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Clifford congruences on an idempotent-surjective R-semigroup

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Abstract. In the paper we describe the least Clifford congruence ξ on an idempotent-surjective R-semigroup, and so we generalize the result of LaTorre (1983). In addition, a characterization of all Clifford congruences on such a semigroup (in particular, on a structurally regular semigroup) is given. Furthermore, we find necessary and sufficient conditions for ξ to be idempotent pure or E-unitary. Moreover, using some earlier result, we give a description of all USG-congruences on an idempotent-surjective semigroup, and so we generalize the result of Howie and Lallement for regular semigroups (1966). Finally, in Section 4 we study the subdirect products of an E-unitary semigroup and a Clifford semigroup.

1. Preliminaries

Whenever possible the notation and conventions of Howie [11, 12] are used. Let S be a semigroup and let $A \subseteq S$. Denote by E_A the set of all *idempotents* of A, that is, $E_A = \{a \in A : a^2 = a\}$, and by $Reg(S)$ the set of all *regular elements* of S, i.e., $Reg(S) = \{a \in S : a \in aSa\}$. We say that S is *regular* if $Reg(S) = S$. More generally, in [10] Hall observed that the set $Reg(S)$ of a semigroup S with $E_S \neq \emptyset$ forms a regular subsemigroup of S if and only if the product of any two idempotents of S is regular. In a such case, S is said to be an *R-semigroup*. Finally, if E_S is a subsemigroup of S, then S is called an *E-semigroup*. Clearly, any E-semigroup is an R-semigroup.

Let S be a semigroup, $a \in S$. The set $W(a) = \{x \in S : x = xax\}$ is called the set of *weak inverses* of a, so the elements of $W(a)$ will be called *weak inverse elements* of a. A semigroup S is said to be *E-inversive* if for every $a \in S$ there is $x \in S$ such that $ax \in E_S$ [21]. Clearly, S is E-inversive iff $W(a) \neq \emptyset$ $(a \in S)$, so if S is E-inversive, then for all $a \in S$ there is $x \in S$ such that $ax, xa \in E_S$. For some interesting results concerning E-inversive semigroups, see [18, 4].

A generalization of the concept of regularity will also prove convenient. Define a semigroup S to be *idempotent-surjective* if whenever ρ is a congruence on S and $a\rho$ is an idempotent of S/ρ, then $a\rho$ contains some idempotent of S [2]. The famous Lallement's Lemma says that all regular semigroups are idempotent-surjective. Finally, it is known that idempotent-surjective semigroups are E-inversive.

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On the other hand, Kopamu defined in [14] a countable family of congruences on a semigroup S, as follows: for each ordered pair of non-negative integers (m,n), he put:

$$\theta_{m,n} = \{(a,b) \in S \times S : (\forall x \in S^m, y \in S^n)\ xay = xby\},$$

and he made the convention that $S^1 = S$ and S^0 denotes the set containing only the empty word. In particular, $\theta_{0,0} = 1_S$. Recall from [14] that if $S/\theta_{m,n}$ is regular for some non-negative integers m,n, then S is *structurally regular*. Kopamu also proved that structurally regular semigroups are idempotent-surjective. Finally, in [8] the author showed that structurally regular semigroups are R-semigroups, and so *every* structurally regular semigroup is an *idempotent-surjective R-semigroup*.

Green's relations on S are denoted by $\mathcal{L}$ ($(a,b) \in \mathcal{L}$ if $Sa \cup \{a\} = Sb \cup \{b\}$), $\mathcal{R}$ ($(a,b) \in \mathcal{R}$ if $aS \cup \{a\} = bS \cup \{b\}$) and $\mathcal{H}$ ($= \mathcal{L} \cap \mathcal{R}$). Denote by $\mathcal{H}_a$ the $\mathcal{H}$-class containing the element a. Notice that Green's Theorem says that in an arbitrary semigroup S either $\mathcal{H}_a\mathcal{H}_a \cap \mathcal{H}_a = \emptyset$ or $\mathcal{H}_a$ is a group.

Recall that a semigroup S is a *semilattice* if $a^2 = a$, $ab = ba$ for all $a,b \in S$. Let $\mathcal{C}$ be some class of semigroups of the same type $\mathcal{T}$ (for example: the class of all groups); call its elements *$\mathcal{C}$-semigroups*. A congruence ρ on a semigroup S is said to be a *$\mathcal{C}$-congruence* if $S/\rho \in \mathcal{C}$. Clearly, the least semilattice congruence η (say) on an arbitrary semigroup S exists. Finally, a semigroup S is a *semilattice S/ρ of groups* if there exists a semilattice congruence ρ on S such that every ρ-class is a group. Since $\mathcal{H} \subseteq \eta$, then a semigroup S is a semilattice S/ρ of groups if and only if $\mathcal{H} = \eta$. Indeed, $\mathcal{H} \subseteq \eta \subseteq \rho$ and evidently $\rho \subseteq \mathcal{H}$. Consequently we have $\mathcal{H} = \eta$. The converse implication follows from Green's Theorem.

Moreover, some preliminaries about group congruences on a semigroup S are needed. A subset A of S is called (respectively) *full*; *reflexive* and *dense* if $E_S \subseteq A$; $(\forall a,b \in S)(ab \in A \Rightarrow ba \in A)$ and $(\forall s \in S)(\exists\, x,y \in S)\ sx, ys \in A$. Also, we define the *closure operator* ω on S by $A\omega = \{s \in S : (\exists\, a \in A)\ as \in A\}$ (where $A \subseteq S$). We shall say that $A \subseteq S$ is *closed* (in S) if $A\omega = A$. Further, a subsemigroup N of a semigroup S is said to be *normal* if it is full, dense, reflexive and closed (if N is normal, then we shall write $N \lhd S$). Finally, if a subsemigroup of S is dense and reflexive, then it is called *quasi-normal*.

