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On fuzzy ordered semigroups

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Abstract. There are two equivalent definitions of a fuzzy right ideal, fuzzy left ideal, fuzzy bi-ideal or fuzzy quasi-ideal f of an ordered semigroup (or a semigroup) S in the bibliography. The first one is based on the fuzzy subset f itself, the other on the multiplication of fuzzy sets and the greatest fuzzy subset of S. Investigations in the existing bibliography are based on the first definition. The present paper serves as an example to show that using the second definition the proofs of the results can be simplified, drastically in some cases, using only the definitions themselves.

1. Introduction and prerequisites

As we have seen in [6], there are two equivalent definitions for each of the following: Fuzzy right ideal, fuzzy left ideal, fuzzy bi-ideal and fuzzy quasi-ideal. These are the following:

Definition 1.1. Let $(S, ., \leqslant)$ be an ordered groupoid. A fuzzy subset f of S is called a *fuzzy right ideal* of $(S, ., \leqslant)$ (or just a *fuzzy right ideal* of S) if
(1) $f(xy) \geqslant f(x)$ for all $x, y \in S$ and
(2) if $x \leqslant y$, then $f(x) \geqslant f(y)$.

Definition 1.2. Let $(S, ., \leqslant)$ be an ordered groupoid. A fuzzy subset f of S is called a *fuzzy right ideal* of S if
(1) $f \circ 1 \preceq f$ and
(2) if $x \leqslant y$, then $f(x) \geqslant f(y)$.

Definition 1.3. Let $(S, ., \leqslant)$ be an ordered groupoid. A fuzzy subset f of S is called a *fuzzy left ideal* of S if
(1) $f(xy) \geqslant f(y)$ for all $x, y \in S$ and
(2) if $x \leqslant y$, then $f(x) \geqslant f(y)$.

Definition 1.4. Let $(S, ., \leqslant)$ be an ordered groupoid. A fuzzy subset f of S is called a *fuzzy left ideal* of S if
(1) $1 \circ f \preceq f$ and
(2) if $x \leqslant y$, then $f(x) \geqslant f(y)$.

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Definition 1.5. Let $(S,.,\leqslant)$ be an ordered semigroup. A fuzzy subset f of S is called a *fuzzy bi-ideal* of S if

(1) $f(xyz) \geqslant \min\{f(x), f(z)\}$ for all $x, y, z \in S$ and

(2) if $x \leqslant y$, then $f(x) \geqslant f(y)$.

Definition 1.6. Let S be an ordered semigroup. A fuzzy subset f of S is called a *fuzzy bi-ideal* of S if

(1) $f \circ 1 \circ f \preceq f$ and

(2) if $x \leqslant y$, then $f(x) \geqslant f(y)$.

Definition 1.7. Let $(S,.,\leqslant)$ be an ordered groupoid. A fuzzy subset f of S is called a *fuzzy quasi-ideal* of S if

(1) if $x \leqslant bs$ and $x \leqslant tc$ for some x, b, s, t, c in S, then $f(x) \geqslant \min\{f(b), f(c)\}$ and

(2) if $x \leqslant y$, then $f(x) \geqslant f(y)$.

Definition 1.8. Let $(S,.,\leqslant)$ be an ordered groupoid. A fuzzy subset f of S is called a *fuzzy quasi-ideal* of S if

(1) $(f \circ 1) \wedge (1 \circ f) \preceq f$ and

(2) if $x \leqslant y$, then $f(x) \geqslant f(y)$.

A fuzzy subset f of $(S,.,\leqslant)$ is said to be a fuzzy right (resp. left) ideal, fuzzy bi-ideal or fuzzy quasi-ideal of $(S,.)$ if the following assertions, respectively hold in $(S,.,\leqslant)$: $f(xy) \geqslant f(x)$ (resp. $f(xy) \geqslant f(y)$); $f(xyz) \geqslant \min\{f(x), f(z)\}$; $x \leqslant bs$ and $x \leqslant tc$ imply $f(x) \geqslant \min\{f(b), f(c)\}$.

Definitions 1.1, 1.3, 1.5 and 1.7 are based on the fuzzy subset f itself while in 1.2, 1.4, 1.6, 1.8 the greatest fuzzy subset 1 of S and the multiplication of fuzzy subsets play an essential role. Investigations in the existing bibliography are based on Definitions 1.1, 1.3, 1.5 and 1.8. Definition 1.7 has been first introduced by Kehayopulu and Tsingelis in [6]. The present paper serves as an example to show that with Definitions 1.2, 1.4, 1.6, 1.8 the proofs of the results can be simplified, drastically is some cases, using only the definitions themselves.

