Quasigroups and Related Systems **20** (2012), 251 − 256

On fuzzy ordered semigroups

Niovi Kehayopulu

Abstract. We characterize the ordered semigroups which are both regular and intra-regular, the completely regular, the quasi-semisimple, and the quasi left (right) regular ordered semigroups in terms of fuzzy sets.

1. For an ordered semigroup S and a subset A of S we denote by $(A]$ the subset of S defined by $(A] := \{t \in S \mid t \leqslant a$ for some $a \in A\}$. An ordered semigroup S is called *regular* if for any $a \in S$ there exists $x \in S$ such that $a \leqslant axa$. It is called *left* (resp. *right*) *regular* if for any $a \in S$ there exists $x \in S$ such that $a \leqslant xa^2$ (resp. $a \leqslant a^2x$). It is called *intra-regular* if for any $a \in S$ there exist $x, y \in S$ such that $a \leqslant xa^2y$. So, an ordered semigroup S is regular (left regular, right regular) if and only if $a \in (aSa]$ ($a \in (Sa^2]$, $a \in (a^2S]$) for all $a \in S$. It is intra-regular if and only if $a \in (Sa^2S]$ for all $a \in S$. Using fuzzy sets, we get the following: An ordered semigroup S is regular if and only if for every fuzzy subset f of S, we have $f \preceq f \circ 1 \circ f$. It is left (resp. right) regular if and only if for every fuzzy subset f of S, we have $f \preceq 1 \circ f^2$ (resp. $f \preceq f^2 \circ 1$). It is intra-regular if and only if for every fuzzy subset f of S, we have $f \preceq 1 \circ f^2 \circ 1$ [2]. An ordered semigroup S is called *completely regular* if at the same time is regular, left regular and right regular. As one can easily see, an ordered semigroup S is completely regular if and only if for every $a \in S$ there exists $x \in S$ such that $a \leq a^2xa^2$. That is, if $a \in (a^2Sa^2]$ for all $a \in S$. Our aim is to show that the definitions of regular, left (right) regular and intra-regular ordered semigroups using fuzzy sets play an essential role in studying the structure of ordered semigroups. In this respect, we prove that an ordered semigroup S is both regular and intra-regular if and only if for every fuzzy subset f of S, we have $f \preceq f \circ 1 \circ f^2 \circ 1 \circ f$. An ordered semigroup S is completely regular if and only if for every fuzzy subset f of S, we have $f \preceq f^2 \circ 1 \circ f^2$. We prove them first in the usual way, then using the equivalent definition of regular, left (right) regular and intra-regular ordered semigroups mentioned above. Comparing the two proofs we see that using the characterizations given in [2] the proofs of the results are drastically simplified.

On the other hand, we characterized in [1] the left (right) quasi-regular and the more general class of semisimple ordered semigroups using similar conditions. An ordered semigroup S is called *left* (resp. *right*) *quasi-regular* if for every $a \in S$ there

2010 Mathematics Subject Classification: 06F05 (08A72).
Keywords: Ordered semigroup, regular, intra-regular, completely regular, fuzzy subset, quasi left (right) regular, quasi-regular, quasi-semisimple.

exist $x, y \in S$ such that $a \leqslant axay$ (resp. $a \leqslant xaya$). Equivalently, if $a \in (aSaS]$ (resp. $a \in (SaSa]$) for all $a \in S$. It is called *semisimple* if for every $a \in S$ there exist $x, y, z \in S$ such that $a \leqslant xayaz$. That is, if $a \in (SaSaS]$ for all $a \in S$. We have already seen in [1] that an ordered semigroup S is left (resp. right) quasi-regular if and only if for every fuzzy subset f of S, we have $f \preceq 1 \circ f \circ 1 \circ f$ (resp. $f \preceq f \circ 1 \circ f \circ 1$); it is semisimple if and only if for every fuzzy subset f of S, we have $f \preceq 1 \circ f \circ 1 \circ f \circ 1$.

A semigroup S (without order) is called *quasi-semisimple* if $a \in SaS$ for every $a \in S$. A semigroup S is called *quasi left* (resp. *right*) *regular* if $a \in Sa$ (resp. $a \in aS$) for every $a \in S$. Keeping in mind the terminology of quasi-semisimple and quasi left (resp. right) regular semigroups given above, in the present paper we first introduce the concepts of quasi-semisimple and quasi left (right) regular ordered semigroups. Then, as a continuation of the paper in [1], we characterize the quasi-semisimple, the quasi left (right) regular and the quasi-regular ordered semigroups in terms of fuzzy sets. Each quasi-regular ordered semigroup is a quasi-semisimple ordered semigroup.