By the *kernel* of a congruence ρ on a semigroup S we shall mean the set $\ker(\rho) = \{x \in S : (x,x^2) \in \rho\}$.

Result 1.1. [5] *Let B be a quasi-normal subsemigroup of a semigroup S. Then the relation $\rho_B = \{(a,b) \in S \times S : (\exists\, x,y \in B)\, ax = yb\}$ is a group congruence on S. Also, $B \subseteq B\omega = \ker(\rho_B)$, and if $B \lhd S$, then $B = \ker(\rho_B)$.*

Conversely, if ρ is a group congruence on S, then there is a normal subsemigroup N of S such that $\rho = \rho_N$ (in fact, $N = \ker(\rho)$). Thus there is an inclusion-preserving bijection between the set of all normal subsemigroups of S and the set of all group congruences on S.

Moreover, the least group congruence on an E-inversive E-semigroup is given by

$$\sigma = \{(a,b) \in S \times S : (\exists\, e,f \in E_S)\ ea = bf\}. \qquad \square$$

Remark 1.2. [5] Let B be a quasi-normal subsemigroup of S. Then:

$$(a, b) \in \rho_B \Leftrightarrow (\exists\, x \in S)\ xa, xb \in B.$$

It is easily seen that if S is an E-inversive semigroup (and so E_S is dense), then there exists the least normal subsemigroup of S. In the light of Result 1.1, every E-inversive semigroup possesses the least group congruence σ.

An inverse semigroup in which the idempotents are central is called a *Clifford* semigroup. Recall that a semigroup S is a Clifford semigroup if and only if it is a semilattice of groups [11]. Observe that if $ab = e \in E_S$, then

$$ba = baa^{-1}a = a^{-1}aba = a^{-1}ea \in E_S.$$

Thus $ab = ba$ (since ab and ba belong to the same subgroup of S), so E_S is reflexive. Further, a semigroup S is called *η-simple* if S has no semilattice congruences except the universal relation. It is well known that every η-class of S is η-simple [20].

Recall from [9] that a full quasi-normal subsemigroup of a semigroup is called *seminormal*.

Finally, we have need the following two results.

Theorem 1.3. *Let ρ be an arbitrary semilattice congruence on an idempotent-surjective R-semigroup S, N be a (semi)normal subsemigroup of S and let $a \in S$. Put $N_a = N \cap a\rho$. Then:*

(*a*) *$a\rho$ is an E-inversive R-semigroup;*

(*b*) *N_a is a (semi)normal subsemigroup of $a\rho$.*

Proof. (*a*). Let $a \in S$ and $e \in E_{a\eta}$. Suppose by way of contradiction that $a\eta$ is not E-inversive. Then the set A of all non E-inversive elements of $a\eta$ is an ideal of $a\eta$. Clearly, $e \notin A$. Consider an equivalence ρ (say) on $a\eta$ induced by the partition: $\{A, a\eta \setminus A\}$ and suppose that there are elements $s, t \in a\eta \setminus A$ such that $st \in A$. Then $fg \in A$ for some idempotents $f, g \in a\eta \setminus A$. Since S is an R-semigroup, then $x = xfgx, fg = fgxfg$ for some $x \in S$. It follows that $x \in a\eta$, so $x \in W(fg)$ in $a\eta$, which contradicted to $fg \in A$. Hence ρ is a semilattice congruence on an η-simple semigroup $a\eta$, a contradiction. Consequently, $A = \emptyset$ (since $e \notin A$), and so $a\eta$ is an E-inversive R-semigroup.

(*b*). The second part of the theorem is a direct consequence of the definition of a (semi)normal subsemigroup and the first part of the theorem. □

Lemma 1.4. *Let B be the least seminormal subsemigroup of an idempotent-surjective semigroup S. If ϕ is an epimorphism of S onto a Clifford semigroup T, then $B\phi = E_T$.*

Proof. Put $A = (E_T)\phi^{-1}$. Clearly, A is a full subsemigroup of S. Thus A is dense. Moreover, if $xy \in A$, then $E_T \ni (xy)\phi = x\phi \cdot y\phi = y\phi \cdot x\phi = (yx)\phi$ (since E_T is reflexive), so $yx \in A$. Hence $B \subseteq A$. Thus $B\phi \subseteq ((E_T)\phi^{-1})\phi \subseteq E_T$. Since S is idempotent-surjective and B is full, then $E_T = (E_S)\phi \subseteq B\phi$. Consequently, $B\phi = E_T$. □

