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# The spectrum of a variety of modular groupoids

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**Abstract.** We prove that the spectrum of the variety of idempotent, right modular and anti-rectangular groupoids consists of all powers of four. We also prove that any finite or countable groupoid anti-isomorphic to a groupoid in that variety is isomorphic to it. Finally, it is proved that, to within isomorphism, there is only one countable groupoid in that variety and that it is isomorphic to a proper subgroupoid of itself.

## 1. Introduction

Kazim and Naseeruddin studied a groupoid variety consisting of what they called *left almost semigroups*, groupoids satisfying the equation $xy \cdot z = zy \cdot x$ [9]. Such groupoids have also been referred to as *left invertive* [5], *Abel-Grassmann's* [8, 10, 11, 12, 14, 15, 16] and *right modular* [7]. Various aspects of these groupoids have been studied over the years, such as partial ordering and congruences [6], inflations [15], idempotent structure [14], zeroids and idempoids [12], structure of unions of groups [10], power groupoids and inclusion classes [11] simplicity [7] and combinatorial chacterization [1].

In this paper we study the variety $I \cap RM \cap AR$ of idempotent, right modular, anti-rectangular groupoids, the collection of groupoids that satisfy the equations $x = x^2$, $xy \cdot z = zy \cdot x$ and $xy \cdot x = y$. These groupoids also satisfy the equation $x \cdot yz = z \cdot yx$ and are therefore modular. They were called *anti-rectangular* AG-*bands* in [14] and are also known, perhaps more commonly, as *affine spaces over* $GF(4)$ [1, 4]. The main result of this paper is that there is, up to isomorphism, exactly one groupoid of order $4^n$ in $I \cap RM \cap AR$ for each $n \in \{0, 1, 2, \ldots\}$ and that there are no finite groupoids in $I \cap RM \cap AR$ of any other orders. We also prove that, up to isomorphism, there is only one countable groupoid in $I \cap RM \cap AR$ and that it is isomorphic to a proper subgroupoid of itself.

## 2. Preliminary definitions, notation and results

We use $G, H, J, \ldots$ to denote groupoids, $xy$ or $x \cdot y$ to denote the product of $x$ on the left with $y$ on the right. For example, $(xy \cdot z) \cdot yz = [(x \cdot y) \cdot z] \cdot (y \cdot z)$. The varieties of *idempotent* and *anti-rectangular* groupoids are denoted by $I$ and $AR$

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and are the collection of groupoids satisfying the equations $x = x^2$ and $xy \cdot x = y$ respectively.

The set of orders of the finite algebras in a groupoid variety $V$ is called the *spectrum of* $V$. We will denote this by $sp(V)$. T. Evans [3] showed that the spectrum of the groupoid variety defined by the equation $xy \cdot yz = y$ is the set $\{n^2 : n \in N\}$. Evans generalised this result and obtained, for each positive integer $n \in N$, a variety of groupoids having as spectrum all $n^{th}$ powers [2]. The main result in this paper, referred to in the introduction above, is that the spectrum of $I \cap RM \cap AR$ is $\{4^n : n \in N \cup \{0\}\}$.

There is another reason to study the structure of groupoids in $I \cap RM \cap AR$. Let $RM$ denote the variety of *right modular* groupoids determined by the equation $xy \cdot z = zy \cdot x$. Protić and Stepanović [14] proved that any idempotent, right modular groupoid $G$ is an idempotent, right modular groupoid $Y_G$ of members of $I \cap RM \cap AR$. In other words,

**Lemma 2.1.** [14, Theorem 2.1]
*If $G \in I \cap RM$, then there exists a groupoid $Y_G \in I \cap RM$ such that $G$ is a disjoint union of groupoids $G_\alpha$ $(\alpha \in Y_G)$, $G_\alpha G_\beta \subseteq G_{\alpha\beta}$ $(\alpha, \beta \in Y_G)$ and $G_\alpha \in I \cap RM \cap AR$ $(\alpha \in Y_G)$.*

So, the finite members of $I \cap RM \cap AR$ are basic building blocks of the finite members of $I \cap RM$. As we shall see, the basic building block of the finite members of $I \cap RM \cap AR$ is the following groupoid $\mathrm{T}_4$ of order 4, called *Traka* 4 in [14]. It is isomorphic to any groupoid generated by any two distinct elements, $a$ and $b$ say, of any member of $I \cap RM \cap AR$ and, therefore, $\mathrm{T}_4 \in I \cap RM \cap AR$ (see Lemma 2.4 below). The multiplication table of $\mathrm{T}_4$ is:

| $\mathrm{T}_4$ | $a$ | $b$ | $ab$ | $ba$ |
|---|---|---|---|---|
| $a$ | $a$ | $ab$ | $ba$ | $b$ |
| $b$ | $ba$ | $b$ | $a$ | $ab$ |
| $ab$ | $b$ | $ba$ | $ab$ | $a$ |
| $ba$ | $ab$ | $a$ | $b$ | $ba$ |

We will also show that if $G \in I \cap RM \cap AR$ and $|G| = 4^n$ then $G$ consists of $4^{n-1}$ disjoint copies of $\mathrm{T}_4$ (see Corollary 3.8). Some of the following results will be used throughout this paper. Several of the proofs are straightforward and are omitted.