As always, denote by 1 the fuzzy subset on S defined by $1(x) = 1$ for every $x \in S$. Recall that if S is regular or intra-regular, then $1 \circ 1 = 1$. If f, g are fuzzy subsets of S such that $f \preceq g$, then for any fuzzy subset h of S we have $f \circ h \preceq g \circ h$ and $h \circ f \preceq h \circ g$. Denote $f^2 := f \circ f$, and by f_a the characteristic function on the set S defined by $f_a(x) = 1$ if $x = a$ and $f_a(x) = 0$ if $x \neq a$ $(a \in S)$. Denote by A_a the subset of $S \times S$ defined by $A_a := \{(x, y) \in S \times S \mid a \leqslant xy\}$.

2. In this section we characterize the ordered semigroups which are both regular and intra-regular and the completely regular ordered semigroups in terms of fuzzy sets. For the following three lemmas we refer to [2].

Lemma 1. *Let $(S, ., \leqslant)$ be an ordered groupoid, f,g fuzzy subsets of S and $a \in S$. The following are equivalent:*

(1) $(f \circ g)(a) \neq 0$.

(2) *There exists $(x, y) \in A_a$ such that $f(x) \neq 0$ and $g(y) \neq 0$.* □

Lemma 2. *Let $(S, ., \leqslant)$ be an ordered groupoid, f a fuzzy subset of S and $a \in S$. The following are equivalent:*

(1) $(f \circ 1)(a) \neq 0$.

(2) *There exists $(x, y) \in A_a$ such that $f(x) \neq 0$.* □

Lemma 3. *Let $(S, ., \leqslant)$ be an ordered groupoid, g a fuzzy subset of S and $a \in S$. The following are equivalent:*

(1) $(1 \circ g)(a) \neq 0$.

(2) *There exists $(x, y) \in A_a$ such that $g(y) \neq 0$.*

Theorem 4. *An ordered semigroup S is both regular and intra-regular if and only if for every fuzzy subset f of S, we have*

$$f \preceq f \circ 1 \circ f^2 \circ 1 \circ f.$$

Proof. $\Longrightarrow$. Let $a \in S$. Since S is regular and intra-regular, there exist $x, y, z \in S$ such that $a \leqslant axa$ and $a \leqslant ya^2z$. Then we have

$$a \leqslant ax(axa) \leqslant ax(ya^2z)xa = (axy)a^2zxa.$$

Since $(axy, a^2zxa) \in A_a$, we have $A_a \neq \emptyset$ and

$$\begin{aligned}(f \circ 1 \circ f^2 \circ 1 \circ f)(a) : &= \bigvee_{(u,v)\in A_a} \min\{(f \circ 1)(u), (f^2 \circ 1 \circ f)(v)\} \\ &\geqslant \min\{(f \circ 1)(axy), (f^2 \circ 1 \circ f)(a^2zxa)\}.\end{aligned}$$

Since $(a, xy) \in A_{axy}$, we have $A_{axy} \neq \emptyset$ and

$$(f \circ 1)(axy) := \bigvee_{(w,t)\in A_{axy}} \min\{f(w), 1(t)\} \geqslant \min\{f(a), 1(xy)\} = f(a).$$

Since $(a^2zx, a) \in A_{a^2zxa}$, we have $A_{a^2zxa} \neq \emptyset$ and

$$(f^2 \circ 1 \circ f)(a^2zxa) : = \bigvee_{(k,h)\in A_{a^2zxa}} \min\{(f^2 \circ 1)(k), f(h)\} \geqslant \min\{(f^2 \circ 1)(a^2zx), f(a)\}.$$

Since $(a^2, zx) \in A_{a^2zx}$, we have $A_{a^2zx} \neq \emptyset$ and

$$(f^2 \circ 1)(a^2zx) : = \bigvee_{(s,g)\in A_{a^2zx}} \min\{f^2(s), 1(g)\} \geqslant \min\{f^2(a^2), 1(zx)\} = f^2(a^2).$$

Since $(a, a) \in A_{a^2}$, we have $A_{a^2} \neq \emptyset$ and

$$(f \circ f)(a^2) : = \bigvee_{(s,g)\in A_{a^2}} \min\{f(s), f(g)\} \geqslant \min\{f(a), f(a)\} = f(a).$$

