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Congruences on completely inverse AG^{**}-groupoids

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Abstract. By a completely inverse AG^{**}-groupoid we mean an inverse AG^{**}-groupoid A satisfying the identity $xx^{-1} = x^{-1}x$, where x^{-1} denotes a unique element of A such that $x = (xx^{-1})x$ and $x^{-1} = (x^{-1}x)x^{-1}$. We show that the set of all idempotents of such groupoid forms a semilattice and the Green's relations $\mathcal{H}, \mathcal{L}, \mathcal{R}, \mathcal{D}$ and $\mathcal{J}$ coincide on A. The main result of this note says that any completely inverse AG^{**}-groupoid meets the famous Lallement's Lemma for regular semigroups. Finally, we show that the Green's relation $\mathcal{H}$ is both the least semilattice congruence and the maximum idempotent-separating congruence on any completely inverse AG^{**}-groupoid.

1. Preliminaries

By an *Abel-Grassmann's groupoid* (briefly an *AG-groupoid*) we shall mean any groupoid which satisfies the identity

$$xy \cdot z = zy \cdot x. \tag{1}$$

Such groupoid is also called a *left almost semigroup* (briefly an *LA-semigroup*) or a *left invertive groupoid* (cf. [2], [3] or [5]). This structure is closely related to a commutative semigroup, because if an AG-groupoid contains a right identity, then it becomes a commutative monoid. Moreover, if an AG-groupoid A with a left zero z is finite, then (under certain conditions) $A \setminus \{z\}$ is a commutative group (cf. [6]).

One can easily check that in an arbitrary AG-groupoid A, the so-called *medial law* is valid, that is, the equality

$$ab \cdot cd = ac \cdot bd \tag{2}$$

holds for all $a, b, c, d \in A$.

Recall from [11] that an *AG-band* A is an AG-groupoid satisfying the identity $x^2 = x$. If in addition, $ab = ba$ for all $a, b \in A$, then A is called an *AG-semilattice*.

Let A be an AG-groupoid and $B \subseteq A$. Denote the set of all idempotents of B by E_B, that is, $E_B = \{b \in B : b^2 = b\}$. From (2) follows that if $E_A \neq \emptyset$, then $E_A E_A \subseteq E_A$, therefore, E_A is an AG-band.

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Further, an AG-groupoid satisfying the identity

$$x \cdot yz = y \cdot xz \tag{3}$$

is said to be an *AG^{**}-groupoid*. Every AG^{**}-groupoid is *paramedial* (cf. [1]), i.e., it satisfies the identity

$$ab \cdot cd = db \cdot ca. \tag{4}$$

Notice that each AG-groupoid with a left identity is an AG^{**}-groupoid (see [1], too). Furthermore, observe that if A is an AG^{**}-groupoid, then (4) implies that if $E_A \neq \emptyset$, then it is an AG-semilattice. Indeed, in this case E_A is an AG-band and $ef = ee \cdot ff = fe \cdot fe = fe$ for all $e, f \in E_A$. Moreover, for $a, b \in A$ and $e \in E_A$, we have

$$e \cdot ab = ee \cdot ab = ea \cdot eb = e(ea \cdot b) = e(ba \cdot e) = ba \cdot ee = ba \cdot e = ea \cdot b,$$

that is,

$$e \cdot ab = ea \cdot b \tag{5}$$

for all $a, b \in A$ and $e \in E_A$. Thus, as a consequence, we obtain

Proposition 1.1. *The set of all idempotents of an AG^{**}-groupoid is either empty or a semilattice.*

We say that an AG-groupoid A with a left identity e is an *AG-group* if each of its elements has a *left inverse* a', that is, for every $a \in A$ there exists $a' \in A$ such that $a'a = e$. It is not difficult to see that such element a' is uniquely determined and $aa' = e$. Therefore an AG-group has exactly one idempotent.