**Lemma 2.2.** [13] *If $G \in RM$, then $G$ satisfies the identity $xu \cdot vy = xv \cdot uy$.*

**Lemma 2.3.** *If $G \in I \cap RM \cap AR$. then $G$ satisfies the identity $x \cdot yz = z \cdot yx$.*

*Proof.* $z \cdot yx = (yx \cdot z) \cdot z = (zx \cdot y) \cdot z = [zx \cdot (zy \cdot z)] \cdot z =$
$= [(z \cdot zy) \cdot (xz)] \cdot z = (z \cdot xz) \cdot (z \cdot zy) = x \cdot [(zy \cdot z) \cdot z] = x \cdot yz.$ □

**Lemma 2.4.** *Let $G \in I \cap RM \cap AR$ with $\{c, d\} \subseteq G$ and $c \neq d$. Then the subgroupoid $\langle c, d\rangle$ of $G$ generated by $c$ and $d$ is isomorphic to $\mathrm{T}_4$. One isomorphism is given by the mapping $c \to a$, $d \to b$, $cd \to ab$ and $dc \to ba$.*

**Lemma 2.5.** *Any two distinct elements of* $\mathrm{T}_4$ *generate* $\mathrm{T}_4$.

**Lemma 2.6.** *Any bijection on* $\mathrm{T}_4$ *is either an isomorphism or an anti-isomorphism. Four-cycles and two-cycles are anti-isomorphisms and the identity mapping, three-cycles and products of two-cycles are isomorphisms.*

**Lemma 2.7.** *Any groupoid anti-isomorphic to* $\mathrm{T}_4$ *is isomorphic to* $\mathrm{T}_4$. *In particular, if* $\Phi : \mathrm{T}_4 \to G$ *is an anti-isomorphism, then the mapping* $a \to \Phi a$, $b \to \Phi b$, $ab \to \Phi(ba)$ *and* $ba \to \Phi(ab)$ *is an isomorphism.*

**Lemma 2.8.** *Suppose that* $H$ *and* $K$ *are subgroupoids of* $G \in I \cap RM \cap AR$ *and that* $H \cong \mathrm{T}_4 \cong K$. *Then either* $H = K$, $H \cap K = \emptyset$ *or* $H \cap K = \{c\}$.

**Notation 2.9.** $G \cong H$ $[G \underset{\leftarrow}{\cong} H]$ will denote that $G$ and $H$ are isomorphic [anti-isomorphic].

**Lemma 2.10.** *If* $G \in I \cap RM \cap AR$ *and* $G \underset{\leftarrow}{\cong} H$, *then* $H \in I \cap RM \cap AR$.

*Proof.* Let $\Phi : G \to H$ be an anti-isomorphism. Then it is straightforward to show that $H$ is an idempotent groupoid that satisfies the equation $xy \cdot x = y$. Let $\{h_1, h_2, h_3\} \subseteq H$. Then there exists $\{g_1, g_2, g_3\} \subseteq G$ such that $h_i = \Phi g_i$, $i \in \{1, 2, 3\}$. Using Lemma 2.3, $h_1 h_2 \cdot h_3 = (\Phi g_1)(\Phi g_2) \cdot (\Phi g_3) = \Phi(g_2 g_1) \cdot (\Phi g_3) = \Phi(g_3 \cdot g_2 g_1) = \Phi(g_1 \cdot g_2 g_3) = \Phi(g_2 g_3) \cdot (\Phi g_1) = (\Phi g_3)(\Phi g_2) \cdot (\Phi g_1) = h_3 h_2 \cdot h_1$ and so $H$ satisfies the equation $xy \cdot z = zy \cdot x$. Hence, $H \in I \cap RM \cap AR$. □

## 3. The structure of finite members of $\mathbf{I} \cap \mathbf{RM} \cap \mathbf{AR}$

We use $G \leq H$ $[G \prec H]$ to denote that $G$ is a subgroupoid [proper subgroupoid] of the groupoid $H$. Recall that $a \in \mathrm{T}_4$.

**Theorem 3.1.** *If* $\mathrm{T}_4 \leq H \prec R$, $R \in I \cap RM \cap AR$ *and* $r \in R - H$, *then* $H_r = H \cup \{rh\}_{h \in H} \cup \{hr\}_{h \in H} \cup \{ar \cdot h\}_{h \in H}$ *is a subgroupoid of* $R$ *and, therefore,* $H_r \in I \cap RM \cap AR$. *If* $H$ *has* $n$ *elements then* $H_r$ *has* $4n$ *elements.*

*Proof.* We will prove that $H_r$ is closed under the multiplication inherited from $R$ and that its multiplication table is as follows:

| $H_r$ | $k$ | $rk$ | $kr$ | $ar \cdot k$ |
|---|---|---|---|---|
| $h$ | $hk$ | $ar \cdot (ka \cdot h)$ | $r \cdot kh$ | $(hk \cdot ah)\, r$ |
| $rh$ | $kh \cdot r$ | $r \cdot hk$ | $ar \cdot (k \cdot ah)$ | $a \cdot hk$ |
| $hr$ | $ar \cdot (ha \cdot kh)$ | $kh$ | $hk \cdot r$ | $r\,(ah \cdot k)$ |
| $ar \cdot h$ | $r\,(h \cdot ka)$ | $(hk \cdot a)\, r$ | $ak \cdot ha$ | $ar \cdot hk$ |

Table 1. The multiplication table for $\{h, k\} \subseteq H$.