It has been announced without proof in [7] that an ordered semigroup $(S,.,\leqslant)$ is intra-regular if and only if for every fuzzy right ideal f, every fuzzy left ideal g and every fuzzy bi-ideal h of $(S,.,\leqslant)$, we have $f \wedge h \wedge g \preceq g \circ h \circ f$ and that it is both regular and intra-regular if and only if for every fuzzy right ideal f, every fuzzy left ideal g and every fuzzy bi-ideal h of $(S,.,\leqslant)$, we have $f \wedge h \wedge g \preceq h \circ f \circ g$. Some more general situations are given in the present paper. According to the present paper, if an ordered semigroup $(S,.,\leqslant)$ is intra-regular, then for every fuzzy right ideal f, every fuzzy left ideal g and every fuzzy bi-ideal h of $(S,.)$, we have $f \wedge h \wedge g \preceq g \circ h \circ f$. If an ordered semigroup $(S,.,\leqslant)$ is both regular and intra-regular, then for every fuzzy right ideal f, every fuzzy subset g and every fuzzy bi-ideal h of $(S,.)$, we have $f \wedge h \wedge g \preceq h \circ f \circ g$. We also prove that if an ordered semigroup $(S,.,\leqslant)$ is regular, then for every fuzzy right ideal f, every fuzzy left ideal g and every fuzzy bi-ideal h of $(S,.)$ we have $f \wedge h \wedge g \preceq f \circ h \circ g$. "Conversely", if for every fuzzy right ideal f, every fuzzy left ideal g and every fuzzy bi-ideal h of $(S, ., \leqslant)$ we have $f \wedge h \wedge g \preceq f \circ h \circ g$, then S is regular. Characterizations of regular and both regular and intra-regular ordered semigroups in terms of fuzzy sets have been also given by Xie in [8].

Let $(S, ., \leqslant)$ be an ordered semigroup. For a subset A of S, denote by $(A]$ the subset of S defined by

$$(A] := \{t \in S \mid t \leqslant a \text{ for some } a \in A\}.$$

A nonempty subset A of $(S, ., \leqslant)$ is called a *left* (resp. *right*) *ideal* of $(S, ., \leqslant)$ (or just of S) if (1) $SA \subseteq A$ (resp. $AS \subseteq A$) and (2) if $a \in A$ and $S \ni b \leqslant a$, then $b \in A$. A is called a *bi-ideal* of S if (1) $ASA \subseteq A$ and (2) if $a \in A$ and $S \ni b \leqslant a$, then $b \in A$. It is called a *quasi-ideal* of S if (1) $(SA] \cap (AS] \subseteq A$ and (2) if $a \in A$ and $S \ni b \leqslant a$, then $b \in A$. A nonempty subset A of $(S, ., \leqslant)$ is said to be a left ideal, right ideal, bi-ideal or quasi-ideal of $(S, .)$ if the relations $SA \subseteq A$, $AS \subseteq A$, $SAS \subseteq A$ or $(AS] \cap (SA] \subseteq A$, respectively hold in S. An ordered semigroup $(S, ., \leqslant)$ is called *regular* if for every $a \in S$ there exists $x \in S$ such that $a \leqslant axa$. Equivalently, if $A \subseteq (ASA]$ for every $A \subseteq S$. It is called *intra-regular* if for every $a \in S$ there exist $x, y \in S$ such that $a \leqslant xa^2y$. Equivalently, if $A \subseteq (SA^2S]$ for every $A \subseteq S$.

