On WIP loops

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Abstract. A weak inverse property loop (WIP loop) is a loop L that satisfies $x(yx)^\rho = y^\rho$ or $(xy)^\lambda x = y^\lambda$ for all $x, y \in L$. In this paper we prove some necessary and sufficient conditions for a WIP loop to be LC, RC, left alternative, right alternative, and C-loop. We also construct infinite families of WIP loops of various orders.

1. Introduction

Let L be a loop with identity element 1, then L will be said to satisfy the *weak inverse property* if whenever three elements x, y, z of L satisfy the relation $xy{\cdot}z = 1$, they also satisfy the relation $x \cdot yz = 1$. The study of weak inverse property loops (WIP loops) was initiated by J. M. Osborn [4] as a class of loops which contains both IP loops and CIP loops. He proved that a WIP loop is a loop which satisfies one of the following equivalent identities

$$x(yx)^\rho = y \quad \text{or} \quad (xy)^\lambda x = y^\lambda.$$

He further proved that the left, middle and right nuclei of a WIP loop coincide. If L is a loop all of whose isotopes have the WIP and N is its nucleus, then N is normal and L/N is a Moufang loop. Isotopy-isomorphy conditions of WIP loops were considered in [2]. We prove some necessary and sufficient conditions for a WIP loop to be LC, RC, left alternative, right alternative, and C-loop in section 3 and construct infinite families of WIP loops of various orders in section 4.

2. Preliminaries

Let L be a loop. Then the sets

$$N_\lambda = \{x \in L : x(yz) = (xy)z \text{ for every } y, z \in L\},$$
$$N_\mu = \{x \in L : y(xz) = (yx)z \text{ for every } y, z \in L\},$$
$$N_\rho = \{x \in L : y(zx) = (yz)x \text{ for every } y, z \in L\}$$

are called the *left nucleus*, *middle nucleus* and *right nucleus* respectively. $N = N_\lambda \cap N_\mu \cap N_\rho$ is called the *nucleus*.

2010 Mathematics Subject Classification: 20N99
Keywords: WIP loop, C-loop, ARIF loop, LC-loop, RC-loop

A loop L is called *left alternative* if $xx \cdot y = x \cdot xy \ \forall x, y \in L$, *right alternative* if $x \cdot yy = xy \cdot y \ \forall x, y \in L$, and *alternative* if it is both left alternative and right alternative.

C-loops are loops satisfying the identity $x(y(yz)) = ((xy)y)z$. Loops satisfying the identity $(xx)(yz)) = (x(xy))z$ are called *LC-loops* and loops satisfying the identity $(xy)(yz) = x(y(zz))z$ are called *RC-loops*. Loops which are both LC-loops and RC-loops are C-loops. *ARIF* loops are defined to be flexible loops satisfying $(zx)(yxy) = (z(xyx))y$.

3. Necessary and sufficient conditions

LC-loops, RC-loops, C-loops, ARIF loops are subclasses of WIP loop. We prove here necessary and sufficient conditions for a WIP loop to satisfy these loops which are its subclasses. We define $L_x : a \longrightarrow xa$, $R_x : a \longrightarrow ax$, $J : x \longrightarrow x^{-1}$ and $P = R_x \circ L_x \ \forall \ x \in L$.

Theorem 3.1. *Let L be a WIP loop of unique inverses. Then $(JP)^n = I$ for any $n \in 2\mathbb{Z}^+$, where $\mathbb{Z}^+$ denotes the set of positive integers.*

Proof. Let $y \in L$. Since $P = R_x \circ L_x$, then for $(JP)^n = I$, where $n \in 2Z^+$. Consider $n = 2$. Then

$$y(JP)^2 = yJPJP = x((x(y^{-1}x))^{-1}x) = x(y^{-1}x)^{-1} = y.$$

Thus $(JP)^2 = I$. Now if any $n \in 2\mathbb{Z}^+$, then $n = 2m$ for some $m \in \mathbb{Z}^+$, so $(JP)^n = (JP)^{2m} = ((JP)^2)^m = (I)^m = I$. □