We will use Lemma 2.2 and Lemma 2.3, together with the fact that $R$ is in $I \cap RM \cap AR$ to calculate the products in rows 2, 3, 4 and 5 of the table.

Row 2: The product in column 2 follows from the fact that $H$ is a subgroupoid of $R$. The product in column 4 follows from Lemma 2.3. For column 3, $h \cdot rk = h \cdot (ar \cdot a)\, k = h \cdot (ka \cdot ar) = ar \cdot (ka \cdot h)$. For column 5, $h \cdot (ar \cdot k) = (h \cdot ar) \cdot hk = (r \cdot ah) \cdot hk = (hk \cdot ah) \cdot r$.

Row 3: The product in column 2 follows from the right modularity of $R$. The product in column 3 follows from Lemma 2.2 and the fact that $R$ is an idempotent groupoid. For the product in column 4, $rh \cdot kr = rk \cdot hr = rk \cdot (ah \cdot a)\, r = rk \cdot (ra \cdot ah) = (r \cdot ra)\,(k \cdot ah) = ar \cdot (k \cdot ah)$. For the product in column 5, $rh \cdot (ar \cdot k) = (r \cdot ar) \cdot hk = a \cdot hk$.

Row 4: For the product in column 2, $hr \cdot k = [h\,(ar \cdot a)]\, k = [k\,(ar \cdot a)]\, h = kh \cdot [(ar \cdot a)\, h] = kh \cdot (ha \cdot ar) = ar\,(ha \cdot kh)$. For the product in column 3, $hr \cdot rk = (rk \cdot r)\, h = kh$. For the product in column 4, $hr \cdot kr = hk \cdot r$. For column 5, $hr \cdot (ar \cdot k) = (h \cdot ar) \cdot rk = (r \cdot ah) \cdot rk = r\,(ah \cdot k)$.

Row 5: For the product in column 2, $(ar \cdot h)\, k = (ar \cdot h)\,(a \cdot ka) = r\,(h \cdot ka)$. For column 3, $(ar \cdot h) \cdot rk = (ar \cdot r) \cdot hk = ra \cdot hk = (hk \cdot a)\, r$. For column 4, $(ar \cdot h) \cdot kr = (hr \cdot a) \cdot kr = (ha \cdot ra) \cdot kr = (ha \cdot k) \cdot a = ak \cdot ha$. The product in column 5 follows from Lemma 2.2 and the fact that $R$ is an idempotent groupoid.

Thus, $H_r$ is closed under the groupoid operation and hence $H_r$ belongs to $I \cap RM \cap AR$.

It is straightforward to show that the sets $H$, $\{rh\}_{h \in H}$, $\{hr\}_{h \in H}$ and $\{ar \cdot h\}_{h \in H}$ are pairwise disjoint sets. Furthermore, it is easy to show that, for $\{h, k\} \subseteq H$, two elements $rh$ and $rk$ [$hr$ and $kr$; $ar \cdot h$ and $ar \cdot k$] are equal if and only if $h = k$. Therefore, if $H$ contains $n$ elements then $H_r$ contains $4n$ elements. □

**Definition 3.2.** We will call $H_r$ the *extension of $H$ by $r$*.

**Corollary 3.3.** $sp\,(I \cap RM \cap AR) = \{4^n : n \in N \cup \{0\}\}$.

**Corollary 3.4.** *A groupoid $G \in I \cap RM \cap AR$ of order $4^n$ has $(n+1)$ generators, $n \in \{0, 1, \dots\}$.*

**Theorem 3.5.** *Suppose that $T_4 \leq H \in I \cap RM \cap AR$ and $r \notin H$. We define pairwise disjoint sets $A = \{rh\}_{h \in H}$, $B = \{hr\}_{h \in H}$ and $C = \{ar \circ h\}_{h \in H}$ such that $A \cap H = B \cap H = C \cap H = \emptyset$. Define $H^r = H \cup A \cup B \cup C$ with a product $\circ$ defined as in Table 2 below. Then $H^r \cong H_r$ and therefore $H^r \in I \cap RM \cap AR$.*

| $H^r$ | $k$ | $rk$ | $kr$ | $ar \circ k$ |
|---|---|---|---|---|
| $h$ | $hk$ | $ar \circ (ka \cdot h)$ | $r(kh)$ | $(hk \cdot ah)r$ |
| $rh$ | $(kh)r$ | $r(hk)$ | $ar \circ (k \cdot ah)$ | $a \cdot hk$ |
| $r$ | $ar \circ (ha \cdot kh)$ | $kh$ | $(hk)r$ | $r(ah \cdot k)$ |
| $ar \circ h$ | $r(h \cdot ka)$ | $(hk \cdot a)r$ | $ak \cdot ha$ | $ar \circ hk$ |

Table 2. The multiplication table for $\circ$ with $\{h, k\} \subseteq H$.

*Proof.* The product $\circ$ is well defined and closed and so $H^r$ is a groupoid. We define a mapping $\Phi : H^r \to H_r$ as follows: for any $h \in H$, $\Phi h = h$, $\Phi(rh) = rh$,

$\Phi(hr) = hr$ and $\Phi(ar \circ h) = ar \cdot h$. It is clear that $\Phi$ is one-to-one and onto $H_r$. We now show that $\Phi$ is a homomorphism. Let $\{x, y\} \subseteq H^r$. There are 16 possible forms $x \circ y$ can take.

