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Some results on E-inversive semigroups

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Abstract. In the paper we study E-inversive semigroups. We show that E-inversive semigroups are M-semigroups and we prove that M-biordered sets arise from E-inversive semigroups. Moreover, some connections between bi-ideals of an E-inversive semigroup S and bi-order ideals, order bi-ideals of the biordered set E_S of S are given. Further, some results of Janet Mills concerning matrix congruences on orthodox semigroups are generalized to E-inversive E-semigroups. Also, we prove that the class of all E-inversive semigroups is structurally closed.

1. Introduction and preliminaries

In the paper we present some results on E-inversive semigroups. The main result of this article is Theorem 2.18 i.e. we show that every M-biordered set arises from some E-inversive semigroup. Our proof of this result is quite simple. Proving this result we used the characterization of the M-set of a semigroup (see Prop. 2.12) and an important Easdown's result (that is, every biordered set comes from some semigroup). Moreover, we can show in a similar way Nambooripad's Theorem (i.e., each regular biordered set comes from some regular semigroup). The proofs of this result were more complicated, see [2, 13]. Also, some equivalent conditions for a semigroup to be E-inversive are given (Corollaries 2.4, 2.11). Further, some connections between bi-ideals of an E-inversive semigroup S and order bi-ideals, bi-order ideals of the biordered set E_S are presented in this work (see Prop. 2.14 and Th. 2.16). Moreover, we give some remarks concerning matrix congruences on E-inversive (E-)semigroups (see Cor. 2.7 and Th. 2.10). Finally, we prove that the class of E-inversive semigroups is structurally closed (Cor. 2.6).

Let S be a semigroup, $a \in S$. The set $W(a) = \{x \in S : x = xax\}$ is called the set of all *weak inverses* of a, and so the elements of $W(a)$ will be called *weak inverse elements* of a. A semigroup S is called *E-inversive* iff for every $a \in S$ there exists $x \in S$ such that $ax \in E_S$, where E_S (or briefly E) is the set of idempotents of S (more generally, if $A \subseteq S$, then E_A denotes the set of all idempotents of A). It is easy to see that a semigroup S is E-inversive if and only if $W(a)$ is nonempty for all $a \in S$. Hence if S is E-inversive, then for every $a \in S$ there is $x \in S$ such that $ax, xa \in E_S$ (see [10, 11]).

Further, by $Reg(S)$ we shall mean the set of *regular elements* of S (an element a of S is called *regular* if $a \in aSa$) and by $V(a) = \{x \in S : a = axa, x = xax\}$ the

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set of all *inverse elements* of a. It is well known that an element a of S is regular iff $V(a) \neq \emptyset$, so a semigroup S is regular iff $V(a) \neq \emptyset$ for every $a \in S$ [6]. Finally, a semigroup S is said to be *eventually regular* if every element of S has a regular power [4]. Clearly, eventually regular semigroups are E-inversive.

In [5] Hall observed that the set $Reg(S)$ of a semigroup S with $E_S \neq \emptyset$ forms a regular subsemigroup of S iff the product of any two idempotents of S is regular. In that case, S is said to be an *R-semigroup.* Also, we say that S is an *E-semigroup* if ${E_S}^2 \subseteq E_S$.

A subsemigroup B of a semigroup S is said to be a *bi-ideal* of S if $BSB \subseteq B$. It is clear that there exists the least bi-ideal (X) containing a nonempty subset X of S. One can easily seen that (X) is of the form: $X \cup X^2 \cup XSX$ [1].

A nonempty subset A of a semigroup S is called a *quasi ideal* iff $AS \cap SA \subseteq A$. Note that every quasi ideal A of S is a bi-ideal of S and each one-sided ideal of S is a quasi ideal of S, so it is a bi-ideal of S. If $\emptyset \neq C \subseteq S$, then $(C \cup SC) \cap (C \cup CS)$ is the smallest quasi ideal of S containing C.

Each subsemigroup eSe of a semigroup S, where $e \in E_S$, will be called a *local subsemigroup* of S. Furthermore, we say that a semigroup S with $E_S \neq \emptyset$ is *locally E-inversive* iff *every* local subsemigroup of S is E-inversive.

By a *rectangular band* we shell mean a semigroup M with the property $aba = a$ for all $a, b \in M$. Note that in that case, $M = E_M$. Also, we say that a congruence ρ on a semigroup S is a *matrix* congruence if S/ρ is a rectangular band [9].