Denote by 1 the fuzzy subset of S defined by $1 : S \to [0, 1] \mid a \to 1$. The fuzzy set 1 is the greatest element in the set of fuzzy subsets of S, that is, $f \preceq 1$ for every fuzzy subset f of S. If S is a regular or an intra-regular ordered semigroup, then we have $1 \circ 1 = 1$. It is well known that an ordered semigroup S is regular if and only if for every fuzzy right ideal f and every fuzzy left ideal g of $(S, ., \leqslant)$, we have $f \wedge g = f \circ g$ equivalently $f \wedge g \preceq f \circ g$ [4]. It is intra-regular if and only if for every fuzzy right ideal f and every fuzzy left ideal g of $(S, ., \leqslant)$, we have $f \wedge g \preceq g \circ f$ [7]. Moreover, an ordered semigroup S is regular if and only if for every fuzzy subset f of S, we have $f \preceq f \circ 1 \circ f$ [6]. It is intra-regular if and only if for every fuzzy subset f of S, we have $f \preceq 1 \circ f^2 \circ 1$ [5]. If $(S, ., \leqslant)$ is an ordered groupoid, f, g fuzzy subsets of $(S, .)$ and $f \preceq g$ then, for any fuzzy subset h of $(S, .)$, we have $f \circ h \preceq g \circ h$ and $h \circ f \preceq h \circ g$ (cf. also [4]). It is also well known that if S is a semigroup or an ordered semigroup, then the multiplication of fuzzy subsets of S is associative (cf. [3]). For the definitions and notations not given in the present paper we refer to [4].

2. Main results

The first theorem characterizes the ordered semigroups which are intra-regular in terms of fuzzy sets. Let us prove it using first the first and then the second definitions.

Theorem 2.1. *Let $(S, ., \leqslant)$ be an ordered semigroup. If S is intra-regular, then for every fuzzy right ideal f, every fuzzy left ideal g and every fuzzy bi-ideal h of*

$(S, .)$ we have

$$f \wedge h \wedge g \preceq g \circ h \circ f.$$

"Conversely", if for every fuzzy right ideal f, every fuzzy left ideal g and every fuzzy bi-ideal h of $(S, ., \leqslant)$ we have $f \wedge h \wedge g \preceq g \circ h \circ f$, then $(S, ., \leqslant)$ is intra-regular.

Proof of Theorem 2.1 using the Definitions 1.1, 1.3, 1.5

We need the following lemmas. As our aim is to compare the two definitions, we would like to mention everything we use in the proofs. In that sense, for the sake of completeness, it is no harm to mention the next lemma related to the real numbers, as well.

Lemma 2.1. *If a, b, c, d, e, f are real numbers, then*
(1) *If $a \geqslant b$ and $c \geqslant d$, then $\min\{a, c\} \geqslant \min\{b, d\}$.*
(2) $\min\{\min\{a, b\}, c\} = \min\{a, b, c\}$.
(3) *If $a \geqslant b$, $c \geqslant d$ and $e \geqslant f$, then $\min\{a, c, e\} \geqslant \min\{b, d, f\}$.*

Lemma 2.2. (cf. also [2; Proposition 2]) *Let $(S, ., \leqslant)$ be an ordered groupoid. If A is a left (resp. right) ideal of $(S, ., \leqslant)$, then the characteristic function f_A is a fuzzy left (resp. fuzzy right) ideal of $(S, ., \leqslant)$. "Conversely", if A is a nonempty set and f_A a fuzzy left (resp. right) ideal of $(S, ., \leqslant)$, then A is a left (resp. right) ideal of $(S, ., \leqslant)$.*

Lemma 2.3. (cf. also [7; Lemma 4]) *Let $(S, ., \leqslant)$ be an ordered semigroup. If B is a bi-ideal of $(S, ., \leqslant)$, then the characteristic function f_B is a fuzzy bi-ideal of $(S, ., \leqslant)$. "Conversely", if B is a nonempty set and f_B a fuzzy bi-ideal of $(S, ., \leqslant)$, then B is a bi-ideal of $(S, ., \leqslant)$.*

Lemma 2.4. [4; Proposition 7] *If S is an ordered groupoid (or groupoid) and $\{A_i \mid i \in I\}$ a nonempty family of subsets of S, then we have*

$$\bigwedge_{i \in I} f_{A_i} = f_{\bigcap\limits_{i \in I} A_i}.$$

Lemma 2.5. *Let S be an ordered semigroup, n a natural number, $n \geqslant 2$ and $\{A_1, A_2, \ldots, A_n\}$ a set of nonempty subsets of S. Then we have*

$$f_{A_1} \circ f_{A_2} \circ \ldots \circ f_{A_n} = f_{(A_1 A_2 \ldots A_n]}.$$

Proof. For $n = 2$ it is true [4; Proposition 8]. Suppose $f_{A_1} \circ f_{A_2} \circ \ldots \circ f_{A_m} = f_{(A_1 A_2 \ldots A_m]}$ for a natural number m, $m \geqslant 2$. Then we have

$$\begin{aligned} f_{A_1} \circ f_{A_2} \circ \ldots \circ f_{A_{m+1}} &= f_{(A_1 A_2 \ldots A_m]} \circ f_{A_{m+1}} = f_{((A_1 A_2 \ldots A_m] A_{m+1}]} \\ &= f_{((A_1 A_2 \ldots A_m) A_{m+1}]} = f_{(A_1 A_2 \ldots A_{m+1}]}. \end{aligned}$$