Let $\{h, k\} \subseteq H$.

Case 1. $x = h$, $y = k$. Then $\Phi(x \circ y) = \Phi(hk) = hk = \Phi h \cdot \Phi k = \Phi x \cdot \Phi y$.

Case 2. $x = h$, $y = rk$. Then $\Phi(x \circ y) = \Phi(h \circ rk) = \Phi(ar \circ ka \cdot h)) = ar(ka \cdot h) = h \cdot rk = \Phi h \cdot \Phi(rk) = \Phi x \cdot \Phi y$.

Case 3. $x = h$, $y = kr$. Then $\Phi(x \circ y) = \Phi(h \circ kr) = \Phi(r \circ kh) = r \cdot kh = h \cdot kr = \Phi h \cdot \Phi(kr) = \Phi x \cdot \Phi y$.

Case 4. $x = h$, $y = ar \circ k$. Then we have $\Phi(x \circ y) = \Phi(h \circ (ar \circ k)) = \Phi((hk \cdot ah)r) = (hk \cdot ah)r = h(ar \cdot k) = \Phi h \cdot \Phi(ar \circ k) = \Phi x \cdot \Phi y$.

Case 5. $x = rh$, $y = k$. Then $\Phi(x \circ y) = \Phi(rh \circ k) = \Phi((kh)r) = kh \cdot r = rh \cdot k = \Phi(rh) \cdot \Phi k = \Phi x \cdot \Phi y$.

Case 6. $x = rh$, $y = rk$. Then $\Phi(x \circ y) = \Phi(rh \circ rk) = \Phi(r(hk)) = r \cdot hk = rh \cdot rk = \Phi(rh) \cdot \Phi(rk) = \Phi x \cdot \Phi y$.

Case 7. $x = rh$, $y = kr$. Then $\Phi(x \circ y) = \Phi(rh \circ kr) = \Phi(ar \circ (k \cdot ah)) = ar \cdot (k \cdot ah) = rh \cdot kr = \Phi(rh) \cdot \Phi(kr) = \Phi x \cdot \Phi y$.

Case 8. $x = rh$, $y = ar \circ k$. Then $\Phi(x \circ y) = \Phi(rh \circ (ar \circ k)) = a \cdot hk = rh \cdot (ar \cdot k) = \Phi(rh) \cdot \Phi(ar \cdot k) = \Phi x \cdot \Phi y$.

Case 9. $x = hr$, $y = k$. Then $\Phi(x \circ y) = \Phi(hr \circ k) = \Phi(ar \circ (ha \cdot kh)) = ar \cdot (ha \cdot kh) = hr \cdot k = \Phi(hr) \cdot \Phi k = \Phi x \cdot \Phi y$.

Case10. $x = hr$, $y = rk$. Then $\Phi(x \circ y) = \Phi(hr \circ rk) = \Phi(kh) = kh = hr \cdot rk = \Phi(hr) \cdot \Phi(rk) = \Phi x \cdot \Phi y$.

Case 11. $x = hr$, $y = kr$. Then $\Phi(x \circ y) = \Phi(hr \circ kr) = \Phi((hk)r) = hk \cdot r = hr \cdot kr = \Phi(hr) \cdot \Phi(kr) = \Phi x \cdot \Phi y$.

Case 12. $x = hr$, $y = ar \circ k$. Then $\Phi(x \circ y) = \Phi(hr \circ (ar \circ k)) = \Phi(r(ah \cdot k)) = r(ah \cdot k) = hr \cdot (ar \cdot k) = \Phi(hr) \cdot \Phi(ar \cdot k) = \Phi x \cdot \Phi y$.

Case 13. $x = ar \cdot h$, $y = k$. Then $\Phi(x \circ y) = \Phi((ar \circ h) \circ k) = \Phi(r(h \cdot ka)) = r(h \cdot ka) = (ar \cdot h) \cdot k = \Phi(ar \circ h) \cdot \Phi k = \Phi x \cdot \Phi y$.

Case 14. $x = ar \circ h$, $y = rk$. Then $\Phi(x \circ y) = \Phi((ar \circ h) \circ rk) = \Phi((hk \cdot a)r) = (hk \cdot a)r = (ar \cdot h) \cdot rk = \Phi(ar \cdot h) \cdot \Phi(rk) = \Phi x \cdot \Phi y$.

Case 15. $x = ar \circ h$, $y = kr$. Then $\Phi(x \circ y) = \Phi((ar \circ h) \circ kr) = ak \cdot ha = (ar \cdot h) \cdot kr = \Phi(ar \cdot h) \cdot \Phi(kr) = \Phi x \cdot \Phi y$.

Case 16. $x = ar \circ h$, $y = ar \circ k$. Then $\Phi(x \circ y) = \Phi((ar \circ h) \circ (ar \circ k)) = \Phi(ar(hk)) = ar \cdot hk = (ar \cdot h) \cdot (ar \cdot k) = \Phi(ar \cdot h) \cdot \Phi(ar \cdot k) = \Phi x \cdot \Phi y$.

Hence, $\Phi$ is an isomorphism and $H^r \cong H_r$. □

**Definition 3.6.** We define $G_0$ as the trivial groupoid, $G_1 = \mathrm{T}_4$ and by induction, $G_n = G_{n-1}^{r_{n-1}}$, $n \geqslant 2$, where $r_n \notin G_n$, $n \geqslant 1$.