Some background material on *biordered sets* will be useful. For a definition of a *biordered set*, its related axioms and concepts see [13, 3, 2]. Let S be a semigroup with $E_S = E \neq \emptyset$. Define

$$\omega^l = \{(e, f) \in E \times E :\ ef = e\}, \quad \omega^r = \{(e, f) \in E \times E :\ fe = e\},$$

$$\leqslant = \omega^l \cap \omega^r, \quad L = \omega^l \cap (\omega^l)^{-1}, \quad R = \omega^r \cap (\omega^r)^{-1},$$

$$D_E = \{(e, f) \in E \times E : ef = e \text{ or } ef = f \text{ or } fe = e \text{ or } fe = f\}.$$

Then the partial algebra E with domain D_E is a biordered set, Th. 1.1 (a1) [13]. It is easy to see that the relation $\leqslant$ is the natural partial order on the set E, and if $e, f \in E$, then $(e, f) \in L$ $[R]$ iff $(e, f) \in \mathcal{L}$ $[\mathcal{R}]$ (in a semigroup S), where $\mathcal{L}, \mathcal{R}$ are Green's relations on S. Furthermore, the relations ω^l and ω^r are quasi-orders on E. For $\rho = \omega^l$ or $\rho = \omega^r$ and any $e \in E$, we put $\rho(e) = \{g \in E : (g, e) \in \rho\}$.

Let E be a biordered set and $e, f \in E$. We define the *M-set* $M(e, f)$ of e, f by $M(e, f) = \omega^l(e) \cap \omega^r(f) = \{g \in E : g = ge = fg\}$. Also, define the *sandwich-set* $S(e, f)$ of e, f [13] by

$$S(e, f) = \{g \in M(e, f) : (\forall h \in M(e, f))\ (eh, eg) \in \omega^r, (hf, gf) \in \omega^l\}.$$

Moreover, we define E to be an *M-biordered set* iff $M(e, f) \neq \emptyset$ for all $e, f \in E$. Let S be a semigroup with $E_S \neq \emptyset$. We say that S is an *M-semigroup* if E_S is an M-biordered set. Finally, a subset F of E_S is called an *order bi-ideal* of E_S iff $M(e, f) \subseteq F$ for all $e, f \in F$.

The following result is probably known:

Lemma 1.1. *Let S be an R-semigroup, $e, f \in E_S$. Then*:

$$S(e,f) = \{g \in M(e,f) : egf = ef\} = \{g \in M(e,f) : g \in V(ef)\} = fV(ef)e.$$

Proof. Denote the above four sets by A, B, C and D, respectively.

If $g \in B$, then $fge = g$, so $efgef = egf = ef, gefg = gg = g$ i.e., $g \in V(ef)$. Thus $B \subset C$.

If $g \in C$, then $g = fge$ and $g \in V(ef)$. Hence $g \in fV(ef)e$. Thus $C \subset D$.

Let $g = fxe$ for some $x \in V(ef)$. Then clearly $g \in M(e,f)$. If $h \in M(e,f)$ (i.e. $fh = h = he$), then $(eg)(eh) = efxeeh = efxe(fh) = (efxef)h = efh = eh$. Thus $(eh, eg) \in \omega^r$, and similarly $(hf, gf) \in \omega^l$, so $g \in A$. Consequently, $D \subset A$.

Finally, let $g \in A, x \in V(ef)$. Then $fxe \in D \subset A$. In particular, $eg \mathcal{R} efxe$ (by the definition of A). Hence

$$egf = e(ge)f = (eg)(ef) = eg(efxef) = (eg \cdot efxe)f = efxef = ef.$$

Thus $g \in B$, as exactly required. □

Let S be an R-semigroup. A subset F of E_S is called a *biorder ideal* if and only if the following two conditions hold:

(i) $(\forall e \in E_S, f \in F)\ e \leqslant f \Longrightarrow e \in F$;

(ii) $(\forall e, f \in F)\ S(e,f) \cap F \neq \emptyset$.

2. The main results

Proposition 2.1. *Let S be a semigroup. The following conditions are equivalent*:

(i) *S is E-inversive;*

(ii) *every bi-ideal of S contains some idempotent of S;*

(iii) *every quasi ideal of S contains some idempotent of S;*

(iv) *every ideal of S contains some idempotent of S.*

Proof. (i) $\Longrightarrow$ (ii). Let B be a bi-ideal of S, $b \in B$ and $x \in W(b^2)$. Then $x = xbbx$. Hence $(bxb)^2 = b(xbbx)b = bxb \in BSB \subseteq B$. Thus $bxb \in E_B$.

(ii) $\Longrightarrow$ (iii) $\Longrightarrow$ (iv). This is evident.