$\square$

Lemma 2.6. [4; Proposition 5] *If S is an ordered groupoid (or groupoid) and A, B subsets of S, then we have*

$$A \subseteq B \Longleftrightarrow f_A \preceq f_B.$$

Taking into account the Proposition 2 and Lemma 2 in [1], one can easily see that the following lemma is satisfied:

Lemma 2.7. *Let $(S, ., \leqslant)$ be an ordered semigroup. If $(S, ., \leqslant)$ is intra-regular, then for every right ideal X, every left ideal Y and every bi-ideal B of $(S, .)$ we have*

$$X \cap B \cap Y \subseteq (YBX].$$

"Conversely", if for every right ideal X, every left ideal Y and every bi-ideal B of $(S, ., \leqslant)$ we have $X \cap B \cap Y \subseteq (YBX]$, then S is intra-regular.

Proof of Theorem 2.1

$\Longrightarrow$. Let f be a fuzzy right ideal, g a fuzzy left ideal, h a fuzzy bi-ideal of $(S, .)$, and $a \in S$. Since $(S, ., \leqslant)$ is intra-regular, there exist $x, y \in S$ such that $a \leqslant xa^2y$. Then we have

$$a \leqslant x(xa^2y)(xa^2y)y = x^2a^2yxa^2y^2,$$

which implies $(x^2a^2yxa, ay^2) \in A_a$ .... $(*)$ and $A_a \neq \emptyset$. Then we have

$$\begin{aligned}((g \circ h) \circ f)(a) :&= \bigvee_{(u,v) \in A_a} \min\{(g \circ h)(u), f(v)\} \text{ (since } A_a \neq \emptyset)\\ &\geqslant \min\{(g \circ h)(x^2a^2yxa), f(ay^2)\} \text{ (by } (*)).\end{aligned}$$

Since $(x^2a, ayxa) \in A_{x^2a^2yxa}$, we have $A_{x^2a^2yxa} \neq \emptyset$, hence

$$\begin{aligned}(g \circ h)(x^2a^2yxa) :&= \bigvee_{(w,t) \in A_a} \min\{(g(w), h(t)\}\\ &\geqslant \min\{g(x^2a), h(ayxa)\}.\end{aligned}$$

Then, by Lemma 2.1(1) and (2), we have

$$\begin{aligned}((g \circ h) \circ f)(a) &\geqslant \min\{\min\{g(x^2a), h(ayxa)\}, f(ay^2)\}\\ &= \min\{g(x^2a), h(ayxa), f(ay^2)\}\\ &= \min\{f(ay^2), h(ayxa), g(x^2a)\}\end{aligned}$$

Since f is a fuzzy right ideal of $(S, .)$, we have $f(ay^2) \geqslant f(a)$. Since h is a fuzzy bi-ideal of $(S, .)$, we have $h(ayxa) \geqslant h(a)$. Since g is a fuzzy left ideal of $(S, .)$, we have $g(x^2a) \geqslant g(a)$. Then, by Lemma 2.1(3), we have

$$((g \circ h) \circ f)(a) \geqslant \min\{f(a), h(a), g(a)\} = (f \wedge h \wedge g)(a).$$

As the multiplication of fuzzy subsets is associative, we obtain $f \wedge h \wedge g \preceq g \circ h \circ f$.

