.* □

For the proof, see [3, Satz 5.8, p. 285].

Theorem 3.2. *Let G be a finite group and $H = S \times L \leq G$, where S is dihedral, L is abelian and* $\gcd(|S|, |L|) = 1$. *If there exist H-connected transversals A and B in G, then G is solvable.*

Proof. Let G be a minimal counterexample. If $H_G > 1$, then we consider G/H_G and its subgroup H/H_G and by using induction or Theorem 2.4, it follows that G/H_G is solvable, hence G is solvable.

Thus we may assume that $H_G = 1$. If H is not maximal in G, then there exists a subgroup T such that $H < T < G$. By Lemma 2.2, $T_G > 1$ and we may consider G/T_G and its subgroup $HT_G/T_G = T/T_G$. It follows that G/T_G is solvable. Since T is solvable by induction, we conclude that G is solvable.

We thus assume that H is a maximal subgroup of G. Let P be a Sylow p-subgroup of L. As $H_G = 1$, we conclude that P is a Sylow p-subgroup of G. From this it follows that L is a Hall subgroup of G. Clearly, $N_G(P) = H = C_G(P)$ and by using the Burnside normal complement theorem there exists a normal p-complement in G for each p that divides $|L|$. Clearly, this means that $G = KL$, where K is normal in G and $\gcd(|K|, |L|) = 1$.

If $1 \neq a \in A$, then $a = yx$, where $y \in L$ and $x \in K$. Then $aK = yK$ and $(aK)^d = K$, where d divides $|L|$. Thus $a^d \in K$, hence $(a^d)^t = 1$, where t divides $|K|$. It follows that $(a^t)^d = 1$, hence $|a^t|$ divides d. Since L is an abelian Hall subgroup of G, we may apply Theorem 3.1 and it follows that $a^t \in L^g$ for some $g \in G$. As L is abelian, $\langle a^t \rangle$ is normal in $\langle a, H^g \rangle = G$. As $H_G = 1$, we conclude that $a^t = 1$. Now there exist integers m and n such that $md + nt = 1$. Thus $a = a^{md+nt} = (a^d)^m (a^t)^n \in K$.

We may conclude that $A \cup B \subseteq K$. Clearly, $S \leq K$ and thus $K = AS = BS$. By Theorem 2.4, K is a solvable group. As $G = KL$, it follows that G is solvable, too. □

Theorem 3.3. *Let G be a finite group and $H = S \times L \leq G$, where S is a nonabelian group of order pq (here $p \neq q$ are prime numbers), L is abelian and* $\gcd(|S|, |L|) = 1$. *If there exist H-connected transversals A and B in G, then G is solvable.*

Proof. The proof is analogous to the proof of Theorem 3.2. We just have to replace Theorem 2.4 by Theorem 2.7 when needed. □

By combining Theorem 2.1 with Theorems 3.2 and 3.3, and by applying the theorem of Vesanen [8], we have the following

Corollary 3.4. *Let Q be a finite loop. If $I(Q) = S \times L$, where S is either dihedral or nonabelian of order pq, L is abelian and* $\gcd(|S|, |L|) = 1$, *then $M(Q)$ is a solvable group and Q is a solvable loop.* □

Remark 3.5. It would be interesting to know if the results of Theorems 3.2 and 3.3 and Corollary 3.4 also hold in the case that L is nilpotent.

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