2. Clifford congruences

Let ε be a semilattice congruence on an idempotent-surjective R-semigroup S. Denote ε-classes of S by S_α, where α's are elements of some set A, and define on A a binary operation $\circ$, as follows: if $a \in S_\alpha, b \in S_\beta$, then

$$\alpha \circ \beta = \gamma \Leftrightarrow ab \in S_\gamma.$$

Clearly, $(A, \circ)$ is a semilattice (isomorphic to S/ϵ), so

$$S = \bigcup\{S_\alpha : \alpha \in A\}$$

is a semilattice A of E-inversive R-semigroups S_α (Theorem 1.3(a)). For any seminormal subsemigroup I of S, put $I_\alpha = I \cap S_\alpha$ ($\alpha \in A$); see Theorem 1.3(b). Then by Result 1.1 and Remark 1.2, for every α, the relation

$$\rho_{I_\alpha} = \{(a, b) \in S_\alpha \times S_\alpha : (\exists\, x \in S_\alpha)\ xa, xb \in I_\alpha\}$$

is a group congruence on S_α. Put $\rho = \bigcup\{\rho_{I_\alpha} : \alpha \in A\}$. We will show that ρ is a congruence on S. Let $(a, b) \in \rho$, say $(a, b) \in \rho_{I_\alpha}$; $c \in S_\beta$. Then $xa, xb \in I_\alpha$ for some $x \in S_\alpha$. Since I_β is dense, then $cz \in I_\beta$ for some $z \in S_\beta$. Notice that $ac, bc, zx \in S_{\alpha\beta}$. Furthermore, $(xa)(cz) \in I_\alpha I_\beta \subseteq I$. Hence $(zx)(ac) \in I$ (since I is reflexive), therefore, $(zx)(ac) \in I \cap S_{\alpha\beta} = I_{\alpha\beta}$. Similarly, $(zx)(bc) \in I_{\alpha\beta}$. This implies that $(ac, bc) \in \rho$, and so ρ is a right congruence on S. By symmetry of the definition of ρ_{I_α}, we conclude that ρ is also a left congruence on S. Thus ρ is a congruence on S and for all $a \in S$, $a\rho = a\rho_{I_\alpha}$ if $a \in S_\alpha$. Put $G_\alpha = S_\alpha/\rho_{I_\alpha}$. Then $S/\rho = \bigcup\{G_\alpha : \alpha \in A\}$ is a semilattice A of groups G_α.

Applying the above construction (of ρ) to the least semilattice congruence η on S and to the least seminormal subsemigroup B of S, we obtain some semilattice of groups congruence on S, say ξ.

Let S be an idempotent-surjective E-semigroup. Then each η-class of S is an E-semigroup. Define on every S_α the least group congruence σ_α (see Result 1.1). Then the relation ξ^*, induced by this partition of S, is a congruence on S. Indeed, if $a\,\xi^*\,b$, say $(a, b) \in \sigma_\alpha$ in S_α; $c \in S_\beta$, then $ea = bf$, where $e, f \in E_{S_\alpha}$, and so $(bcc^*b^*e)ac = bc(c^*{\cdot}b^*bf{\cdot}c)$ for every $b^* \in W_{S_\alpha}(b)$, $c^* \in W_{S_\beta}(c)$. The expressions in the parentheses belong to E_S. Further, $bcc^*b^*e, c^*b^*bfc \in S_{\alpha\beta}$, $ac, bc \in S_{\alpha\beta}$. Hence ξ^* is a right congruence on S. By symmetry, ξ^* is a left congruence on S. Thus S/ξ^* is a semilattice of groups.

Finally, we will show that ξ is the least Clifford congruence on an idempotent-surjective R-semigroup S. Let ρ be any congruence on S such that S/ρ is a semilattice A of groups, say $S/\rho = \bigcup\{G_\alpha : \alpha \in A\}$; $\rho^\natural$ be the natural homomorphism of S onto S/ρ and φ be the canonical morphism of S/ρ onto A, defined by $(a\rho)\varphi = \alpha$ if $a\rho \in G_\alpha$. The composition map $\Phi = \rho^\natural\varphi$ is a morphism of S onto A, so $\Phi\Phi^{-1}$, where $a(\Phi\Phi^{-1})b$ if and only if $a\rho, b\rho \in G_\alpha$ for some $\alpha \in A$, is a semilattice congruence on S. Thus $\eta \subseteq \Phi\Phi^{-1}$. Suppose that $a\,\xi\,b$. Then $a\,\eta\,b$ and $xa = by$ for some

$x, y \in a\eta \cap B$, where B is the least seminormal subsemigroup of S. Since x, y, a, b lie in the same η-class, then they belong to the same $\Phi\Phi^{-1}$-class, so $x\rho, y\rho, a\rho, b\rho$ lie in G_α ($\alpha \in A$). Since $x, y \in B$, then $x\rho, y\rho \in E_{S/\rho}$ (Lemma 1.4), so $x\rho = y\rho = 1_{G_\alpha}$ (the identity of the group G_α). It follows that

$$a\rho = (x\rho)(a\rho) = (xa)\rho = (by)\rho = (b\rho)(y\rho) = b\rho.$$

Consequently, $\xi \subseteq \rho$, as required.

Observe that if S is an E-semigroup, then $x, y \in E_S$ (by the definition of ξ^*), so obviously $x\rho = y\rho = 1_{G_\alpha} \in E_{S/\rho}$. Thus $\xi^* \subseteq \rho$.

Note that $\xi, \xi^* \subseteq \eta \cap \sigma$ and denote by $B_{a\eta}$ the intersection of $a\eta$ and B ($a \in S$).

We have just shown the following theorem.

Theorem 2.1. *The least Clifford congruence on an idempotent-surjective R-semigroup S is given by*

$$\xi = \{(a, b) \in \eta : (\exists\, x, y \in B_{a\eta})\ xa = by\}. \qquad \square$$

Remark 2.2. In the light of Remark 1.2,

$$\xi = \{(a, b) \in \eta : (\exists\, x \in a\eta)\ xa, xb \in B_{a\eta}\}.$$

Corollary 2.3. *The least Clifford congruence on an idempotent-surjective E-semigroup S is given by*

$$\xi^* = \{(a, b) \in \eta : (\exists\, e, f \in E_{a\eta})\ ea = bf\}. \qquad \square$$

Note also that we have proved the first part of the following theorem which is new for regular semigroups (and it is probably new even for inverse semigroups).