**Corollary 3.7.** *Any finite member of* $I \cap RM \cap AR$ *is isomorphic to* $G_n$ *for some* $n \in \{0, 1, 2 \ldots\}$. *If* $G \in I \cap RM \cap AR$ *and* $|G| = 4^n$, *then* $G \cong G_n$.

**Corollary 3.8.** *For $n \in N$, $G_n$ is a disjoint union of groupoids $G_\alpha$ with $G_\alpha G_\beta \subseteq G_{\alpha\beta}$ and $G_\alpha \cong G_{n-1}$, $\alpha, \beta \in \mathrm{T}_4$. Therefore, $G_n$ is a disjoint union of $4^{n-1}$ copies of $\mathrm{T}_4$.*

## 4. The countable member of $\mathbf{I} \cap \mathbf{RM} \cap \mathbf{AR}$

In this section we show that, to within isomorphism, there is precisely one countable member of $I \cap RM \cap AR$. This result will follow from the following construction of such a groupoid.

**Construction 4.1.** Let $H = \bigcup_{n=1}^{\infty} G_n$, with the $G_n$'s as in Definition 3.6. Define a product $*$ on $H$ as follows. If $\{u, v\} \subseteq H$ with $u \in G_{n_u} - G_{n_u - 1}$ and $v \in G_{n_v} - G_{n_v - 1}$ then $u * v$ is defined as the product of $u$ and $v$ in $G_{\max\{n_u, n_v\}}$.

**Theorem 4.2.** *$H$ in Construction 4.1 is countable and $H \in I \cap RM \cap AR$.*

*Proof.* Clearly $*$ is well defined and $H$ is closed with respect to $*$. By Theorem 3.5, $G_n \in I \cap RM \cap AR$, $n \in N$, and since $\max\{\max\{n_u, n_v\}, n_w\} = \max\{\max\{n_w, n_v\}, n_u\}$, it follows easily that $H \in I \cap RM \cap AR$. Since each $G_n$, $n \in N$, is countable, so is $H$. □

**Theorem 4.3.** *A countable $K \in I \cap RM \cap AR$ is isomorphic to $H$ in Construction 4.1.*

*Proof.* Let $K = \bigcup_{n=1}^{\infty} \{y_n\}$, with $y_i = y_j$ if and only if $i = j$. Define $K_0 = \emptyset$, $K_1 = \{y_1, y_2, y_1y_2, y_2y_1\}$ and $R_1 = K - K_1$. Define $K_2 = K_1^{y_{t_1}}$, where $t_1$ is the minimum of the subscripts of the $y_n$'s in $R_1$. Define $R_2 = K - K_2$ and $K_3 = K_2^{y_{t_2}}$, where $t_2$ is the minimum subscript of the $y_n$'s in $R_2$. In general, by induction we define $R_n = K - K_n$ and $K_{n+1} = K_n^{y_{t_n}}$, where $t_n$ is the minimum subscript of the $y_n$'s in $R_n$. Then every $y_n$ must eventually appear in some $K_t$ and therefore $K = \bigcup_{n=0}^{\infty} K_n$. Note that if $\{h, k\} \subseteq K$, with $h \in K_n - K_{n-1}$ and $k \in K_m - K_{m-1}$, then the product $hk$ in $K$ equals the product $hk$ in $K_M$, where $M = \max\{n, m\}$.

By Lemma 2.4, $K_1 \cong G_1 = \mathrm{T}_4$. Call this isomorphism $\Phi_1$. Note that $\Phi_1(y_1) = a$, $\Phi_1(y_2) = b$, $\Phi_1(y_1y_2) = ab$ and $\Phi_1(y_2y_1) = ba$.

Now by induction we define $\Phi_n : K_n \to G_n$, $n \geqslant 2$, as follows. Firstly, $\Phi_n = \Phi_{n-1}$ on $K_{n-1}$. Then for $k \in K_n - K_{n-1}$ we define

$$\Phi_n(y_{t_{n-1}}k) = r_{n-1} * (\Phi_{n-1}k), \quad \Phi_n(ky_{t_{n-1}}) = (\Phi_{n-1}k) * r_{n-1} \text{ and}$$
$$\Phi_n((y_1y_{t_{n-1}})k) = ((\Phi_{n-1}y_1) * r_{n-1}) * (\Phi_{n-1}k).$$

We now prove by induction on $n$ that $\Phi_n$ is an isomorphism ($n \geqslant 2$). Assume that for $1 \leqslant t \prec n$, $\Phi_t$ is an isomorphism and $\Phi_t y_1 = a$. Then the fact that $\Phi_n$ is one-to-one and onto $G_n$ follows from the definition of $\Phi_n$ and the fact that $\Phi_{n-1}$ is one-to-one and onto $G_{n-1}$. The fact that $\Phi_n(xy) = (\Phi_n x)(\Phi_n y)$ for any $\{x, y\} \subseteq K_n$ follows from the definition of product in $K_n$ and $G_n$ (see Tables 3 and 4 below) and the facts that $\Phi_{n-1}$ is an isomorphism and $\Phi_{n-1}y_1 = a$. We leave the straightforward details of these calculations to the reader.