(iv) $\Longrightarrow$ (i). Let $a \in S$. By assumption SaS has at least one idempotent, that is, $xay = e$ for some $x, y \in S$, $e \in E_S$, so $exaye = e$. Hence $yexayex = yex$. Thus $yex \in W(a)$. □

Lemma 2.2. *Every E-inversive semigroup S is locally E-inversive.*

Proof. Let $a \in eSe$, where $e \in E_S$, $x \in W(a)$. Then $x = xax = x(eae)x$. It follows that $exe = (exe)a(exe)$. Thus $exe \in W(a)$ in eSe, as exactly required. □

Corollary 2.3. *Every bi-ideal of an E-inversive semigroup S is E-inversive. Hence a semigroup S is E-inversive if and only if every bi-ideal of S is E-inversive.*

Proof. Let B be a bi-ideal of S and $b \in B$. By Proposition 2.1, B contains some idempotent of S, say e. By Lemma 2.2, $eSe \in BSB \subseteq B$ is E-inversive and so $(ebe)y \in E_{eSe}$ for some $y \in eSe$. Hence $(eb)(ey) \in E_{eSe}$, say $(eb)(ey) = f$, where $ey \in e(eSe) = eSe$. Therefore $f(eb)eyf = f$, so $eyf(eb)eyf = eyf$. We conclude that there exists $x \in W(eb)$ in B (for example: $x = (ey)f \in (eSe)(eSe) \subseteq B$), so $x = xebx$. Thus $(xe)b(xe) = xe$ and $xe \in Be \subseteq B$. Consequently, B is E-inversive (remark that even $xe = eyfe \in eSe$). □

Let a semigroup S (with $E_S \neq \emptyset$) be locally E-inversive, $b \in S$ and $e \in E_S$. Consider the least bi-ideal, say B, of S containing the set $\{e, b\}$. Note that $(e) \subseteq B$ i.e., $eSe \subseteq B$. From the proof of Corollary 2.3 and from Lemma 2.2 we obtain:

Corollary 2.4. *A semigroup is E-inversive if and only if it is locally E-inversive.*

In [7] S. Kopamu defined a countable family of congruences on a semigroup S, as follows: for each ordered pair of non-negative integers (m, n), he put:

$$\theta_{m,n} = \{(a, b) \in S \times S : (\forall x \in S^m, y \in S^n)\ xay = xby\},$$

and he made the convention that $S^1 = S$ and S^0 denotes the set containing the empty word. In particular, $\theta_{0,0}$ is the identity relation on S. Let $\mathcal{C}$ be a class of semigroups of the same type $\mathcal{T}$ (for example: the class of E-inversive semigroups); call its elements *$\mathcal{C}$-semigroups*. A semigroup S is called a *structurally $\mathcal{C}$-semigroup* if $S/\theta_{m,n} \in \mathcal{C}$ for some integers $m, n \geq 0$. Further, denote by $\mathcal{SC}$ the class of all structurally $\mathcal{C}$-semigroups. It is clear that $\mathcal{C} \subseteq \mathcal{SC}$. Finally, we say that the class $\mathcal{C}$ is *structurally closed* if $\mathcal{C} = \mathcal{SC}$ [8].

Lemma 2.5. *Every structurally E-inversive semigroup is locally E-inversive.*

Proof. Let S be a structurally E-inversive semigroup, say $S/\theta_{m,n}$ is E-inversive; $a \in eSe$, where $e \in E_S$. Since the class of E-inversive semigroups is closed under homomorphic images, then we may suppose that m, n are both positive integers. Moreover, $a = eae$, $(x, xax) \in \theta_{m,n}$ for some $x \in S$. Hence $e^m x e^n = e^m xaxe^n$, that is, $exe = exaxe = ex(eae)xe$ and so $exe = (exe)a(exe)$. Therefore $exe \in W(a)$ in the semigroup eSe. Consequently, S is locally E-inversive. □

Combining the above lemma with Corollary 2.4 we obtain the following:

Corollary 2.6. *The class of all E-inversive semigroups is structurally closed.* □

By the *trace* $\mathrm{tr}\rho$ of a congruence ρ on a semigroup S we mean $\rho \cap (E_S \times E_S)$.

Corollary 2.7. *If ρ is a matrix congruence on an E-inversive semigroup S, then every ρ-class of S is E-inversive.*

Moreover, every matrix congruence on an E-inversive semigroup is uniquely determined by its trace.

Proof. The first part follows from Corollary 2.3 and the following easy observation: if A is any ρ-class of S, where ρ is a matrix congruence on S, then A is a bi-ideal.