Theorem 2.4. *Let ε be an arbitrary semilattice congruence on an idempotent-surjective R-semigroup S and let A be a seminormal subsemigroup of S. Then the relation*

$$\rho_{A,\varepsilon} = \{(a, b) \in \varepsilon : (\exists\, x, y \in a\varepsilon \cap A)\ xa = by\}$$

is a Clifford congruence on S.

Conversely, if ρ is a Clifford congruence on S, then there exists a semilattice congruence ε on S and a seminormal subsemigroup A of S such that $\rho = \rho_{A,\varepsilon}$.

Proof. Let ρ be a semilattice of groups congruence on S. Since S/ρ is a semilattice of groups, then the least semilattice congruence on S/ρ is $\mathcal{H}^{S/\rho}$. Define a relation ε on S, as follows: $(a, b) \in \varepsilon$ if and only if $(a\rho, b\rho) \in \mathcal{H}^{S/\rho}$. Then $\mathcal{H}^{S/\rho} = \varepsilon/\rho$. It follows that ε is a semilattice congruence on S, since $(S/\rho)/\mathcal{H}^{S/\rho} \cong S/\varepsilon$. Next, put

$$A = \bigcup\{e\rho : e \in E_S\}.$$

Since S is idempotent-surjective and $E_{S/\rho}$ is a subsemigroup of S/ρ, then A is a semigroup. Obviously, A is full. Finally, A is reflexive, since $E_{S/\rho}$ is reflexive.

Consequently, A is a seminormal subsemigroup of S. Further, note that $\rho \subseteq \varepsilon$, and consider an arbitrary ρ-class $e\rho$, where $e \in E_S$. Let $x \in (e\rho)\omega$ in $e\varepsilon$ (in particular, $(x\rho, e\rho) \in \mathcal{H}^{S/\rho}$). Then $ax \in e\rho$ for some $a \in e\rho$. Hence

$$e\rho = (a\rho)(x\rho) = (e\rho)(x\rho) = x\rho,$$

because $(x\rho, e\rho) \in \mathcal{H}^{S/\rho}$. Thus $e\rho$ is closed in $e\varepsilon$. Since $A \cap e\varepsilon = e\rho$ for every $e \in E_S$, then $\rho = \rho_{A,\varepsilon}$, as required. □

A congruence ρ on a semigroup S is called *idempotent pure* if $e\rho \subseteq E_S$ for all $e \in E_S$. Note that if S is idempotent-surjective, then ρ is idempotent pure if and only if $\ker(\rho) = E_S$.

Let $\mathcal{E}$ be the relation on a semigroup S induced by the partition $\{E_S, S \setminus E_S\}$. Then $\mathcal{E}^\flat$ is the greatest idempotent pure congruence on S. Put $\tau = \mathcal{E}^\flat$. Then [12]

$$\tau = \{(a, b) \in S \times S : (\forall x, y \in S^{(1)})\ xay \in E_S \Leftrightarrow xby \in E_S\},$$

where $S^{(1)}$ denotes the semigroup obtained from S by adjoining the identity 1.

Recall from [5] that an E-inversive semigroup S is *E-unitary* if and only if E_S is closed in S.

The following result will be useful.

Result 2.5. [5, 7] *Let S be an idempotent-surjective semigroup. Then the following conditions are equivalent*:

(a) *S is E-unitary*;

(b) $\ker(\sigma) = E_S$;

(c) *every idempotent pure congruence on S is E-unitary*;

(d) *there exists an idempotent pure E-unitary congruence on S*;

(e) $\sigma = \tau$. □

The following theorem gives necessary and sufficient conditions for ξ to be idempotent pure. Note that the condition (c) is new even for regular semigroups.

Theorem 2.6. *Let S be an idempotent-surjective R-semigroup. Then the following conditions are equivalent*:

(a) *ξ is idempotent pure*;

(b) *each η-class of S is an E-unitary E-inversive subsemigroup of S*;

(c) $\xi = \eta \cap \tau$.

Proof. (a) $\Longleftrightarrow$ (b). It follows from the construction of ξ and Result 2.5 (see (b)).

(a) $\Longrightarrow$ (c). Let ξ be idempotent pure, that is, $\xi \subseteq \tau$. Then evidently $\xi \subseteq \eta \cap \tau$. Conversely, let $a(\eta \cap \tau)b$. Take any weak inverse x of a in $a\eta$. Then $(xa, xb) \in \tau$, where $xa \in E_{a\eta}$. Since $xb \in a\eta$, then $xb \in E_{a\eta}$. Thus $(a, b) \in \xi$ (by Remark 2.2).

(c) $\Longrightarrow$ (a). This is trivial. □

Corollary 2.7. *Let S be an idempotent-surjective R-semigroup. Then ξ is idempotent pure if and only if S is a semilattice of E-unitary E-inversive semigroups.* □

Moreover, we have the following theorem.