$\Longleftarrow$. Let X be a right ideal, Y a left ideal, and B a bi-ideal of $(S,.,\leqslant)$. By Lemma 2.7 it is enough to prove that $X \cap B \cap Y \subseteq (YBX]$. By Lemmas 2.2 and 2.3, f_X is a fuzzy right ideal, f_Y a fuzzy left ideal and f_B a fuzzy bi-ideal of $(S,.,\leqslant)$. By hypothesis, we have $f_X \wedge f_B \wedge f_Y \preceq f_Y \circ f_B \circ f_X$. By Lemma 2.4, we have $f_X \wedge f_B \wedge f_Y = f_{X \cap B \cap Y}$. By Lemma 2.5, $f_Y \circ f_B \circ f_X = f_{(YBX]}$. Hence we have $f_{X \cap B \cap Y} \preceq f_{(YBX]}$. Then, by Lemma 2.6, $X \cap B \cap Y \subseteq (YBX]$. □

Proof of Theorem 2.1 using the Definitions 1.2, 1.4, 1.6

$\Longrightarrow$. Let f be a fuzzy right ideal, g a fuzzy left ideal, h a fuzzy bi-ideal of $(S,.)$. Since $f \wedge h \wedge g$ is a fuzzy subset of S and S is intra-regular, we have

$$\begin{aligned} f \wedge h \wedge g &\preceq 1 \circ (f \wedge h \wedge g)^2 \circ 1 = 1 \circ (f \wedge h \wedge g) \circ (f \wedge h \wedge g) \circ 1 \\ &\preceq 1 \circ 1 \circ (f \wedge h \wedge g)^2 \circ 1 \circ 1 \circ (f \wedge h \wedge g)^2 \circ 1 \circ 1 \\ &= 1 \circ (f \wedge h \wedge g) \circ (f \wedge h \wedge g) \circ 1 \circ (f \wedge h \wedge g) \circ (f \wedge h \wedge g) \circ 1 \\ &\preceq (1 \circ g) \circ (h \circ 1 \circ h) \circ (f \circ 1) \\ &\preceq g \circ h \circ f. \end{aligned}$$

$\Longleftarrow$. Let f be a fuzzy right ideal and g a fuzzy left ideal of $(S,.,\leqslant)$. Since 1 is a fuzzy right ideal and f a fuzzy bi-ideal of $(S,.,\leqslant)$, by hypothesis, we have $f \wedge g = 1 \wedge f \wedge g \preceq g \circ f \circ 1 \preceq g \circ f$, so S is intra-regular. □

The next theorem characterizes the ordered semigroups which are both regular and intra-regular using fuzzy sets.

Theorem 2.2. *Let $(S,.,\leqslant)$ be an ordered semigroup. If S is both regular and intra-regular, then for every fuzzy right ideal f, every fuzzy subset g and every fuzzy bi-ideal h of $(S,.)$ we have*

$$f \wedge h \wedge g \preceq h \circ f \circ g.$$

"Conversely", if for every fuzzy right ideal f, every fuzzy left ideal g and every fuzzy bi-ideal h of $(S,.,\leqslant)$ we have $f \wedge h \wedge g \preceq h \circ f \circ g$, then S is both regular and intra-regular.

Proof of Theorem 2.2 using the Definitions 1.1, 1.3, 1.5

In addition to Lemmas 2.1–2.6 mentioned above, we need the following lemma.

Lemma 2.8. (cf. also [1; Proposition 3]) *Let $(S,.,\leqslant)$ be an ordered semigroup. If $(S,.,\leqslant)$ is both regular and intra-regular, then for every right ideal X, every subset Y and every bi-ideal B of $(S,.)$ we have*

$$X \cap B \cap Y \subseteq (BXY].$$

"Conversely", if for every right ideal X, every left ideal Y and every bi-ideal B of $(S,.,\leqslant)$ *we have* $X\cap B\cap Y\subseteq (BXY]$, *then* $(S,.,\leqslant)$ *is both regular and intra-regular.*

Proof of Theorem 2.2

$\Longrightarrow$. Let f be a fuzzy right ideal of $(S,.)$, g a fuzzy subset of S, h a fuzzy bi-ideal of $(S,.)$, and $a\in S$. Since S is regular, there exists $x\in S$ such that $a\leqslant axa$. Since S is intra-regular, there exist $z,y\in S$ such that $a\leqslant za^2y$. Then we have

$$a\leqslant ax(axa)\leqslant ax(za^2y)xa=axza^2yxa,$$

$(axza^2yx,a)\in A_a$, $A_a\neq\emptyset$, and

$$\begin{aligned}((h\circ f)\circ g)(a):&=\bigvee_{(u,v)\in A_a}\min\{(h\circ f)(u),g(v)\}\\&\geqslant\min\{(h\circ f)(axza^2yx),g(a)\}.\end{aligned}$$