Corollary 3.2. *$(JP)^n = I$ for all $n \in \mathbb{Z}^+$ if the loop is a WIP loop of exponent 2.*

Proof. Let L be a WIP loop of exponent 2. Then

$$\begin{aligned} y(JP) &= y^{-1}R_x \circ L_x = x(y^{-1}x) \\ &= x(y^{-1}x)^{-1} \qquad \text{since } L \text{ is of exponent 2} \\ &= y^{-1} \qquad \text{since } L \text{ is a WIP loop} \\ &= y. \end{aligned}$$

Thus $JP = I$ and hence $(JP)^n = I$ for all $n \in \mathbb{Z}^+$ if the loop is a WIP loop of exponent 2. □

Next we prove necessary and sufficient conditions for a WIP loop to be left alternative, and right alternative.

Theorem 3.3. *Let L be a WIP loop. Then L is left alternative if and only if $L_x = R_x J L_{x^2} JP$.*

Proof. Let L be a WIP loop satisfying $L_x = R_xJL_{x^2}JP$. Then

$$\begin{aligned} L_x &= R_xJL_{x^2}JP \\ JR_x^{-1}J &= R_xJL_{x^2}JP \quad \text{since } L_x = JR_x^{-1}J \\ R_x^{-1}J &= L_x^{-1}L_{x^2}JP \quad \text{since } L_x^{-1} = JR_xJ \\ L_xR_x^{-1}P &= L_{x^2}(JP)^2 \\ L_xR_x^{-1}R_xL_x &= L_{x^2}I \quad \text{by Theorem 3.1} \\ L_xL_x &= L_{x^2} \end{aligned}$$

Conversely, if is $x(xy) = x^2y$ for all $x, y \in L$, then $L_xL_x = L_{x^2}$ for all $x \in$L. Thus $L_xL_xP^{-1} = L_{x^2}P^{-1}$. From this, by Theorem 3.1, we obtain $L_xR_x^{-1} = L_{x^2}(JP)^2P^{-1}$, i.e., $R_x^{-1} = L_x^{-1}L_{x^2}JPJ$. The last, by left and right cancellation of J, implies $L_x = R_xJL_{x^2}JP$. □

Theorem 3.4. *Let L be a WIP loop. Then L is right alternative if and only if* $R_x = PJR_{x^2}JL_x$.

Proof. If L satisfies $R_x = PJR_{x^2}JL_x$, then

$$\begin{aligned} JR_xJ &= JPJR_{x^2}JL_xJ \quad \text{by multiplication of both sides by J} \\ PL_x^{-1} &= PJPJR_{x^2}R_x^{-1} \quad \text{by multiplication of both sides by P} \\ R_xR_x &= R_{x^2}. \end{aligned}$$

Conversely, let L be right alternative. Then $R_xR_x = R_{x^2}$. Hence $P^{-1}R_xR_x = P^{-1}R_{x^2}$. Thus $L_x^{-1}IR_x = P^{-1}R_{x^2}$, which implies $L_x^{-1}R_x = IP^{-1}R_{x^2}$, and consequently $R_x = PJR_{x^2}JL_x$. □

Theorem 3.5. *A WIP loop L is an LC loop if and only if it satisfies the identity* $JL_{x^2}T_z = L_zT_xJPL_z$, *where* $T_x = R_x^{-1}L_x$.

Proof. Let L be an LC loop. Then $xx \cdot yz = (x \cdot xy)z$, which implies $R_zL_{x^2} = L_xL_xR_z$. Thus $R_zL_{x^2}T_z = L_xL_xR_zT_z$, whence, putting $L_x^{-1} = JR_xJ$, we obtain $JL_{x^2}T_z = L_zR_x^{-1}L_xJR_xJJL_xL_z$. Thus $JL_{x^2}T_z = L_zT_xJPL_z$.