Theorem 2.8. *Let S be an idempotent-surjective R-semigroup. Then the following conditions are equivalent*:
(*a*) *S is E-unitary*;
(*b*) *ξ is an idempotent pure E-unitary congruence on S*;
(*c*) *for every $a \in S$, $a\eta$ is E-unitary and $\sigma_{a\eta} = \sigma_S \cap (a\eta \times a\eta)$.*

Proof. $(a) \Longrightarrow (b)$. If S is E-unitary, then each η-class of S is also E-unitary and so, by Theorem 2.6, ξ is idempotent pure. Hence by Result 2.5, ξ is E-unitary.

$(b) \Longrightarrow (a)$. This follows from Result 2.5.

$(a) \Longrightarrow (c)$. Let $a \in S$. It is clear that $a\eta$ is E-unitary. Also, if $(a, b) \in \sigma$, then $ab^* \in E_S$ for all $b^* \in W(b)$, so if $(a, b) \in \sigma \cap (a\eta \times a\eta)$, then $ab^* \in E_{a\eta}$ for all $b^* \in W(b)$ in $a\eta$. Thus $ab^*b \in E_{a\eta}b$. It follows that $(a, b) \in \sigma_{a\eta}$. Therefore $\sigma \cap (a\eta \times a\eta) \subseteq \sigma_{a\eta}$. The converse inclusion is obvious.

$(c) \Longrightarrow (a)$. Let $e \in E_S$, $x \in a\eta$, where $a \in S$. Choose $f \in E_{a\eta}$ and suppose that $(x, e) \in \sigma_S$. Clearly, $(e, f) \in \sigma_S$. Hence $(x, f) \in \sigma_S \cap (a\eta \times a\eta) = \sigma_{a\eta}$. Thus $x \in E_S$, so S is E-unitary (by Result 2.5). □

The next result gives some equivalent conditions for ξ to be E-unitary, when ξ is idempotent pure.

Corollary 2.9. *Let an idempotent-surjective R-semigroup S be a semilattice of an E-unitary E-inversive semigroups. Then the following conditions are equivalent*:
(*a*) *S is E-unitary*;
(*b*) $\xi = \eta \cap \sigma$;
(*c*) *ξ is E-unitary*;
(*d*) *for every $a \in S$, $\sigma_{a\eta} = \sigma_S \cap (a\eta \times a\eta)$.*

Proof. $(a) \Longrightarrow (b)$. The main assumption of the corollary implies that ξ is idempotent pure (Corollary 2.7). Hence $\xi = \eta \cap \tau$ (Theorem 2.6). Since S is E-unitary, then $\tau = \sigma$ (Result 2.5). Thus $\xi = \eta \cap \sigma$.

$(b) \Longrightarrow (c)$. The congruences η and σ are both E-unitary. Therefore $\xi = \eta \cap \sigma$ is also E-unitary.

$(c) \Longrightarrow (a)$. The assumptions imply that the congruence ξ is idempotent pure and E-unitary. Thus S is E-unitary (Result 2.5).

$(a) \Longleftrightarrow (d)$. It is a consequence of Theorem 2.8. □

Finally, we have the following corollary.

Corollary 2.10. *In any E-unitary idempotent-surjective semigroup S,*

$$\xi \cap \mathcal{H} = 1_S.$$

If in addition E_S forms a semilattice, then

$$\xi \cap \mathcal{L} = \xi \cap \mathcal{R} = 1_S.$$

Proof. This follows from Theorem 5.5 [5], since $\xi \subseteq \sigma$. □

3. USG-congruences

A semigroup S is said to be a USG-*semigroup* if it is an E-unitary Clifford semigroup. Recall from [13] that if S is a USG-semigroup, then $\sigma \cap \eta = 1_S$.

Remark that if a semigroup is a subdirect product of a group and a semilattice, then it is an E-semigroup.

Theorem 3.1. *In any idempotent-surjective semigroup S, $\sigma \cap \eta = 1_S$ if and only if S is a* USG-*semigroup.*

Proof. Let $\sigma \cap \eta = 1_S$. Then S is a subdirect product of the group S/σ and the semilattice S/η, so S is an idempotent-surjective E-semigroup. In particular, the least Clifford congruence ξ exists on S. Also, $\xi \subseteq \sigma \cap \eta$ and so $\xi = 1_S$. Hence S is a semilattice of groups. Thus $\mathcal{H} = \eta$. Let $(x, e) \in \sigma$ (where $x \in S, e \in E_S$). Then (since $x \in \mathcal{H}_f$ for some $f \in E_S \subseteq e\sigma$) $(x, f) \in \sigma \cap \mathcal{H} = \sigma \cap \eta = 1_S$, so $x = f \in E_S$. Consequently, S is E-unitary. □

If ρ, υ are two congruences on S such that $\rho \subseteq \upsilon$, then the map $\varphi : S/\rho \to S/\upsilon$, $(a\rho)\varphi = a\upsilon$ $(a \in S)$, is a well-defined epimorphism between these semigroups. Denote its kernel $\varphi\varphi^{-1}$ by

$$\upsilon/\rho = \{(a\rho, b\rho) \in S/\rho \times S/\rho : a\,\upsilon\,b\}.$$

Then $(S/\rho)/(\upsilon/\rho) \cong S/\upsilon$. Also, each congruence α on S/ρ is of the form υ/ρ, where $\upsilon \supseteq \rho$ is a congruence on S. Indeed, the relation υ, defined on S by: $a\,\upsilon\,b$ if and only if $(a\rho, b\rho) \in \alpha$, is a congruence on S such that $\rho \subseteq \upsilon$ and $\alpha = \upsilon/\rho$. Finally, let $\rho \subseteq \upsilon_1, \upsilon_2$ (where υ_1, υ_2 are congruences on S). Then $(\upsilon_1/\rho) \cap (\upsilon_2/\rho) = (\upsilon_1 \cap \upsilon_2)/\rho$, and $(\upsilon_1 \cap \upsilon_2)/\rho = 1_{S/\rho}$ implies that $\rho = \upsilon_1 \cap \upsilon_2$.