Thus

$$\begin{aligned}(f \circ 1 \circ f^2 \circ 1 \circ f)(a) &\geqslant \min\{(f \circ 1)(axy), (f^2 \circ 1 \circ f)(a^2zxa)\} \\ &\geqslant \min\{f(a), \min\{(f^2 \circ 1)(a^2zx)\}, f(a)\}\} \\ &\geqslant \min\{f(a), \min\{f^2(a^2), f(a)\}\} \\ &\geqslant \min\{f(a), \min\{f(a), f(a)\}\} \\ &= \min\{f(a), f(a)\} = f(a).\end{aligned}$$

$\Longleftarrow$. Let $a \in S$. Since f_a is a fuzzy set in S, by hypothesis, we have $1 = f_a(a) \leqslant (f_a \circ 1 \circ f_a^2 \circ 1 \circ f_a)(a)$. Since $f_a \circ 1 \circ f_a^2 \circ 1 \circ f_a$ is a fuzzy set in S, we have $(f_a \circ 1 \circ f_a^2 \circ 1 \circ f_a)(a) \leqslant 1$. Thus we have $(f_a \circ 1 \circ f_a^2 \circ 1 \circ f_a)(a) = 1$. By Lemma 1, there exists $(x, y) \in A_a$ such that $(f_a \circ 1)(x) \neq 0$ and $(f_a^2 \circ 1 \circ f_a)(y) \neq 0$. Since $(f_a \circ 1)(x) \neq 0$, by Lemma 2, there exists $(u, v) \in A_x$ such that $f_a(u) \neq 0$. Since $(f_a^2 \circ 1 \circ f_a)(y) \neq 0$, by Lemma 1, there exists $(w, t) \in A_y$ such that $(f_a^2 \circ 1)(w) \neq 0$ and $f_a(t) \neq 0$. Since $(f_a^2 \circ 1)(w) \neq 0$, by Lemma 2, there exists $(k, h) \in A_w$ such that $f_a^2(k) \neq 0$. Since $(f_a \circ f_a)(k) \neq 0$, by Lemma 1, there exists $(s,g) \in A_k$ such that $f_a(s) \neq 0$ and $f_a(g) \neq 0$. Since $f_a(u) \neq 0$, we have $f_a(u) = 1$, and $u = a$. Since $f_a(t) \neq 0$, $t = a$; since $f_a(s) \neq 0$, $s = a$; since $f_a(g) \neq 0$, $g = a$. Thus we have $a \leqslant xy \leqslant (uv)(wt) \leqslant uv(kh)t \leqslant uv(sg)ht = ava^2ha$, from which $a \leqslant a(va^2h)a$ and $a \leqslant (av)a^2(ha)$, where the elements va^2h and av, ha are in S. So S is regular and intra-regular. □

Second proof

$\Longrightarrow$. Let f be a fuzzy set on S. Since S is regular, we have $f \preceq f \circ 1 \circ f$; since S is intra-regular, $f \preceq 1 \circ f^2 \circ 1$. Thus we have

$$f \preceq f \circ 1 \circ (f \circ 1 \circ f) \preceq f \circ 1 \circ (1 \circ f^2 \circ 1) \circ 1 \circ f = f \circ 1 \circ f^2 \circ 1 \circ f.$$

$\Longleftarrow$. Let f be a fuzzy set on S. By hypothesis, we have

$$f \preceq f \circ 1 \circ f^2 \circ 1 \circ f \preceq f \circ 1 \circ f,\ 1 \circ f^2 \circ 1,$$

so S is both regular and intra-regular. □

Theorem 5. *An ordered semigroup S is completely regular if and only if for every fuzzy subset f of S we have*

$$f \preceq f^2 \circ 1 \circ f^2.$$

Proof. $\Longrightarrow$. Let $a \in S$. Since S is completely regular, there exists $x \in S$ such that $a \leqslant a^2xa^2$. Since $(a^2xa, a) \in A_a$, we have $A_a \neq \emptyset$, and

$$(f^2 \circ 1 \circ f^2)(a) := \bigvee_{(u,v)\in A_a} \min\{(f^2 \circ 1 \circ f)(u), f(v)\} \geqslant \min\{(f^2 \circ 1 \circ f)(a^2xa), f(a)\}.$$

Since $(a^2x, a) \in A_{a^2xa}$, we have $A_{a^2xa} \neq \emptyset$, and

$$(f^2 \circ 1 \circ f)(a^2xa) := \bigvee_{(w,t)\in A_{a^2xa}} \min\{(f^2 \circ 1)(w), f(t)\} \geqslant \min\{(f^2 \circ 1)(a^2x), f(a)\}.$$