| $K_n{=}K_{n-1}^{y_{t_{n-1}}}$ | $m$ | $y_{t_{n-1}}m$ | $my_{t_{n-1}}$ | $y_1y_{t_{n-1}} \cdot m$ |
|---|---|---|---|---|
| $l$ | $lm$ | $y_1y_{t_{n-1}}{\cdot}(my_1{\cdot}l)$ | $y_{t_{n-1}} \cdot ml$ | $(lm{\cdot}y_1l){\cdot}y_{t_{n-1}}$ |
| $y_{t_{n-1}}l$ | $ml \cdot y_{t_{n-1}}$ | $y_{t_{n-1}} \cdot lm$ | $y_1y_{t_{n-1}}{\cdot}(m{\cdot}y_1l)$ | $y_1 \cdot lm$ |
| $ly_{t_{n-1}}$ | $y_1y_{t_{n-1}}{\cdot}(ly_1{\cdot}ml)$ | $ml$ | $lm \cdot y_{t_{n-1}}$ | $y_{t_{n-1}}{\cdot}(y_1l{\cdot}m)$ |
| $y_1y_{t_{n-1}} \cdot l$ | $y_{t_{n-1}} \cdot (l \cdot my_1)$ | $(lm \cdot y_1) \cdot y_{t_{n-1}}$ | $y_1l \cdot my_1$ | $y_1y_{t_{n-1}} \cdot lm$ |

Table 3. The multiplication table for $\{l, m\} \subseteq K_{n-1}$.

| $G_n = G_{n-1}^{r_{n-1}}$ | $k$ | $r_{n-1}k$ | $kr_{n-1}$ | $ar_{n-1} \cdot k$ |
|---|---|---|---|---|
| $h$ | $hk$ | $ar_{n-1}{\cdot}(ka{\cdot}h)$ | $r_{n-1}(kh)$ | $(hk{\cdot}ah)r_{n-1}$ |
| $r_{n-1}h$ | $(kh)r_{n-1}$ | $r_{n-1}(hk)$ | $ar_{n-1}{\cdot}(k{\cdot}ah)$ | $a \cdot hk$ |
| $hr_{n-1}$ | $ar_{n-1}{\cdot}(ha{\cdot}kh)$ | $kh$ | $(hk)r_{n-1}$ | $r_{n-1}(ah{\cdot}k)$ |
| $ar_{n-1}{\cdot}h$ | $r_{n-1}(h \cdot ka)$ | $(hk \cdot a)r_{n-1}$ | $ak \cdot ha$ | $ar_{n-1}{\cdot}hk$ |

Table 4. The multiplication table for $\{h, k\} \subseteq G_{n-1}$.

So every $\Phi_n : K_n \to G_n$ is an isomorphism.

We now define $\Phi : K \to H$ as follows: for $x \in K_n - K_{n-1}$, $\Phi x = \Phi_n x$. Note that if $x \in K_n - K_{n-1}$ and $M \geqslant n$ then, since $K_n \subseteq K_{n+1} \subseteq \ldots \subseteq K_{M-1}$ and $\Phi_t = \Phi_{t-1}$ on $K_{t-1}$, $t \in N - \{1\}$, $\Phi_M = \Phi_n$ on $K_n$. Then for any $\{x, y\} \subseteq K$, with $x \in K_n - K_{n-1}$ and $y \in K_m - K_{m-1}$, $\Phi(xy) = \Phi_M(xy) = (\Phi_M x)(\Phi_M y) = (\Phi_n x)(\Phi_m y) = (\Phi x)(\Phi y)$, where $M = \max\{n, m\}$. Using the definition of the $\Phi_n$'s it is straightforward to prove that $\Phi$ is one-to-one and onto $H$. So, $H \cong K$. □

**Corollary 4.4.** *A countable member of $I \cap RM \cap AR$ is a union of a countable number of disjoint, isomorphic copies of* $\mathrm{T}_4$.

**Corollary 4.5.** *A countable member of $I \cap RM \cap AR$ is isomorphic to a proper subgroupoid of itself.*

*Proof.* Consider $H$ in Construction 4.1. Let $J_1 = \{a, ar_1, r_1a, r_1\}$. For $1 \prec n$ define $J_n$ by induction as $J_n = J_{n-1}^{r_n}$. Then $J = \bigcup_{n=1}^{\infty} J_n$, with the multiplication inherited from $H$, is a proper, countable subgroupoid of $H$. By Theorem 4.3, $J$ and $H$ are isomorphic. □

It follows from Lemma 2.10, Corollary 3.7 and Theorem 4.3 that:

**Corollary 4.6.** *If $G \in I \cap RM \cap AR$, $G$ is finite or countable and $G \underset{\leftarrow}{\cong} H$, then $G \cong H$.*

# 5. Smallest $(\mathbf{W},\mathbf{W})$ groupoids in RM – AR

**Definition 5.1.** A groupoid $G$ is called a *groupoid* $Y_G$ of groupoids $G_\alpha$, $\alpha \in Y_G$ if $G$ is a disjoint union of the groupoids $G_\alpha$ and $G_\alpha G_\beta \subseteq G_{\alpha\beta}$, $\alpha, \beta \in Y_G$. If $a \in G_\alpha$, then $G_a$ will denote $G_\alpha$.

In Definition 5.1, if $Y_G \in U$ and $G_\alpha \in V$ $(\alpha \in Y_G)$ for some groupoid varieties $U$ and $V$, then $G$ is called a $(U,V)$*-groupoid*.