Since $(axza,ayx)\in A_{axza^2yx}$, we have $A_{axza^2yx}\neq\emptyset$, and

$$\begin{aligned}(h\circ f)(axza^2yx):&=\bigvee_{(w,t)\in A_{xza^2yx}}\min\{h(w),f(t)\}\\&\geqslant\min\{h(axza),f(ayx)\}.\end{aligned}$$

Hence we obtain

$$\begin{aligned}((h\circ f)\circ g)(a)&\geqslant\min\{\min\{h(axza),f(ayx)\},g(a)\}\\&=\min\{h(axza),f(ayx),g(a)\}\end{aligned}$$

Since h is a fuzzy bi-ideal, f a fuzzy right ideal and g a fuzzy subset of S, we obtain

$$((h\circ f)\circ g)(a)\geqslant\min\{h(a),f(a),g(a)\}=(f\wedge h\wedge g)(a).$$

$\Longleftarrow$. Let X be a right ideal, Y a left ideal and B a bi-ideal of $(S,.,\leqslant)$. Since f_X is a fuzzy right ideal, f_Y a fuzzy left ideal and f_B a fuzzy bi-ideal of $(S,.,\leqslant)$, by hypothesis, we have $f_X\wedge f_B\wedge f_Y\preceq f_B\circ f_X\circ f_Y$. Then $f_{X\cap B\cap Y}\preceq f_{(BXY]}$, and $X\cap B\cap Y\subseteq (BXY]$. By Lemma 2.8, S is both regular and intra-regular. $\square$

Proof of Theorem 2.2 using the Definitions 1.2, 1.4, 1.6

$\Longrightarrow$. Since S is both regular and intra-regular, for any fuzzy subset f of S, we have $f\preceq f\circ 1\circ f^2\circ 1\circ f$. Indeed: Since S is regular, we have $f\preceq f\circ 1\circ f$. Since S is intra-regular, we have $f\preceq 1\circ f^2\circ 1$. Thus we have

$$\begin{aligned}f&\preceq(f\circ 1\circ f)\circ 1\circ f\preceq f\circ 1\circ(1\circ f^2\circ 1)\circ 1\circ f\\&=f\circ 1\circ f^2\circ 1\circ f.\end{aligned}$$

Let now f be a fuzzy right ideal, g a fuzzy subset and h a fuzzy bi-ideal of $(S,.)$. Since $f \wedge h \wedge g$ is a fuzzy subset of S, we have

$$\begin{aligned} f \wedge h \wedge g &\preceq (f \wedge h \wedge g) \circ 1 \circ (f \wedge h \wedge g) \circ (f \wedge h \wedge g) \circ 1 \circ (f \wedge h \wedge g) \\ &\preceq (h \circ 1 \circ h) \circ (f \circ 1) \circ g \\ &\preceq h \circ f \circ g. \end{aligned}$$

$\Longleftarrow$. Let f be a fuzzy right and g a fuzzy left ideal of $(S,.,\leqslant)$. Since 1 is a fuzzy right ideal of $(S,.,\leqslant)$ and f a fuzzy bi-ideal of $(S,.,\leqslant)$, by hypothesis, we have $f \wedge g = 1 \wedge f \wedge g \preceq f \circ 1 \circ g \preceq f \circ g$, and S is regular. Since g is a fuzzy bi-ideal and 1 a fuzzy left ideal of $(S,.,\leqslant)$, by hypothesis, we have $f \wedge g = f \wedge g \wedge 1 \preceq g \circ f \circ 1 \preceq g \circ f$, and S is intra-regular. $\square$

We finally characterize the ordered semigroups which are regular in terms of fuzzy sets.

Theorem 2.3. *Let $(S,.,\leqslant)$ be an ordered semigroup. If S is regular, then for every fuzzy right ideal f, every fuzzy left ideal g and every fuzzy bi-ideal h of $(S,.)$ we have*

$$f \wedge h \wedge g \preceq f \circ h \circ g.$$

"Conversely", if for every fuzzy right ideal f, every fuzzy left ideal g and every fuzzy bi-ideal h of $(S,.,\leqslant)$ we have $f \wedge h \wedge g \preceq f \circ h \circ g$, then S is regular.

Proof of Theorem 2.3 using the Definitions 1.1, 1.3, 1.5

In addition to Lemmas 2.1–2.6, we need the following lemma.