We show the second part. Let ρ_1, ρ_2 be matrix congruences on an E-inversive semigroup S, $\mathrm{tr}\rho_1 \subset \mathrm{tr}\rho_2$, $e \in E_S$. If $a \in e\rho_1$, then there exists $x \in W(a)$ in $e\rho_1$. Hence $ax(\mathrm{tr}\rho_1)e(\mathrm{tr}\rho_1)xa$ and so $ax(\mathrm{tr}\rho_2)e(\mathrm{tr}\rho_2)xa$. Therefore we get $a\ \rho_2\ axxa\ \rho_2\ e$ i.e., $a \in e\rho_2$. Thus $\rho_1 \subset \rho_2$. Consequently, if $\mathrm{tr}\rho_1 = \mathrm{tr}\rho_2$, then $\rho_1 = \rho_2$. □

Remark 2.8. The second part of the above corollary generalizes Theorem 2.1 [9]. One can modify all results of J. Mills in Section 2 of [9] for E-inversive E-semigroups. Denote by ψ the least matrix congruence on a semigroup S. It is clear that the interval $[\psi,\ S \times S]$ consists of all matrix congruences on S and it is a complete sublattice of the lattice of all congruences on S. Denote it by $\mathcal{MC}(S)$. Moreover, if S is an E-semigroup, then the symbol $\mathcal{MC}(E_S)$ means the complete lattice of matrix congruences on E_S.

For terminology and elementary facts about lattices the reader is referred to the book [14] (Section I.2). The following result will be useful (see Lemma I.2.8 and Exercise I.2.15 (iii) in [14]):

Lemma 2.9. *If φ is an order isomorphism of a lattice L onto a lattice M, then φ is a lattice isomorphism. Moreover, every lattice ismomorhism of complete lattices is a complete lattice isomorphism.* □

In particular, the following theorem is valid (see Theorems 2.5, 2.6 and Corollary 2.7 in [9]):

Theorem 2.10. *Let S be an E-inversive E-semigroup. Suppose also that the least matrix congruence on E_S can be extended to a matrix congruence on S. Then each matrix congruence on E_S can be extended uniquely to a matrix congruence on S. In fact, if it is the case, then for any matrix congruence ρ_E on E_S, the relation ρ defined on S by:*

$$(a,b) \in \rho \iff (\exists\ e, f \in E_S)\ (a\psi e)\rho_E(f\psi b)$$

is the unique matrix congruence on S which extends ρ_E. Thus there is an inclusion-preserving bijection θ between the lattice $\mathcal{MC}(S)$ and the lattice $\mathcal{MC}(E_S)$. In fact, θ is defined by:

$$\theta : \rho \to tr\rho$$

for every $\rho \in \mathcal{MC}(S)$. Furthermore, θ^{-1} is an inclusion-preserving bijection, too (by the proof of the second part of Corollary 2.7), so θ is an order isomorphism of the lattice $\mathcal{MC}(S)$ onto the lattice $\mathcal{MC}(E_S)$. Consequently, θ is a complete lattice isomorphism between the complete lattices $\mathcal{MC}(S)$ and $\mathcal{MC}(E_S)$, respectively.

Also, ρ is a matrix congruence on an E-inversive E-semigroup S if and only if $tr\rho$ is a matrix congruence on E_S and every ρ-class of S contains some idempotent of S. □

Clearly, every semigroup S is an ideal (of S) and so S is a bi-ideal. Also, if A is a left [right or bi-] ideal of S, $a \in A$, then the principle left [right or bi-] ideal of S containing a is contained in A. Thus by Proposition 2.1 and Corollary 2.3 we obtain the following:

Corollary 2.11. *Let S be a semigroup. The following conditions are equivalent:*

(*i*) *S is E-inversive;*
(*ii*) *every left [right] (principle) ideal of S contains some idempotent of S;*
(*iii*) *every (principle) ideal of S contains some idempotent of S;*
(*iv*) *every (principle) quasi ideal of S contains some idempotent of S;*
(*v*) *every (principle) bi-ideal of S contains some idempotent of S;*
(*vi*) *every (principle) bi-deal of S is E-inversive;*
(*vii*) *every (principle) quasi ideal of S is E-inversive;*
(*viii*) *every (principle) left [right] ideal of S is E-inversive;*
(*ix*) *every (principle) ideal of S is E-inversive.* □

Proposition 2.12. *Every E-inversive semigroup S is an M-semigroup. In fact,*

$$M(e, f) = fW(ef)e$$

for all $e, f \in E_S$.