In this section $W$ will denote the variety $I \cap RM \cap AR$.

Looking closely at Lemma 2.1, it is natural to wonder whether a right modular $(W,W)$-groupoid is anti-rectangular and, hence, a member of $W$. The converse statement is trivial, since any $G \in W$ is a groupoid $Y_G = G$ of trivial members of $W$. However, there is a $(W,W)$-groupoid $G \in RM - AR$. In fact we find a right modular $(W,W)$-groupoid $G$ of order 16, which is the minimal order for a right modular $(W,W)$-groupoid that is not anti-rectangular, as we proceed to prove. We also prove that $G$ is unique up to isomorphism and that any right modular $(W,W)$-groupoid $K \notin AR$ contains an isomorphic copy of $G$.

**Lemma 5.2.** *If $K \in RM$ is a groupoid $Y_K$ of groupoids $K_\alpha$, $\alpha \in Y_K$, with $Y_K \in W$ and $K_\alpha \in W$ $(\alpha \in Y_K)$, then*

1) *$K$ is cancellative,*
2) *for any $\{a,b\} \subseteq K$, $|K_a| = |K_b|$,*
3) *for any $\{a,b\} \subseteq K$, $ab \cdot a = b$ if and only if $ba \cdot b = a$.*

*Proof.* 1) Suppose that $a \in K_\alpha = K_a$, $b \in K_\beta = K_b$ and $c \in K_\gamma = K_c$. If $ca = cb$, then $\gamma\alpha = \gamma\beta$ and, since $Y_K$ is cancellative, $\alpha = \beta$. Then $ab \cdot a = b$ and $bc = (ab \cdot a)c = ca \cdot ab = cb \cdot ab = (ab \cdot b)c = ba \cdot c$.

Hence, $(ca \cdot c)b = bc \cdot ca = (ba \cdot c) \cdot ca = (ca \cdot c) \cdot ba$. But since $\{b, ba, ca \cdot c\} \subseteq K_\beta$, and $K_\beta$ is cancellative, $b = ba$. Therefore $b = ba = bb$. So $a = b$. Dually, if $ac = bc$, then $a = b$. Therefore $K$ is cancellative.

2) Now let $c \in K_\alpha = K_a$. Then $ab \cdot c \in K_\beta$. Since $K$ is cancellative $|K_\alpha| \leqslant |K_\beta|$. Dually $|K_\beta| \leqslant |K_\alpha|$ and so $|K_\alpha| = |K_\beta|$.

3) Note that $ab \cdot a = a \cdot ba$ and so we can write $aba$ to denote $ab \cdot a$. If $aba = b$, then $ba \cdot b = a((bab)a) = a((ba)(aba)) = a((ba)b)$. But $\{a, bab\} \subset K_a$ and $K_a$ is cancellative. Hence $a = bab$. Dually, $bab = a$ implies $aba = b$. □

Now suppose that $K \in RM$ is a groupoid $Y_K$ of groupoids $K_\alpha$ $(\alpha \in Y_K)$, with $Y_K \in W$ and $K_\alpha \in W$, $(\alpha \in Y_K)$. If $K$ is not anti-rectangular, then it follows from Lemma 5.2 that there is a set $\{a,b,c,d\} \subseteq K$ with $aba = d \neq b$, $bab = c \neq a$, $\{a, c, ac, ca\} \subseteq K_a$, $\{b, d, bd, db\} \subseteq K_b$, $ab \neq cd$ and $ba \neq dc$.

It follows from Lemma 2.4 and the fact that $K$ is a groupoid $Y_K$ of groupoids $K_\alpha$, $(\alpha \in Y_K)$, with $Y_K \in W$ and $K_\alpha \in W$ that $\{a, c, ac, ca\} = G_a$, $\{b, d, bd, db\} = G_b$, $\{ab, cd, ab \cdot cd, cd \cdot ab\} = G_{ab}$ and $\{ba, dc, ba \cdot dc, dc \cdot ba\} = G_{ba}$ are disjoint, isomorphic copies of $T_4$ contained in $K_a$, $K_b$, $K_{ab}$ and $K_{ba}$ respectively. We proceed to demonstrate that the union $G = \bigcup G_g$, $g \in \{a, b, ab, ba\}$, of these four copies of $\mathrm{T}_4$ is a subgroupoid of $K$ and is a groupoid $\mathrm{T}_4$ of groupoids $G_g$.

Recall that $K \in I \cap RM$ is cancellative. We have $ab \cdot a = d$. Then $ab \cdot c = cb \cdot a = (bab \cdot b)a = (b \cdot ba)a = aba \cdot b = db$, $ab \cdot ac = (aba)(ab \cdot c) = aba \cdot (cb \cdot a) = d \cdot db = bd$ and $ab \cdot ca = (ab \cdot c) \cdot aba = db \cdot d = b$. We have shown that $G_b = (ab)G_a$.

Similarly we can calculate that $G_{ab} = G_a b$ and $G_{ba} = bG_a$.