Conversely, if L satisfies $JL_{x^2}T_z = L_zT_xJPL_z$, then also $JR_zL_{x^2}R_z^{-1} = T_xJP$, which implies $R_zL_{x^2} = L_xL_xR_z$. Hence, L is an LC loop. □

Theorem 3.6. [2, Theorem 4.2]
A loop L (WIP loop) is a C-Loop if and only if $R_x = PJR_{x^2}JL_x$ *and* $JL_{x^2}T_z = L_zT_xJPL_z$. □

4. Various constructions of WIP loops

Here we give the construction of infinite families of non-associative WIP loops by extensions of loops.

Lemma 4.1. *Let $\mu : G \times G \to A$ be a factor set. Then (G, A, μ) is a WIP loop if and only if*

$$\mu(h, h^{-1}) + \mu(g, g^{-1}h^{-1}) = \mu(h, g) + \mu(hg, g^{-1}h^{-1}) \tag{D}$$

for all $g, h \in G$.

Proof. The loop (G, A, μ) is a WIP loop iff $(g, a)[(h, b)(g, a)]^{-1} = (h, b)^{-1}$ hold for every $g, h \in G$ and every $a, b \in A$. Straight forward calculation with (A) shows that this happens iff (D) holds. □

We call a factor set μ satisfies (A) and (D) a *W-factor set.* We now use a particular W-factor set to construct the above-mentioned families of WIP loops.

Proposition 4.2. *Let $n \geqslant 2$ be an integer and let A be an abelian group of order n with respect to addition with neutral element 0 and $\alpha \in A$ be an element of order bigger than 2. Let $G = \{1, x, x^2\}$ be the cyclic group of order 3 with respect to multiplication with neutral element 1. Define $\mu : G \times G \to A$ by*

$$\mu(h, g) = \begin{cases} \alpha & \text{if } (h, g) = (x, x), \\ 0 & \text{otherwise.} \end{cases}$$

Then (G, A, μ) is a non-alternative (hence non-associative) commutative WIP loop with $N = \{(1, a) : a \in A\}$.

Proof. The map μ is clearly a factor set. To show that (G, A, μ) is a WIP loop, we verify (D). Since μ is a factor set, there is nothing to prove when $g = 1$. Assume that $g = x$. Then (D) becomes $\mu(h, h^{-1}) + \mu(x, x^2h^{-1}) = \mu(h, x) + \mu(x, x^2h^{-1})$. If $h = 1$, then $\mu(1, 1) + \mu(x, x^2) = \mu(1, x) + \mu(x, x^2)$ and both sides of this equation are equal to 0. If $h = x$, then$\mu(x, x^2) + \mu(x, x) = \mu(x, x) + \mu(x, x)$ and both sides of this equation are equal to α. Assume $h = x^2$, then $\mu(x^2, x) + \mu(x, 1) = \mu(x^2, x) + \mu(1, xx)$ and both sides of this equation are equal to 0. Next assume that $g = x^2$, then (D) becomes $\mu(h, h^{-1}) + \mu(x^2, xh^{-1}) = \mu(h, x^2) + \mu(hx^2, xh^{-1})$. If $h = 1$, then both sides of this equation are equal to 0. Assume $h = x$, then both sides of this equation are equal to 0, Assume $h = x^2$, then $\mu(x^2, x) + \mu(x^2, x^2) = \mu(x^2, x^2) + \mu(x, x^2)$ and both sides of this equation are equal to 0. Since $\alpha \neq 0$, we have that, $(x, a)(x, a) \cdot (x^2, a) \neq (x, a) \cdot (x, a)(x^2, a)$. Thus (G, A, μ) is non-alternative and hence non-associative. Also neither $(x, a) \in N$ nor $(x^2, a) \in N$ for all $a \in A$. Also we have that $(1, a)((h, b)(g, c)) = ((1, a)(h, b))(g, c)$ for all $h, g \in G$ and $a, b, c \in A$. Which implies that $(1, a)$ belongs to nucleus. Thus $\{(1, a); a \in A\}$ is the nucleus of the loop (G, A, μ). □

Corollary 4.3. *For each natural number n there exists a non-alternative commutative WIP loop having nucleus of order n.*

Proof. It remains to show that there exist non-alternative commutative WIP loop having nucleus of order 1. This requirement is fulfilled by the following example. □

Example 4.4. A commutative, non-alternative WIP loop of order 10 having trivial nucleus.