Lemma 2.9. (cf. also [1; Proposition 1]) *Let $(S,.,\leqslant)$ be an ordered semigroup. If S is regular, then for every right ideal X, every left ideal Y and every bi-ideal B of $(S,.)$ we have*

$$X \cap B \cap Y \subseteq (XBY].$$

"Conversely", if for every right ideal X, every left ideal Y and every bi-ideal B of $(S,.,\leqslant)$ we have $X \cap B \cap Y \subseteq (XBY]$, then S is regular.

Proof of Theorem 2.3

$\Longrightarrow$. Let f be a fuzzy right ideal, g a fuzzy left ideal, h a fuzzy bi-ideal of $(S,.)$, and $a \in S$. Then $a \leqslant axa \leqslant (axa)x(axa)$ for some $x \in S$. Then $(axaxa, xa) \in A_a$, and

$$\begin{aligned} ((f \circ h) \circ g)(a) := &\bigvee_{(u,v) \in A_a} \min\{(f \circ h)(u), g(v)\} \\ &\geqslant \min\{(f \circ h)(axaxa), g(xa)\}. \end{aligned}$$

Since $(ax, axa) \in A_{axaxa}$, we have

$$(f \circ h)(axaxa) := \bigvee_{(w,t)\in A_{axaxa}} \min\{(f(w), h(t)\} \geqslant \min\{f(ax), h(axa)\}.$$

Then we have

$$\begin{aligned}((f \circ h) \circ g)(a) &\geqslant \min\{\min\{f(ax), h(axa)\}, g(xa)\}\\ &= \min\{f(ax), h(axa), g(xa)\}\\ &\geqslant \min\{f(a), h(a), g(a)\}\\ &= (f \wedge h \wedge g)(a).\end{aligned}$$

Hence we obtain $f \wedge h \wedge g \preceq f \circ h \circ g$.
$\Longleftarrow$. Let X be a right ideal, Y a left ideal, and B a bi-ideal of $(S, ., \leqslant)$. Then f_X is a fuzzy right ideal, f_Y a fuzzy left ideal and f_B a fuzzy bi-ideal of $(S, ., \leqslant)$. By hypothesis, we have $f_X \wedge f_B \wedge f_Y \preceq f_X \circ f_B \circ f_Y$. Since $f_X \wedge f_B \wedge f_Y = f_{X\cap B\cap Y}$ and $f_Y \circ f_B \circ f_X = f_{(YBX]}$, we have $f_{X\cap B\cap Y} \preceq f_{(YBX]}$. Then, by Lemma 2.9, $X \cap B \cap Y \subseteq (XBY]$, and S is regular. □

Proof of Theorem 2.3 using the Definitions 1.2, 1.4, 1.6

$\Longrightarrow$. Let f be a fuzzy right ideal, g a fuzzy left ideal, h a fuzzy bi-ideal of $(S, .)$. Since S is regular and $f \wedge h \wedge g$ a fuzzy subset of S, we have

$$\begin{aligned}f \wedge h \wedge g &\preceq (f \wedge h \wedge g) \circ 1 \circ (f \wedge h \wedge g)\\ &\preceq (f \wedge h \wedge g) \circ 1 \circ (f \wedge h \wedge g) \circ 1 \circ (f \wedge h \wedge g) \circ 1 \circ (f \wedge h \wedge g)\\ &\preceq (f \circ 1) \circ (h \circ 1 \circ h) \circ (1 \circ g)\\ &\preceq f \circ h \circ g\end{aligned}$$

$\Longleftarrow$. Let f be a fuzzy right ideal and g a fuzzy left ideal of $(S, ., \leqslant)$. Since 1 is a fuzzy bi-ideal of $(S, ., \leqslant)$, by hypothesis, we have $f \wedge g = f \wedge 1 \wedge g \preceq f \circ (1 \circ g) \preceq f \circ g$, and S is regular. □

As a conclusion, we have the following

Theorem. *An ordered semigroup S is intra-regular (resp. regular) if and only if for every fuzzy right ideal f, every fuzzy left ideal g and every fuzzy bi-ideal h of S we have $f \wedge h \wedge g \preceq g \circ h \circ f$ (resp. $f \wedge h \wedge g \preceq f \circ h \circ g$). It is both regular and intra-regular if and only if for every fuzzy right ideal f, every fuzzy left ideal g and every fuzzy bi-ideal h of S, we have $f \wedge h \wedge g \preceq h \circ f \circ g$.*

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