Note that if a class $\mathcal{C}$ of semigroups is closed under homomorphic images and the least $\mathcal{C}$-congruence $\rho_S^{\mathcal{C}}$ on a semigroup S exists, then the interval $[\rho_S^{\mathcal{C}}, S \times S]$ consists of all $\mathcal{C}$-congruences on S and is a complete sublattice of $\mathcal{C}(S)$.

Theorem 3.2. *Let $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$ be some classes of semigroups; $\rho_A^{\mathcal{C}_1}$, $\rho_A^{\mathcal{C}_2}$ be the least $\mathcal{C}_1$-congruence, $\mathcal{C}_2$-congruence on any semigroup A, respectively, such that $A \in \mathcal{C}_3$ if and only if $\rho_A^{\mathcal{C}_1} \cap \rho_A^{\mathcal{C}_2} = 1_A$. Then the intersection of a $\mathcal{C}_1$-congruence and a $\mathcal{C}_2$-congruence on a semigroup S is a $\mathcal{C}_3$-congruence. Conversely, every $\mathcal{C}_3$-congruence on S can be expressed in this way.*

Proof. Let ρ_i be a $\mathcal{C}_i$-congruence on S (for $i = 1, 2$). Put $\rho = \rho_1 \cap \rho_2$ and observe that ρ_1/ρ is a $\mathcal{C}_1$-congruence, ρ_2/ρ is a $\mathcal{C}_2$-congruence on S/ρ. Since $(\rho_1/\rho) \cap (\rho_2/\rho)$ is the identity relation on S/ρ, then $\rho_{S/\rho}^{\mathcal{C}_1} \cap \rho_{S/\rho}^{\mathcal{C}_2} = 1_{S/\rho}$. Thus $S/\rho \in \mathcal{C}_3$, and so $\rho = \rho_1 \cap \rho_2$ is a $\mathcal{C}_3$-congruence on S.

Conversely, let ρ be any $\mathcal{C}_3$-congruence on S, $\rho_1/\rho = \rho_{S/\rho}^{\mathcal{C}_1}$, $\rho_2/\rho = \rho_{S/\rho}^{\mathcal{C}_2}$, where $\rho \subseteq \rho_1, \rho_2$. Then ρ_i is a $\mathcal{C}_i$-congruence on S (for $i = 1, 2$). Furthermore,

$$(\rho_1 \cap \rho_2)/\rho = \rho_{S/\rho}^{\mathcal{C}_1} \cap \rho_{S/\rho}^{\mathcal{C}_1} = 1_{S/\rho}.$$

Thus $\rho = \rho_1 \cap \rho_2$, as required. □

Remark 3.3. One can modify Theorem 3.2 for any type of a universal algebra.

The following theorem describes all USG-congruences on idempotent-surjective semigroups.

Theorem 3.4. *The intersection of a group congruence ν and a semilattice congruence γ on an idempotent-surjective semigroup S is a* USG*-congruence.*

Conversely, any USG*-congruence ρ on S can be expressed in this way, and ν, γ are uniquely determined by ρ.*

Proof. Note that the class of all idempotent-surjective semigroups is closed under homomorphic images. All assertions of the theorem except a uniqueness follows from Theorems 3.1, 3.2 (see the proof of Theorem 3.2).

Let $\rho = \nu_1 \cap \gamma_1 = \nu_2 \cap \gamma_2$, where ν_i is a group congruence and γ_i is a semilattice congruence on S $(i = 1, 2)$, and let $(a, b) \in \gamma_1$. Since $\gamma_1 \cap \gamma_2$ is a band congruence, then there are $e, f \in E_S$ such that $(a, e) \in \gamma_1 \cap \gamma_2$, $(e, f) \in \nu_1$ and $(f, b) \in \gamma_1 \cap \gamma_2$. In fact, $(e, f) \in \gamma_1 \cap \nu_1 = \gamma_2 \cap \nu_2 \subseteq \gamma_2$. Hence $(a, b) \in \gamma_2$. Thus $\gamma_1 \subseteq \gamma_2$. Similarly, we obtain the opposite inclusion, so $\gamma_1 = \gamma_2$. Put $\gamma_1 = \gamma_2 = \gamma$. Let $(a, b) \in \nu_1$. Then $(aab, abb) \in \nu_1 \cap \gamma \subseteq \nu_2$. Hence $(a, b) \in \nu_2$ (by cancellation), therefore, $\nu_1 \subseteq \nu_2$. By symmetry, $\nu_2 \subseteq \nu_1$. Consequently, $\nu_1 = \nu_2$, as required. □

Corollary 3.5. *The relation $\sigma \cap \eta$ is the least* USG*-congruence on an arbitrary idempotent-surjective semigroup S.* □

Corollary 3.6. *An idempotent-surjective semigroup is a subdirect product of a group and a semilattice if and only if it is a* USG*-semigroup.*

Proof. Let $S \subseteq G \times Y$ be a subdirect product of a group G and a semilattice Y. Then the two projection maps induce on S a group congruence and a semilattice congruence. The intersection of these congruences is the equality relation on S. Thus $\sigma \cap \eta = 1_S$, so S is a USG-semigroup (Theorem 3.1).