$\cdot$	0	1	2	3	4	5	6	7	8	9
0	0	1	2	3	4	5	6	7	8	9
1	1	0	3	2	5	4	8	9	6	7
2	2	3	0	1	6	7	4	5	9	8
3	3	2	1	0	8	9	7	6	4	5
4	4	5	6	8	1	0	9	2	7	3
5	5	4	7	9	0	1	2	8	3	6
6	6	8	4	7	9	2	3	0	5	1
7	7	9	5	6	2	8	0	3	1	4
8	8	6	9	4	7	3	5	1	2	0
9	9	7	8	5	3	6	1	4	0	2

Example 4.5. The smallest group A satisfying the assumption of Proposition 4.2 is the cyclic group $\{0,1\}$ of order 2. The construction of Proposition 4.2 with $\alpha = 1$ yields the smallest non-alternative commutative WIP loop of order 6.

$\cdot$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	0	3	2	5	4
2	2	3	5	4	0	1
3	3	2	4	5	1	0
4	4	5	0	1	2	3
5	5	4	1	0	3	2

Proposition 4.6. *Let $n \geqslant 3$ be an integer and let A be an abelian group of order n with respect to addition with neutral element 0 and $\alpha \in A$ be an element of order bigger than 2. Let $G = \{1, u, v, w\}$ denotes the Klein group with respect to multiplication with neutral element 1. Define $\mu : G \times G \to A$ by*

$$\mu(x,y) = \begin{cases} \alpha & \text{if } \ (x,y) \in \{(u,v),(v,w),(w,u)\}, \\ 0 & \text{otherwise.} \end{cases}$$

Then (G, A, μ) is a non-alternative, non-commutative WIP loop with nucleus $N = \{(1,a) \,:\, a \in A\}$.

Proof. The map μ is clearly a factor set. To show that (G, A, μ) is a WIP loop, we verify (D). Since μ is a factor set, there is nothing to prove when $g = 1$. Assume that $g = u$, then (D) becomes $\mu(h, h^{-1}) + \mu(u, uh^{-1}) = \mu(h, u) + \mu(hu, uh^{-1})$. If $h = 1$, then both sides of this equation are equal to 0. Assume $h = v$, then $\mu(v, v) + \mu(u, w) = \mu(v, u) + \mu(w, w)$ and both sides of this equation are equal to 0. Assume $h = w$, then $\mu(w, w) + \mu(u, v) = \mu(w, u) + \mu(v, v)$ and both sides of this equation are equal to α. Next assume that $g = v$, then (D) becomes $\mu(h, h^{-1}) + \mu(v, vh^{-1}) = \mu(h, v) + \mu(hv, vh^{-1})$. If $h = 1$, then both sides of this equation are equal to 0. Assume $h = u$, $\mu(u, u) + \mu(v, w) = \mu(u, v) + \mu(w, w)$ and both sides of this equation are equal to α. Assume $h = v$, then $\mu(v, v) + \mu(v, 1) =$

$\mu(v,v)+\mu(1,1)$ both sides of this equation are equal to 0. Assume $h=w$, then $\mu(w,w)+\mu(v,u)=\mu(w,v)+\mu(u,u)$ and both sides of this equation are equal to 0. Next assume that $g=w$, then (D) becomes $\mu(h,h^{-1})+\mu(w,wh^{-1})=\mu(h,w)+\mu(hw,wh^{-1})$. If $h=1$, then both sides of this equation are equal to 0. Assume $h=u$, then this equation is equal to $\mu(u,u)+\mu(w,v)=\mu(u,w)+\mu(v,v)$ and both sides of this equation are equal to 0. Assume $h=v$, then $\mu(v,v)+\mu(w,u)=\mu(v,w)+\mu(u,u)$ and both sides of this equation are equal to α. Assume $h=w$, then $\mu(w,w)+\mu(w,1)=\mu(w,w)+\mu(1,1)$ and both sides of this equation are equal to 0. Since $\alpha\neq 0$, and we have that, $(u,a)(u,a)\cdot(v,a)\neq(u,a)\cdot(u,a)(v,a)$ also we have that, $(w,a)(u,a)\cdot(u,a)\neq(w,a)\cdot(u,a)(u,a)$. Thus (G,A,μ) is non-alternative and hence non-associative. Also $(u,a),(v,a),(w,a)\notin N$ for all $a\in A$. Also we have that $(1,a)((h,b)(g,c))=((1,a)(h,b))(g,c)$ for all $h,g\in G$ and $a,b,c\in A$. Which implies that $(1,a)$ belongs to the nucleus. Thus $\{(1,a):a\in A\}$ is the nucleus of the loop (G,A,μ). □