Let A be an arbitrary groupoid, $a \in A$. Denote by $V(a)$ the set of all *inverses* of a, that is,

$$V(a) = \{a^* \in A : a = aa^* \cdot a,\ a^* = a^*a \cdot a^*\}.$$

An AG-groupoid A is called *regular* (in [1] it is called *inverse*) if $V(a) \neq \emptyset$ for all $a \in A$. Note that AG-groups are of course regular AG-groupoids, but the class of all regular AG-groupoids is vastly more extensive than the class of all AG-groups. For example, every AG-band A is regular, since $a = aa \cdot a$ for all $a \in A$. In [1] it has been proved that in any regular AG^{**}-groupoid A we have $|V(a)| = 1$ $(a \in A)$, so we call it an *inverse AG^{**}-groupoid*. In this case, we denote a unique inverse of $a \in A$ by a^{-1}. Notice that $(ab)^{-1} = a^{-1}b^{-1}$ for all $a, b \in A$. Further, one can prove that in an inverse AG^{**}-groupoid A, we have $aa^{-1} = a^{-1}a$ if and only if $aa^{-1}, a^{-1}a \in E_A$ (cf. [1]).

Many authors studied various congruences on some special classes of AG^{**}-groupoids and described the corresponding quotient algebras as semilattices of some subgroupoids (see for example [1, 5, 7, 8, 9, 10]). Also, in [1, 9] the authors studied congruences on inverse AG^{**}-groupoids satisfying the identity $xx^{-1} =$

$x^{-1}x$. We will be called such groupoids *completely inverse AG^{**}-groupoids.* A simple example of such AG^{**}-groupoid is an AG-group. In the light of Proposition 1.1, the set of all idempotents of any completely inverse AG^{**}-groupoid forms a semilattice.

A nonempty subset B of a groupoid A is called a *left ideal* of A if $AB \subseteq B$. The notion of a *right ideal* is defined dually. Also, B is said to be an *ideal* of A if it is both a left and right ideal of A. It is clear that for every $a \in A$ there exists the least left ideal of A containing the element a. Denote it by $L(a)$. Dually, $R(a)$ is the least right ideal of A containing the element a. Finally, $J(a)$ denotes the least ideal of A containing $a \in A$.

In a similar way as in semigroup theory we define the *Green's equivalences* on an AG-groupoid A by putting:

$$\begin{aligned} a\,\mathcal{L}\,b &\iff L(a) = L(b),\\ a\,\mathcal{R}\,b &\iff R(a) = R(b),\\ a\,\mathcal{J}\,b &\iff J(a) = J(b),\\ \mathcal{H} = \mathcal{L} \cap \mathcal{R}, &\quad \mathcal{D} = \mathcal{L} \vee \mathcal{R}. \end{aligned}$$

2. The main results

Let A be a completely inverse AG^{**}-groupoid. Then

$$a = (aa^{-1})a \in Aa$$

for every $a \in A$.

Proposition 2.1. *Let A be a completely inverse AG^{**}-groupoid, $a \in A$. Then*:
(a) $aA = Aa$;
(b) $aA = L(a) = R(a) = J(a)$;
(c) $\mathcal{H} = \mathcal{L} = \mathcal{R} = \mathcal{D} = \mathcal{J}$;
(d) $aA = (aa^{-1})A$;
(e) $aA = a^{-1}A$;
(f) $eA = fA$ *implies* $e = f$ *for all* $e, f \in E_A$.

Proof. (a). Let $b \in A$. Then

$$ab = (aa^{-1})a \cdot b = ba \cdot aa^{-1} = ba \cdot a^{-1}a = aa \cdot a^{-1}b = (a^{-1}b \cdot a)a \in Aa.$$

Thus $aA \subseteq Aa$. Also,

$$ba = b \cdot (aa^{-1})a = aa^{-1} \cdot ba = ab \cdot a^{-1}a = ab \cdot aa^{-1} = a(ab \cdot a^{-1}) \in aA,$$

so $Aa \subseteq aA$. Consequently, $aA = Aa$.