Proof. Let $g \in M(e, f)$, where $e, f \in E_S$. Then $g = fge$. Also, $gefg = gg = g$ and so $g \in W(ef)$. Consequently, $g \in fW(ef)e$.

Conversely, if $g = fxe$ for some $x \in W(ef)$, then $gg = f(xefx)e = fxe = g$. Hence $g \in E_S$. Clearly, $g = ge = fg$. Thus $g \in M(e, f)$, as required. □

Remark 2.13. The free monoids are M-semigroups but they are not E-inversive. Note that in [4] Edwards shows that eventually regular semigroups are M-semigroups and gives an example of an M-biordered set which does not arise from eventually regular semigroups.

In the following three results are presented some connections between bi-ideals of an E-inversive semigroup S and order bi-ideals, bi-order ideals of the biordered set E_S.

Proposition 2.14. *Let S be an R-semigroup. Then F is an order bi-ideal of E_S if and only if F is a biorder ideal of E_S.*

Proof. Let F be an order bi-ideal of E_S. Then $S(g, h) \subseteq M(g, h) \subseteq F$ for every $g, h \in F$, so $S(g, h) \cap F = S(g, h) \neq \emptyset$, since S is an R-semigroup (Lemma 1.1). Also, if $e \in E_S$, then for every $f \in F$ such that $e \leqslant f$ (i.e., $e = ef = fe$) we have $e \in W(f)$. Consequently, $e = fef \in fW(ff)f = M(f, f) \subseteq F$. Therefore F is a biorder ideal of E_S.

The proof of the opposite implication is similar to the proof of Theorem 1 [1] and is omitted. □

Lemma 2.15. *Let B be a bi-ideal of an E-inversive semigroup S. Then E_B is an order bi-ideal of E_S.*

Proof. Let B be a bi-ideal of $S, g, h \in E_B, e \in M(g,h)$. Then $e = hxg$ for some $x \in W(gh)$ (Proposition 2.12), so $e \in BSB \subseteq B$ i.e., $e \in E_B$. Thus $M(g,h) \subseteq E_B$ for all $g, h \in E_B$. Consequently, E_B is an order bi-ideal of E_S. □

The following theorem generalizes Theorem 2 [1].

Theorem 2.16. *Let S be an E-inversive semigroup and B be a bi-ideal of S. Then E_B is an order bi-ideal of E_S. Also, $A = E_BSE_B$ is an E-inversive bi-ideal of S such that $E_A = E_B$.*

Conversely, if F is an order bi-ideal of E_S, then $B = FSF$ is an E-inversive bi-ideal of S such that $E_B = F$.

Proof. Indeed, E_B is an order bi-ideal of E_S. It is clear that A is a bi-ideal of S and so A is E-inversive (Corollary 2.3). Also, $E_A = E_B$, since $BSB \subseteq B$.

We may show in a similar way the second part of the theorem. □

Finally, we show that every M-biordered set E arises from some E-inversive semigroup. Firstly, we have need the following important Easdown's Theorem:

Theorem 2.17. (Corollary from Theorem 3.3 [3]) *Every biordered set comes from some semigroup.* □

We say that an element a of a semigroup is *E-inversive* if $W(a) \neq \emptyset$.

The following theorem is the main result of the paper.

Theorem 2.18. *Each M-biordered set E arises from some E-inversive semigroup.*

Proof. Let E be an M-biordered set. By Easdown's Theorem there exists some semigroup S with $E_S = E$. Since E_S is M-biordered, then $M(e,f)$ is nonempty for all $e, f \in E_S$, so by Proposition 2.12, $W(ef) \neq \emptyset$ for all $e, f \in E_S$. We show that the set T (say) of all E-inversive elements of S forms an E-inversive subsemigroup of S. Clearly, $E_S \subset T$ and so $T \neq \emptyset$. Moreover, if $W(a), W(b)$ are nonempty, then $xa, by \in E_S$ for some $x, y \in S$. Thus $W(xaby) \neq \emptyset$ and so $s = sxabys$ for some $s \in S$. It follows that $ysx = ysx(ab)ysx$. Therefore $W(ab) \neq \emptyset$. We conclude that E is the set of idempotents of an E-inversive semigroup T (since if $t \in T$ and $x \in W(t)$ in S, then $x \in Reg(S) \subset T$, so $x \in W(t)$ in T). □

Remark 2.19. A biordered set E is called *regular* if $S(e,f) \neq \emptyset$ for all $e, f \in E$. By Hall's result, Easdown's Theorem and Lemma 1.1 we obtain Nambooripad's Theorem [13]:

Theorem 2.20. *Every regular biordered set comes from some regular semigroup.*

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