Corollary 4.7. *For each $n\geqslant 1$ there exists a non-alternative non-commutative WIP loop having nucleus of order n.*

Proof. It remains to show that there exist a non-alternative non-commutative WIP loop having nuclei of order 1 and 2. The first requirement follows by Example 4.8 while the second requirement follows by Example 4.9. □

Example 4.8. A non-alternative non-commutative WIP loop having nucleus of order 1.

$\cdot$	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	0	4	6	2	7	3	5
2	2	7	5	0	3	1	4	6
3	3	5	0	4	6	2	7	1
4	4	6	3	1	7	0	5	2
5	5	3	7	2	0	6	1	4
6	6	4	1	7	5	3	2	0
7	7	2	6	5	1	4	0	3

Example 4.9. A non-alternative non-commutative WIP loop having nucleus of order 2.

$\cdot$	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	1	2	0	5	6	4	3
2	2	0	1	6	5	3	4
3	3	6	5	4	0	1	2
4	4	5	6	0	3	2	1
5	5	3	4	2	1	6	0
6	6	4	3	1	2	0	5

Example 4.10. The smallest group A satisfying the assumption of Proposition 4.6 is the cyclic group $\{0,1,2\}$. The construction of Proposition 4.6 with $\alpha = 1$ yields the smallest non-alternative commutative WIP loop of order 12.

·	0	1	2	3	4	5	6	7	8	9	10	11
0	0	1	2	3	4	5	6	7	8	9	10	11
1	1	2	0	4	5	3	7	8	6	10	11	9
2	2	0	1	5	3	4	8	6	7	11	9	10
3	3	4	5	0	1	2	11	9	10	6	7	8
4	4	5	3	1	2	0	9	10	11	7	8	6
5	5	3	4	2	0	1	10	11	9	8	6	7
6	6	7	8	9	10	11	0	1	2	5	3	4
7	7	8	6	10	11	9	1	2	0	3	4	5
8	8	6	7	11	9	10	2	0	1	4	5	3
9	9	10	11	8	6	7	3	4	5	0	1	2
10	10	11	9	6	7	8	4	5	3	1	2	0
11	11	9	10	7	8	6	5	3	4	2	0	1

GAP gives these extra informations about the above WIP loop. It is (1) power associative, (2) not Moufang, (3) neither automorphic nor anti-automorphic, (4) neither left nor right Bol.

Proposition 4.11. *Let $n \geqslant 3$ be an integer and let A be an abelian group of order n with respect to addition with neutral element 0 and $\alpha \in A$ be an element of order bigger than 2. Let $G = \{1, u, v, w\}$ denotes the Klein group with respect to multiplication with neutral element 1. Define $\mu : G \times G \to A$ by*

$$\mu(x,y) = \begin{cases} \alpha & \textit{if } \ (x,y) \in \{(u,v),(v,u),(u,w),(w,u),(v,w),(w,v)\}, \\ 0 & \textit{otherwise.} \end{cases}$$

Then (G, A, μ) is a non-alternative, commutative WIP loop with nucleus $N = \{(1,a) : a \in A\}$.