Since $(a^2, x) \in A_{a^2x}$, we have $A_{a^2x} \neq \emptyset$, and

$$(f^2 \circ 1)(a^2x) := \bigvee_{(k,h)\in A_{a^2x}} \min\{f^2(k), 1(h)\} \geqslant \min\{f^2(a^2), 1(x)\} = f^2(a^2).$$

Since $(a, a) \in A_{a^2}$, we have $A_{a^2} \neq \emptyset$, and

$$(f \circ f)(a^2) := \bigvee_{(s,g)\in A_{a^2}} \min\{f(s), f(g)\} \geqslant \min\{f(a), f(a)\} = f(a).$$

Then

$$\begin{aligned}(f^2 \circ 1 \circ f^2)(a) &\geqslant \min\{(f^2 \circ 1 \circ f)(a^2xa), f(a)\}\\ &\geqslant \min\{\min\{(f^2 \circ 1)(a^2x), f(a)\}, f(a)\}\\ &\geqslant \min\{\min\{f^2(a^2), f(a)\}, f(a)\}\\ &\geqslant \min\{\min\{f(a), f(a)\}, f(a)\} = f(a).\end{aligned}$$

Thus $f \preceq f^2 \circ 1 \circ f^2$.

$\Longleftarrow$. Let $a \in S$. For the characteristic function f_a, by hypothesis, we have $1 = f_a(a) \leqslant (f_a^2 \circ 1 \circ f_a^2)(a)$. Since $f_a^2 \circ 1 \circ f_a^2$ is a fuzzy subset of S, we have $(f_a^2 \circ 1 \circ f_a^2)(a) \leqslant 1$. Thus we have $(f_a^2 \circ 1 \circ f_a^2)(a) = 1$. By Lemma 1, there exists $(x, y) \in A_a$ such that $(f_a^2 \circ 1)(x) \neq 0$ and $f_a^2(y) \neq 0$. Since $(f_a^2 \circ 1)(x) \neq 0$, by Lemma 2, there exists $(u, v) \in A_x$ such that $f_a^2(u) \neq 0$. Since $(f_a \circ f_a)(y) \neq 0$, by Lemma 1, there exists $(w, t) \in A_y$ such that $f_a(w) \neq 0$ and $f_a(t) \neq 0$. Since $(f_a \circ f_a)(u) \neq 0$, by Lemma 1, there exists $(k, h) \in A_u$ such that $f_a(k) \neq 0$ and $f_a(h) \neq 0$. Since $f_a(w) \neq 0$, we have $f_a(w) = 1$, and so $w = a$. Since $f_a(t) \neq 0$, $f_a(k) \neq 0$, $f_a(h) \neq 0$, we have $t = k = h = a$. Thus we have

$$a \leqslant xy \leqslant (uv)y \leqslant uv(wt) \leqslant (kh)vwt = a^2va^2,$$

where $v \in S$, so S is completely regular. □

Second proof

$\Longrightarrow$. Let f be a fuzzy set on S. Since S is completely regular, we have $f \preceq f \circ 1 \circ f$, $f \preceq f^2 \circ 1$ and $f \preceq 1 \circ f^2$. Then we have

$$f \preceq f \circ 1 \circ f \preceq (f^2 \circ 1) \circ 1 \circ (1 \circ f^2) = f^2 \circ 1 \circ f^2.$$

$\Longleftarrow$. Let f be a fuzzy set on S. By hypothesis, we have

$$f \preceq f \circ f \circ 1 \circ f \circ f \preceq f \circ 1 \circ f,\ f^2 \circ 1,\ 1 \circ f^2,$$

so S is regular, left regular and right regular. □

3. In this section, we characterize the quasi-semisimple, the quasi left (right) regular and the quasi-regular ordered semigroups using fuzzy sets.

Definition 6. An ordered semigroup $(S, ., \leqslant)$ is called *quasi-semisimple* if, for every $a \in S$ we have $a \in (SaS]$. That is, for every $a \in S$ there exist $x, y \in S$ such that $a \leqslant xay$.