We can then calculate the Cayley table consisting of the 256 products of pairs of elements of $G$. In order to have sufficient space to show the Cayley table we define the following two ordered 16–tuples as equal:

$$(1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16) = (a, c, ac, ca, b, d, bd, db, ab, cd, ab \cdot cd, cd \cdot ab, ba, dc, ba \cdot dc, dc \cdot ba).$$

| **G** | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 3 | 4 | 2 | 9 | 11 | 12 | 10 | 16 | 14 | 13 | 15 | 6 | 8 | 7 | 5 |
| 2 | 4 | 2 | 1 | 3 | 12 | 10 | 9 | 11 | 13 | 15 | 16 | 14 | 7 | 5 | 6 | 8 |
| 3 | 2 | 4 | 3 | 1 | 10 | 12 | 11 | 9 | 15 | 13 | 14 | 16 | 5 | 7 | 8 | 6 |
| 4 | 3 | 1 | 2 | 4 | 11 | 9 | 10 | 12 | 14 | 16 | 12 | 13 | 8 | 6 | 5 | 7 |
| 5 | 13 | 15 | 16 | 14 | 5 | 7 | 8 | 6 | 2 | 4 | 3 | 1 | 12 | 10 | 9 | 11 |
| 6 | 16 | 14 | 13 | 15 | 8 | 6 | 5 | 7 | 3 | 1 | 2 | 4 | 9 | 11 | 12 | 10 |
| 7 | 14 | 16 | 15 | 13 | 6 | 8 | 7 | 5 | 1 | 3 | 4 | 2 | 11 | 9 | 10 | 12 |
| 8 | 15 | 13 | 14 | 16 | 7 | 5 | 6 | 8 | 4 | 2 | 1 | 3 | 10 | 12 | 11 | 9 |
| 9 | 6 | 8 | 7 | 5 | 13 | 15 | 16 | 14 | 9 | 11 | 12 | 10 | 4 | 2 | 1 | 3 |
| 10 | 7 | 5 | 6 | 8 | 16 | 14 | 13 | 15 | 12 | 10 | 9 | 11 | 1 | 3 | 4 | 2 |
| 11 | 5 | 7 | 8 | 6 | 14 | 16 | 15 | 13 | 10 | 12 | 11 | 9 | 3 | 1 | 2 | 4 |
| 12 | 8 | 6 | 5 | 7 | 15 | 13 | 14 | 16 | 11 | 9 | 10 | 12 | 2 | 4 | 3 | 1 |
| 13 | 9 | 11 | 12 | 10 | 2 | 4 | 3 | 1 | 8 | 6 | 5 | 7 | 13 | 15 | 16 | 14 |
| 14 | 12 | 10 | 9 | 11 | 3 | 1 | 2 | 4 | 5 | 7 | 8 | 6 | 16 | 14 | 13 | 15 |
| 15 | 10 | 12 | 11 | 9 | 1 | 3 | 4 | 2 | 7 | 5 | 6 | 8 | 14 | 16 | 15 | 13 |
| 16 | 11 | 9 | 10 | 12 | 4 | 2 | 1 | 3 | 6 | 8 | 7 | 5 | 15 | 13 | 14 | 16 |

Table 5.

| **G** | $h$ | $(ab) \cdot h$ | $hb$ | $bh$ |
|---|---|---|---|---|
| $g$ | $gh$ | $[c(g \cdot ah)]\,b$ | $b\,[(a \cdot hg)c]$ | $(ab) \cdot (ga \cdot h)$ |
| $(ab) \cdot g$ | $b(ca \cdot hg)$ | $(ab) \cdot (gh)$ | $cg \cdot ha$ | $(gh \cdot a)b$ |
| $gb$ | $(ab) \cdot (ha \cdot gh)$ | $b(hg \cdot ca)$ | $(gh)b$ | $h \cdot (ag \cdot c)$ |
| $bg$ | $(hg)b$ | $h \cdot gc$ | $(ab) \cdot (g \cdot ch)$ | $b(gh)$ |

Table 6. The multiplication table for $\{g, h\} \subseteq G_a = \{a, c, ac, ca\}$.

Table 6 is derived using calculations obtained from Table 5. Notice that Table 6 yields the following Cayley table in set theoretic notation:

| **G** | $G_a$ | $G_b = (ab)G_a$ | $G_{ab} = G_a b$ | $G_{ba} = bG_a$ |
|---|---|---|---|---|
| $G_a$ | $G_a$ | $G_{ab}$ | $G_{ba}$ | $G_b$ |
| $G_b = (ab)G_a$ | $G_{ba}$ | $G_b$ | $G_a$ | $G_{ab}$ |
| $G_{ab} = G_a b$ | $G_b$ | $G_{ba}$ | $G_{ab}$ | $G_a$ |
| $G_{ba} = bG_a$ | $G_{ab}$ | $G_a$ | $G_b$ | $G_{ba}$ |

Table 7.

Note that the subscripts of the $G_g's$, $g \in \{a, b, ab, ba\}$, multiply in exactly the same way as the elements of $\mathrm{T}_4$. The fact that $G \in RM$ follows from the fact that $G \leq K$ and $K \in I \cap RM \subseteq RM$. This proves that $G$ is a right modular groupoid

$T_4$ of groupoids $G_g$, where each $G_g \cong T_4$. Note however that $\{a, b, ab, ba\}$ is not even a subgroupoid of $G$! We have therefore proved:

**Theorem 5.3.** *$G \in I \cap RM$ and $G$ is a groupoid $T_4$ of (four) isomorphic copies of $T_4$. However $G \notin W$. Also, if $(W, W)$-groupoid $K \in RM - AR$, then $K$ contains an isomorphic copy of $G$.*

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