Proof. The map μ is clearly a factor set. To show that (G, A, μ) is a WIP loop, we verify (D). Since μ is a factor set, there is nothing to prove when $g = 1$. Assume that $g = u$, then (D) becomes $\mu(h,h^{-1}) + \mu(u, uh^{-1}) = \mu(h,u) + \mu(hu, uh^{-1})$. If $h = 1$, then $\mu(h,h^{-1}) + \mu(u,u) = \mu(1,u) + \mu(u,u)$ both sides of this equation are equal to 0. Assume $h = u$ then $\mu(u,u) + \mu(u,1) = \mu(u,u) + \mu(1,1)$ both sides of this equation are equal to 0. Assume $h = v$, then $\mu(v,v) + \mu(u,w) = \mu(v,u) + \mu(w,w)$ and both sides of this equation are equal to α. Assume $h = w$, then $\mu(w,w) + \mu(u,v) = \mu(w,u) + \mu(v,v)$ and both sides of this equation are equal to α. Next assume that $g = v$, then (D) becomes $\mu(h,h^{-1}) + \mu(v, vh^{-1}) = \mu(h,v) + \mu(hv, vh^{-1})$. If $h = 1$, then $\mu(1,1) + \mu(v,v) = \mu(1,v) + \mu(v,v)$ and both sides of this equation are equal to 0. Assume $h = u$, then $\mu(u,u) + \mu(v,w) = \mu(u,v) + \mu(w,w)$ and both sides of this equation are equal to α. Assume $h = v$, then $\mu(v,v) + \mu(v,1) = \mu(v,v) + \mu(1,1)$ both sides of this equation are equal to 0. Assume $h = w$, then $\mu(w,w) + \mu(v,u) = \mu(w,v) + \mu(u,u)$ and both sides of this equation are equal to α. Next assume that $g = w$, then (D) becomes $\mu(h, h^{-1}) + \mu(w, wh^{-1}) = \mu(h, w) + \mu(hw, wh^{-1})$. If $h = 1$, then $\mu(1,1) + \mu(w,w) = \mu(1,w) + \mu(w,w)$ both sides of this equation are equal to 0. Assume $h = u$, then $\mu(u,u) + \mu(w,v) = \mu(u,w) + \mu(v,v)$ and both sides of this equation are equal to α. Assume $h = v$, then $\mu(v,v) + \mu(w,u) = \mu(v,w) + \mu(u,u)$ and both sides of this equation are equal to α. Assume $h = w$, then $\mu(w,w) + \mu(w,1) = \mu(w,w) + \mu(1,1)$ and both sides of this equation are equal to 0. Since $\alpha \neq 0$, and we have that, $(u,a)(u,a) \cdot (v,a) \neq (u,a) \cdot (u,a)(v,a)$. Also $(w,a)(u,a) \cdot (u,a) \neq (w,a) \cdot (u,a)(u,a)$. Thus (G, A, μ) is non-alternative and hence non-associative. Also $(u,a), (v,a), (w,a) \notin N$ for all $a \in A$. Also we have that $(1,a)((h,b)(g,c)) = ((1,a)(h,b))(g,c)$ for all $h, g \in G$ and $a, b, c \in A$. Which implies that $(1,a)$ belongs to the nucleus. Thus $\{(1,a) : a \in A\}$ is the nucleus of the loop (G, A, μ). □

Example 4.12. The smallest group A satisfying the assumption of Proposition 4.11 is the cyclic group $\{0, 1, 2\}$. The construction of Proposition 4.11 with $\alpha = 1$ then yields the smallest non-alternative commutative WIP loop of order 12.

$\cdot$	0	1	2	3	4	5	6	7	8	9	10	11
0	0	1	2	3	4	5	6	7	8	9	10	11
1	1	2	0	4	5	3	7	8	6	10	11	9
2	2	0	1	5	3	4	8	6	7	11	9	10
3	3	4	5	0	1	2	11	9	10	8	6	7
4	4	5	3	1	2	0	9	10	11	6	7	8
5	5	3	4	2	0	1	10	11	9	7	8	6
6	6	7	8	11	9	10	0	1	2	5	3	4
7	7	8	6	9	10	11	1	2	0	3	4	5
8	8	6	7	10	11	9	2	0	1	4	5	3
9	9	10	11	8	6	7	5	3	4	0	1	2
10	10	11	9	6	7	8	3	4	5	1	2	0
11	11	9	10	7	8	6	4	5	3	2	0	1

GAP [3] gives these extra informations about the above WIP loop. It is (1) power associative, (2) not automorphic inverse property loop, (2) neither LC-loop nor RC-loop.