Theorem 7. *An ordered semigroup $(S, ., \leqslant)$ is quasi-semisimple if and only if for every fuzzy subset f of S, we have $f \preceq 1 \circ f \circ 1$.*

Proof. $\Longrightarrow$. Let f be a fuzzy subset of S and $a \in S$. Since S is quasi-semisimple, there exist $x, y \in S$ such that $a \leqslant xay$. Then $(x, ay) \in A_a$, $A_a \neq \emptyset$ and

$$(1 \circ f \circ 1)(a) := \bigvee_{(u,v) \in A_a} \min\{1(u), (f \circ 1)(v)\} \geqslant \min\{1(x), (f \circ 1)(ay)\} = (f \circ 1)(ay).$$

Since $(a, y) \in A_{ay}$, we have $A_{ay} \neq \emptyset$ and

$$(f \circ 1)(ay) := \bigvee_{(w,t) \in A_{ay}} \min\{f(w), 1(t)\} \geqslant \min\{f(a), 1(y)\} = f(a).$$

Thus we have $(1 \circ f \circ 1)(a) \geqslant (f \circ 1)(ay) \geqslant f(a)$, and so $f \preceq 1 \circ f \circ 1$.

$\Longleftarrow$. Let $a \in S$. Since f_a is a fuzzy subset of S, by hypothesis, we have

$$1 = f_a(a) \leqslant (1 \circ f_a \circ 1)(a).$$

Since $1 \circ f_a \circ 1$ is a fuzzy subset of S, we have $(1 \circ f_a \circ 1)(a) \leqslant 1$. Then we have $(1 \circ f_a \circ 1)(a) = 1$. Since $(1 \circ (f_a \circ 1))(a) \neq 0$, by Lemma 3, there exists $(x, y) \in A_a$ such that $(f_a \circ 1)(y) \neq 0$. Then, by Lemma 2, there exists $(u, v) \in A_y$ such that $f_a(u) \neq 0$. Then $f_a(u) = 1$, and $u = a$. Finally, $a \leqslant xy \leqslant x(uv) = xav \in SaS$, so $a \in (SaS]$, and S is quasi-semisimple. □

Definition 8. An ordered semigroup $(S, ., \leqslant)$ is called *quasi left regular* if, for every $a \in S$ we have $a \in (Sa]$. That is, for every $a \in S$ there exists $x \in S$ such that $a \leqslant xa$. It is called *quasi right regular* if, for every $a \in S$ we have $a \in (aS]$, and *quasi-regular* if it is both left quasi regular and right quasi regular.

Theorem 9. *An ordered semigroup $(S, ., \leqslant)$ is quasi left regular if and only if for every fuzzy subset f of S, we have $f \preceq 1 \circ f$.*

Proof. $\Longrightarrow$. Let f be a fuzzy subset of S and $a \in S$. Since S is quasi left regular, there exists $x \in S$ such that $a \leqslant xa$. Then $(x, a) \in A_a$, $A_a \neq \emptyset$ and

$$(1 \circ f)(a) := \bigvee_{(u,v) \in A_a} \min\{1(u), f(v)\} \geqslant \min\{1(x), f(a)\} = f(a).$$

Thus we have $f \preceq 1 \circ f$.

$\Longleftarrow$. Let $a \in S$. Since f_a and $1 \circ f_a$ are fuzzy subsets of S, by hypothesis, we have $1 = f_a(a) \leqslant (1 \circ f_a)(a) \leqslant 1$, so $(1 \circ f_a)(a) = 1$. Since $(1 \circ f_a)(a) \neq 0$, by Lemma 3, there exists $(x, y) \in A_a$ such that $f_a(y) \neq 0$. Then $f_a(y) \neq 1$, and $y = a$. Thus we have $a \leqslant xy = xa \in Sa$, and $a \in (Sa]$. □

In a similar we prove the following:

Theorem 10. *An ordered semigroup $(S, ., \leqslant)$ is quasi right regular if and only if for every fuzzy subset f of S, we have $f \preceq f \circ 1$.* □

Corollary 11. *A quasi-regular ordered semigroup is quasi-semisimple.*

Proof. Let f be a fuzzy subset of S. Since S is quasi left regular, by Theorem 9, we have $f \preceq 1 \circ f$. Since S is quasi right regular, by Theorem 10, we have $f \preceq f \circ 1$. Then we have $f \preceq 1 \circ f \preceq 1 \circ (f \circ 1) = 1 \circ f \circ 1$. By Theorem 7, S is quasi-semisimple. □

References

[1] **N. Kehayopulu**, *Characterization of left quasi-regular and semisimple ordered semigroups in terms of fuzzy sets*, Int. J. Algebra **6** (2012), 747 − 755.

[2] **N. Kehayopulu and M. Tsingelis**, *Characterization of some types of ordered semigroups in terms of fuzzy sets*, Lobachevskii J. Math. **29** (2008), 14 − 20.

Received May 23, 2012