(b). Obviously, it is sufficient to show that $aA = Aa$ is an ideal of A. Let $x = ab \in aA$ and $c \in A$. Then we have $cx = c(ab) = a(cb) \in aA$ and $xc = (ab)c = (cb)a \in Aa = aA$.

(c). It follows from (b).

(d). Let $b \in A$. Then $ab = (aa^{-1})a \cdot b = ba \cdot aa^{-1} \in A(aa^{-1}) = (aa^{-1})A$, that is, $aA \subseteq (aa^{-1})A$. Furthermore, $(aa^{-1})b = (ba^{-1})a \in Aa = aA$. Thus $(aa^{-1})A \subseteq aA$. Consequently, the condition (d) holds.

(e). By (d), $aA = (aa^{-1})A = (a^{-1}a)A = (a^{-1}(a^{-1})^{-1})A = a^{-1}A$.

(f). Let $e, f \in E_A$ and $eA = fA$. Then $e \in fA$, that is, $e = fa$ for some $a \in A$. Hence $fe = f(fa) = (ff)a$ (by Proposition 1.1), and so $fe = e$. Similarly, $ef = f$. Since E_A is a semilattice, $e = f$. □

Corollary 2.2. *Let A be a completely inverse AG^{**}-groupoid. Then each left ideal of A is also a right ideal of A, and vice versa. In particular,*

$$L \cap R = LR$$

for every (left) ideal L and every (right) ideal R.

Proof. Let L be a left ideal of A and $l \in L$. Then $lA = Al \subseteq L$. It follows that

$$L = \bigcup \{lA : l \in L\}.$$

Since each component lA of the above set-theoretic union is a right ideal of A, then L is itself a right ideal of A. Similar arguments show that every right ideal of A is a left ideal.

Clearly, $LR \subseteq L \cap R$. Conversely, if $a \in L \cap R$, then $a = (aa^{-1})a \in LR$. Hence $L \cap R = LR$. □

Let A be a completely inverse AG^{**}-groupoid. Denote by $\mathcal{H}_a$ the equivalence $\mathcal{H}$-class containing the element $a \in A$. We say that $\mathcal{H}_a \leq \mathcal{H}_b$ if and only if $aA \subseteq bA$.

The following theorem is the main result of this paper.

Theorem 2.3. *If ρ is a congruence on a completely inverse AG^{**}-groupoid A and $a\rho \in E_{A/\rho}$ $(a \in A)$, then there exists $e \in E_{a\rho}$ such that $\mathcal{H}_e \leq \mathcal{H}_a$.*

Proof. Let ρ be a congruence on A, $a \in A$ and $a\rho a^2$. We know that there exists $x \in A$ such that $a^2 = a^2x \cdot a^2$, $x = xa^2 \cdot x$ and $a^2x = xa^2 \in E_A$. Notice that

$$a^2x \cdot aa = a(a^2x \cdot a) = a(xa^2 \cdot a) = a(aa^2 \cdot x) = aa^2 \cdot ax = a^2 \cdot a^2x = a^2 \cdot xa^2,$$

i.e., $a^2 = a^2 \cdot xa^2$. Put $e = a \cdot xa$. Then $e \,\rho\, (a^2 \cdot xa^2) = a^2 \,\rho\, a$. Hence $e \in a\rho$. Also,

$$e^2 = (a \cdot xa)(a \cdot xa) = a((a \cdot xa) \cdot xa) = a(ax \cdot (xa \cdot a)) = a(ax \cdot a^2x).$$

Further,

$$ax \cdot a^2x = ax \cdot xa^2 = a^2x \cdot xa = xa^2 \cdot xa = (xa^2 \cdot x)a$$

by (5), since $xa^2 \in E_A$. Hence $ax \cdot a^2x = xa$. Consequently,

$$e^2 = a \cdot xa = e \in E_A.$$

Thus, $e \in E_{a\rho}$.