Proposition 4.13. *Let $n \geqslant 2$ be an integer and let A be an abelian group of order n with respect to addition with neutral element 0 and $\alpha \in A$ be an element of order bigger than 2. Let $G = \{1, x, x^2, x^3, x^4\}$ be the Cyclic group of order 5 with respect to multiplication with neutral element 1. Define $\mu : G \times G \to A$ by*

$$\mu(h, g) = \begin{cases} \alpha & \text{if } (h, g) \in \{(x^2, x^2), (x, x^2), (x^2, x)\}, \\ 0 & \text{otherwise.} \end{cases}$$

Then (G, A, μ) is a non-alternative commutative WIP loop with nucleus $N = \{(1, a) : a \in A\}$.

Proof. The map μ is clearly a factor set. To show that (G, A, μ) is a WIP loop, we verify (D). Since μ is a factor set, there is nothing to prove when $g = 1$. Assume that $g = x$, then (D) becomes $\mu(h, h^{-1}) + \mu(x, x^4h^{-1}) = \mu(h, x) + \mu(hx, x^4h^{-1})$. If $h = 1$, then $\mu(h, h^{-1}) + \mu(x, x^4h^{-1}) = \mu(h, x) + \mu(hx, x^4h^{-1})$ and both sides of this equation equals to 0. $h = x$, then $\mu(x, x^4) + \mu(x, x^3) = \mu(x, x) + \mu(x^2, x^3)$ then both sides of this equation are equal to 0,Assume $h = x^2$, then $\mu(x^2, x^3) + \mu(x, x^2) = \mu(x^2, x) + \mu(x^3, x^2)$ and both sides of this equation are equal to α. Assume $h = x^3$, then $\mu(x^3, x^2) + \mu(x, x) = \mu(x^3, x) + \mu(x^4, x)$ and both sides of this equation are equal to 0 Assume $h = x^4$, then $\mu(x^4, x) + \mu(x, 1) = \mu(x^4, x) + \mu(1, 1)$and both sides of this equation are equal to 0 assume that $g = x^2$, then (D) becomes $\mu(h, h^{-1}) + \mu(x^2, x^3h^{-1}) = \mu(h, x^2) + \mu(hx^2, x^3h^{-1})$. If $h = 1$, then $\mu(1, 1) + \mu(x^2, x^3) = \mu(1, x^2) + \mu(x^2, x^3)$ and both sides of this equation equals to 0. Assume $h = x$, then $\mu(x, x^4) + \mu(x^2, x^2) = \mu(x, x^2) + \mu(x^3, x^2)$ then both sides of this equation are equal to α, Assume $h = x^2$, then $\mu(x^2, x^3) + \mu(x^2, x) = \mu(x^2, x^2) + \mu(x^4, x)$ and both sides of this equation are equal to α. Assume $h = x^3$, then $\mu(x^3, x^2) + \mu(x^2, 1) = \mu(x^3, x^2) + \mu(1, 1)$ and both sides of this equation are equal to 0. Assume $h = x^4$, then $\mu(x^4, x) + \mu(x^2, x^4) = \mu(x^4, x^2) + \mu(x, x^4)$ and both sides of this equation are equal to 0. Assume that $g = x^3$, then $\mu(h, h^{-1}) + \mu(x^3, x^2h^{-1}) = \mu(h, x^3) + \mu(hx^3, x^2h^{-1})$. If $h = 1$, then $\mu(1, 1) + \mu(x^3, x^2) = \mu(1, x^3) + \mu(x^3, x^2)$ and both sides of this equation equals to 0. Assume $h = x$, then this equation equals to $\mu(x, x^4) + \mu(x^3, x) = \mu(x, x^3) + \mu(x^4, x)$ then both sides of this equation are equal to 0, Assume $h = x^2$, then $\mu(x^2, x^3) + \mu(x^3, 