The converse implication is clear. □

Lemma 3.7. *Let S be an E-unitary idempotent-surjective semigroup. Then S/ξ is a* USG*-semigroup.*

Proof. Let S be E-unitary. Then every η-class of S is E-unitary, too. In the light of Theorem 2.6, ξ is idempotent pure. Hence ξ is E-unitary (Corollary 2.9). Thus S/ξ is a USG-semigroup. □

One can show without difficulty that the least E-unitary congruence π on an arbitrary E-inversive semigroup exists.

Lemma 3.8. *Let S be an idempotent-surjective R-semigroup. Then the relation*

$$(\xi \vee \pi)/\pi$$

is the least Clifford congruence on S/π.

Proof. Indeed, $S/(\xi \vee \pi)$ is a Clifford semigroup, so $(\xi \vee \pi)/\pi$ is a semilattice of groups congruence on S/π, since $S/(\xi \vee \pi) \cong (S/\pi)/((\xi \vee \pi)/\pi)$. On the other hand, if α is a semilattice of groups congruence on S/π, then $\alpha = \rho/\pi$, where $\pi \subseteq \rho$. Since $(S/\pi)/(\rho/\pi) \cong S/\rho$, then ρ is a Clifford congruence on S, so $\pi, \xi \subseteq \rho$. Hence $\xi \vee \pi \subseteq \rho$. Thus $(\xi \vee \pi)/\pi \subseteq \rho/\pi = \alpha$, as required. □

Theorem 3.9. *In any idempotent-surjective R-semigroup S,*

$$\sigma \cap \eta = \xi \vee \pi.$$

Proof. We have just seen that $S/(\xi \vee \pi) \cong (S/\pi)/((\xi \vee \pi)/\pi)$. By Lemmas 3.7, 3.8, $(S/\pi)/((\xi \vee \pi)/\pi)$ is an E-unitary semilattice of groups and so $S/(\xi \vee \pi)$ is also an E-unitary semilattice of groups. Thus $\xi \vee \pi$ is a USG-congruence on S. Moreover, $\xi \subseteq \sigma \cap \eta$ and $\pi \subseteq \sigma \cap \eta$. Hence $\xi \vee \pi \subseteq \sigma \cap \eta$. Thus $\xi \vee \pi = \sigma \cap \eta$ (because $\sigma \cap \eta$ is the least USG-congruence on S). □

Corollary 3.10. *In any E-unitary idempotent-surjective semigroup,*

$$\xi = \sigma \cap \eta.$$ □

4. The condition $\pi \cap \xi = 1_S$

In this section we characterize those idempotent-surjective R-semigroups S which are a subdirect product of an E-unitary semigroup and a Clifford semigroup, i.e., those semigroups S for which $\pi \cap \xi$ is the identity relation. Since E-unitary semigroups and Clifford semigroups are both E-semigroups, then S are E-semigroups, too.

In [2] Edwards defined the relation μ on a semigroup S by

$$(a, b) \in \mu \iff \begin{cases} (x\,\mathcal{L}\,ax \text{ or } x\,\mathcal{L}\,bx) \Longrightarrow ax\,\mathcal{H}\,bx, \\ (x\,\mathcal{R}\,xa \text{ or } x\,\mathcal{R}\,xb) \Longrightarrow xa\,\mathcal{H}\,xb, \end{cases}$$

where x is an arbitrary element of $Reg(S)$. Furthermore, he proved in [3] that μ is the maximum *idempotent-separating* congruence on an arbitrary idempotent-surjective semigroup S (that is, $\mu \cap (E_S \times E_S) = 1_S$).

Recall that a semigroup S is:

- *fundamental* if $\mu = 1_S$ [1];
- *η-simple* if $\eta = S \times S$ [20].

Note that if an E-inversive semigroup S is η-simple, then the least Clifford congruence ξ coincides with σ. Indeed, let ρ be a Clifford congruence on S. Since S/ρ is a Clifford semigroup, then the least semilattice congruence on S/ρ is $\mathcal{H}$. Define a relation ε on S, as follows: $(a, b) \in \varepsilon$ if $(a\rho)\mathcal{H}(b\rho)$. Then $\mathcal{H} = \varepsilon/\rho$, so ε is a semilattice congruence on S, since $(S/\rho)/\mathcal{H} \cong S/\varepsilon$. Thus $(a\rho)\mathcal{H}(b\rho)$ for all $a, b \in S$. Consequently, S/ρ is a group.

Recall that π denotes the least E-unitary congruence on an E-inversive semigroup. Clearly, $\pi \subseteq \sigma$ (the least group congruence).

From the last two paragraphs we obtain the following corollary.

Corollary 4.1. *Let S be an η-simple E-inversive semigroup. Then S is E-unitary if and only if $\pi \cap \xi = 1_S$.* □

Proposition 4.2. *Let S be an idempotent-surjective R-semigroup, $\pi \cap \xi = 1_S$. Then S is a semilattice of (η-simple) E-unitary E-inversive semigroups.*

Proof. It is sufficient to show that every η-class of S is E-unitary. Let $a \in S$. Then the restriction of π to $a\eta$ is an E-unitary congruence on $a\eta$ and the restriction of ξ to $a\eta$ is a group congruence on $a\eta$. From the assumption of the proposition follows that the intersection of these two congruences is the identity relation on $a\eta$, so the intersection of the least E-unitary congruence and the least Clifford congruence on $a\eta$ is also the identity relation. In the light of Corollary 4.1, $a\eta$ is E-unitary. □

Theorem 4.3. *Let S be a fundamental idempotent-surjective R-semigroup. Then $\pi \cap \xi = 1_S$ if and only if S is E-unitary.*

Proof. Let $\pi \cap \xi = 1_S$; $e, f \in E_S$. If $(e, f) \in \pi$, then $(e, f) \in \eta$. Hence $(e, f) \in \xi$. Thus $e = f$, so $\pi \subseteq \mu = 1_S$. Consequently, S is E-unitary.