Finally, let $b \in A$. Then $eb = (a \cdot xa)b = (b \cdot xa)a \in Aa = aA$, therefore, $eA \subseteq aA$, so $\mathcal{H}_e \leq \mathcal{H}_a$. □

We say that a congruence ρ on a groupoid A is *idempotent-separating* if $e\rho f$ implies that $e = f$ for all $e, f \in E_A$. Furthermore, ρ is a *semilattice* congruence if A/ρ is a semilattice. Finally, A is said to be a *semilattice A/ρ of AG-groups* if ρ is a semilattice congruence and every ρ-class of A is an AG-group.

Corollary 2.4. *Let A be a completely inverse AG^{**}-groupoid. Then*:

(a) *$\mathcal{H}$ is the least semilattice congruence on A*;

(b) *$\mathcal{H}$ is the maximum idempotent-separating congruence on A*;

(c) *A is a semilattice $A/\mathcal{H} \cong E_A$ of AG-groups $\mathcal{H}_e$ $(e \in E_A)$.*

Proof. (a). Let $aA = bA$ and $c, x \in A$. Then $x \cdot ca = c \cdot xa$. On the other hand,

$$xa \in Aa = aA = bA = Ab,$$

i.e., $xa = yb$, where $b \in A$, so $x \cdot ca = c \cdot yb = y \cdot cb \in A(cb)$. Thus $A(ca) \subseteq A(cb)$. By symmetry, we conclude that $A(ca) = A(cb)$. Moreover, $a = yb$ for some $y \in A$. Hence $ac \cdot x = xc \cdot a = xc \cdot yb = bc \cdot yx \in (bc)A$. Thus $(ac)A \subseteq (bc)A$. In a similar way we can obtain the converse inclusion, so $(ac)A = (bc)A$. Consequently, $\mathcal{H}$ is a congruence (by Proposition 2.1 (b)). In the light of Proposition 2.1 (d), every $\mathcal{H}$-class contains an idempotent of A. This implies that $A/\mathcal{H}$ is a semilattice, that is, $\mathcal{H}$ is a semilattice congruence on A.

Suppose that there is a semilattice congruence ρ on A such that $\mathcal{H} \nsubseteq \rho$. Then the relation $\mathcal{H} \cap \rho$ is a semilattice congruence which is properly contained in $\mathcal{H}$, and so not every $(\mathcal{H} \cap \rho)$-class contains an idempotent of A, since each $\mathcal{H}$-class contains exactly one idempotent (Proposition 2.1 (f)), a contradiction with Theorem 2.3. Consequently, $\mathcal{H}$ must be the least semilattice congruence on A.

(b). By (a) and Proposition 2.1 (f), $\mathcal{H}$ is an idempotent-separating congruence on A. On the other hand, if ρ is an idempotent-separating congruence on A and $(a, b) \in \rho$, then $(a^{-1}, b^{-1}) \in \rho$, so $(aa^{-1}, bb^{-1}) \in \rho$. Hence $aa^{-1} = bb^{-1}$. Let $x \in A$. Then

$$xa = x(aa^{-1} \cdot a) = x(bb^{-1} \cdot a) = bb^{-1} \cdot xa = (xa \cdot b^{-1})b \in Ab.$$

Thus $Aa \subseteq Ab$. By symmetry, we conclude that $Aa = Ab$. Consequently, $a\,\mathcal{H}\,b$ (Proposition 2.1 (b)), that is, $\rho \subseteq \mathcal{H}$, as required.