1) = \mu(x^2, x^3) + \mu(1, 1)$ and both sides of this equation are equal to 0. Assume $h = x^3$, then $\mu(x^3, x^2) + \mu(x^3, x^4) = \mu(x^3, x^3) + \mu(x, x^4)$ and both sides of this equation are equal to 0. Assume $h = x^4$, then $\mu(x^4, x) + \mu(x^3, x^3) = \mu(x^4, x^3) + \mu(x^2, x^3)$ and both sides of this equation are equal to 0, Assume that $g = x^4$, then (D) becomes $\mu(h, h^{-1}) + \mu(x^4, xh^{-1}) = \mu(h, x^4) + \mu(hx^4, xh^{-1})$. If $h = 1$, then $\mu(1, 1) + \mu(x^4, x) = \mu(1, x^4) + \mu(x^4, x)$ both sides of this equation equals to 0. Assume $h = x$, then $\mu(x, x4) + \mu(x^4, 1) = \mu(x, x^4) + \mu(1, 1)$ and both sides of this equation are equal to 0, Assume $h = x^2$, then $\mu(x^2, x^3) + \mu(x^4, x^4) = \mu(x^2, x^4) + \mu(x, x^4)$ and both sides of this equation are equal to 0. Assume $h = x^3$, then $\mu(x^3, x^2) + \mu(x^3, x^4) = \mu(x^3, x^3) + \mu(x, x^4)$ and both sides of this equation are equal to 0. Assume $h = x^4$, then $\mu(x^4, x) + \mu(x^4, x^2) = \mu(x^4, x^4) + \mu(x^3, x^2)$ and both sides of this equation are equal to 0. Since $\alpha \neq 0$, we have that, $(x^3, a) \cdot (x^2, a)(x^2, a) \neq (x3, a)(x^2, a) \cdot (x^2, a)$. Also $(x^2, a) \cdot (x, a)(x^3, a) \neq (x, 3a+\alpha) = (x^2, a)(x, a) \cdot (x^3, a)$. Thus (G, A, μ) is non-alternative and hence non-associative WIP loop. Also neither $(x, a), (x^2, a), (x^3, a) \in N$ for all $a \in A$. Similarly $(x^4, a) \notin A$. Also we have that $(1, a)((h, b)(g, c)) = ((1, a)(h, b))(g, c)$ for all $h, g \in G$ and $a, b, c \in A$. Which implies that $(1, a)$ belongs to the nucleus. Thus $\{(1, a);\, a \in A\}$ is the nucleus of the loop (G, A, μ). □

Example 4.14. The smallest group A satisfying the assumption of Proposition 4.13 is the cyclic group $\{0, 1, 2\}$ of order 3. The construction of Proposition 4.13 with $\alpha = 1$ yields the smallest non-alternative commutative WIP loop of order 10.

·	0	1	2	3	4	5	6	7	8	9
0	0	1	2	3	4	5	6	7	8	9
1	1	0	3	2	5	4	7	6	9	8
2	2	3	4	5	7	6	8	9	0	1
3	3	2	5	4	6	7	9	8	1	0
4	4	5	7	6	9	8	0	1	2	3
5	5	4	6	7	8	9	1	0	3	2
6	6	7	8	9	0	1	2	3	4	5
7	7	6	9	8	1	0	3	2	5	4
8	8	9	0	1	2	3	4	5	6	7
9	9	8	1	0	3	2	5	4	7	6

GAP shows that the following properties do not hold in this WIP loop: (1) automorphic inverse property, (2) anti-automorphic inverse property, (3) LC, (4) RC, (5) left Bol, (6) right Bol, (7) Moufang, (8) power alternative, (9) power associative, (10) left nuclear square, (13) right nuclear square, (14) left inverse and (15) right inverse property.

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Received February 4, 2011

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