The converse implication is trivial. □

Remark 4.4. The above theorem is valid for any $\mathcal{C}$-congruence ρ (instead of π) contained in η (i.e., if we replace in the theorem π by ρ, then we must replace "E-unitary" with "$\mathcal{C}$-semigroup").

Recall from [7] that (for idempotent-surjective semigroups) every congruence of the interval $[\pi, \sigma]$ is E-unitary. Also, $\ker(\rho) = \ker(\pi)$ for every $\rho \in [\pi, \sigma]$.

We have mentioned above that the class of idempotent-surjective semigroups is closed under homomorphic images. Using Hall's observation, one can prove without difficulty that the class of all idempotent-surjective R-semigroups possess this property. It is also known that the class of all structurally regular semigroups is closed under taking homomorphic images [14].

For regular semigroups S, $\mu \cap \tau = 1_S$. The next theorem gives necessary and sufficient conditions for $\pi \cap \xi$ to be the identity relation on idempotent-surjective R-semigroups S such that $\mu \cap \tau = 1_S$ (in particular, the theorem is valid, too, for structurally regular semigroups having this additional property).

Remark 4.5. Using Lemma 1.2 [17], Janet Mills proved for orthodox semigroups a similar result to the next theorem (see Theorem 3.5 [17]). However, the proof of her lemma is not correct (see [6]). Moreover, in [6] using different methods, the author showed the theorem of Mills (with a very important additional condition). Finally, notice that the implication "$(f) \Rightarrow (g)$" in the following theorem is proved in a different way than the corresponding implication in [6].

Theorem 4.6. *If S is an idempotent-surjective R-semigroup such that $\mu \cap \tau = 1_S$, then the following conditions are equivalent:*

(a) $\pi \cap \xi = 1_S$;

(b) *S is a semilattice of E-unitary E-inversive semigroups and $\pi \subseteq \mu$;*

(c) *S is a semilattice of E-unitary E-inversive semigroups and $\pi \subseteq \mu \cap \sigma \subseteq \sigma$;*

(d) *S is a semilattice of E-unitary E-inversive semigroups and the congruence $\mu \cap \sigma$ is E-unitary;*

(e) *S is a semilattice of E-unitary E-inversive semigroups and at least one idempotent-separating congruence on S (say ρ) is E-unitary;*

(f) *S is a subdirect product of an E-unitary idempotent-surjective semigroup and a Clifford semigroup;*

(g) *S is a semilattice of E-unitary E-inversive semigroups and the relation $\mathcal{H} \cap \sigma$ is E-unitary congruence on S.*

Proof. (a) $\Longrightarrow$ (b). This implication follows directly from Proposition 4.2 and from the proof of Theorem 4.3.

(b) $\Longrightarrow$ (c). This is clear, since $\pi \subseteq \sigma$.

(c) $\Longrightarrow$ (d). In that case, $\mu \cap \sigma \in [\pi, \sigma]$, so $\mu \cap \sigma$ is E-unitary.

(d) $\Longrightarrow$ (e). This is evident.

(e) $\Longrightarrow$ (a). In such case, $\pi \subseteq \rho \subseteq \mu$. Hence $\pi \cap \xi \subseteq \mu \cap \xi = \mu \cap (\eta \cap \tau)$ (see Corollary 2.7 and Theorem 2.6). Thus $\pi \cap \xi \subseteq \mu \cap \tau = 1_S$.

(a) $\Longrightarrow$ (f). This is clear.

(f) $\Longrightarrow$ (g). Suppose that S is a subdirect product of an E-unitary idempotent-surjective semigroup A and a Clifford semigroup T. Notice that $(a,t)(\mathcal{H}\cap\sigma)(b,w)$ in S if and only if $(a,b) \in \mathcal{H} \cap \sigma$ in A and $(t,w) \in \mathcal{H} \cap \sigma = \eta \cap \sigma$ in T, i.e., if and only if $a = b$ (Theorem 5.5 [5]) and $(t,w) \in \eta \cap \sigma$ in T. This implies that $\mathcal{H} \cap \sigma$ is a congruence on S. Finally, we will show that the congruence $\mathcal{H} \cap \sigma$ is E-unitary. Let

$$(e,g)(a,t)(\mathcal{H} \cap \sigma)(f,h),$$

where $(e,g), (f,h) \in E_S$, then $ea = f$ and $(gt, h) \in \mathcal{H} \cap \sigma$ in T. It follows that

$$a \in E_A \quad \& \quad t \in \ker(\sigma_T).$$

Hence

$$(t,i) \in \mathcal{H}^T \cap \sigma_T$$

for some $i \in E_T$, since T is a semilattice of groups. Consequently,

$$(a,t)(\mathcal{H} \cap \sigma)(a,i),$$

where $(a,i) \in E_S$, so $\mathcal{H} \cap \sigma$ is E-unitary.

(g) $\Longrightarrow$ (e). This is evident. □

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