(c). We show that every $\mathcal{H}$-class of A is an AG-group. In view of the above and Proposition 2.1 $(d),(e)$, each $\mathcal{H}$-class is an AG^{**}-groupoid. Consider an arbitrary $\mathcal{H}$-class $\mathcal{H}_e$ $(e \in E_A)$. Let $a \in \mathcal{H}_e$. Then $aa^{-1} \in \mathcal{H}_e$. Hence $aa^{-1} = e$ and so $ea = a$, that is, e is a left identity of $\mathcal{H}_e$. Since $a^{-1}a = e$ and $a^{-1} \in \mathcal{H}_e$, then $\mathcal{H}_e$ is an AG-group. Obviously, $A/\mathcal{H} \cong E_A$. Consequently, A is a semilattice $A/\mathcal{H} \cong E_A$ of AG-groups $\mathcal{H}_e$ $(e \in E_A)$. □

We say that an ideal K of a groupoid A is the *kernel* of A if K is contained in every ideal of A. If in addition, K is an AG-group, then it is called the *AG-group kernel* of A. Finally, a congruence ρ on A is said to be an *AG-group* congruence if A/ρ is an AG-group.

Corollary 2.5. *Let A be a completely inverse AG^{**}-groupoid. If e is a zero of E_A, then $\mathcal{H}_e = eA$ is the AG-group kernel of A and the map $\varphi : A \to eA$ given by $a\varphi = ea$ $(a \in A)$ is an epimorphism such that $x\varphi = x$ for all $x \in eA$.*

Proof. Obviously, $\mathcal{H}_e \subseteq eA$. Conversely, if $x = ea \in eA$, then

$$xx^{-1} = ea \cdot ea^{-1} = ee \cdot aa^{-1} = e.$$

In a view of Proposition 2.1 (d), $x \in \mathcal{H}_e$. Consequently, $\mathcal{H}_e = eA$. If I is an ideal of A, then clearly $E_I \neq \emptyset$. Let $i \in E_I$. Then $e = ei \in E_I$. Hence $a = ea \in I$ for all $a \in \mathcal{H}_e$, so $\mathcal{H}_e \subseteq I$. Thus $\mathcal{H}_e = eA$ is the AG-group kernel of A. Also, for all $a, b \in A$, $(a\varphi)(b\varphi) = (ea)(eb) = (ee)(ab) = e(ab) = (ab)\varphi$, i.e., φ is a homomorphism of A into eA. Evidently, φ is surjective. Finally, $\varphi_{|eA} = 1_{eA}$ (by Proposition 1.1). □

Corollary 2.6. *Let A be a completely inverse AG^{**}-groupoid. If e is a zero of E_A, then*

$$\sigma = \{(a,b) \in A \times A : ea = eb\}$$

is the least AG-group congruence on A and $A/\sigma \cong \mathcal{H}_e$.

Proof. It is clear that σ is an AG-group congruence on A induced by φ (defined in the previous corollary). If ρ is also an AG-group congruence on A and $a\,\sigma\,b$, then $(e\rho)(a\rho) = (e\rho)(b\rho)$. By cancellation, $a\,\rho\,b$ and so $\sigma \subseteq \rho$. Obviously, $A/\sigma \cong \mathcal{H}_e$. □

Remark 2.7. Let I be an ideal of a completely inverse AG^{**}-groupoid A. The relation $\rho_I = (I \times I) \cup 1_A$ is a congruence on A. If e is a zero of E_A, then $\mathcal{H}_e$ is an ideal of A and $\sigma \cap \rho_{\mathcal{H}_e} = 1_A$. It follows that A is a subdirect product of the group $\mathcal{H}_e$ and the completely inverse AG^{**}-groupoid $A/\mathcal{H}_e$. Note that we may think about $A/\mathcal{H}_e$ as a groupoid $B = (A \setminus \mathcal{H}_e) \cup \{e\}$ with zero e, where all products $ab \in \mathcal{H}_e$ are equal e. In fact, $fg = e$ in A $(f, g \in E_A)$ if and only if $\mathcal{H}_f\mathcal{H}_g \subseteq \{e\} = \mathcal{H}_e$ in B.

Obviously, in any finite completely inverse AG^{**}-groupoid A, the semilattice E_A has a zero.

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