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The varieties of Bol-Moufang quasigroups defined by a single operation

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Abstract. A quasigroup identity is said to be of *Bol-Moufang* type if it involves three variables, two of which occur once on each side and one of which appears twice; moreover, the order in which the variables appear is the same on both sides, and there is only one binary operation in the identity. Answering a question of Drapál, we classify all varieties of quasigroups of Bol-Moufang type where the operation involved is $*$, $/$, or $\backslash$, determining all inclusions among these and providing all necessary counterexamples. This work extends that of Phillips and Vojtěchovský, who described the relationships among the 26 varieties obtained when the operation is $*$. We find that 52 varieties, distinct from each other and from the aforementioned 26, are obtained when one allows $/$ or $\backslash$ as the operation. We determine all inclusions among these varieties, furnishing all necessary counterexamples to complete the classification.

1. Introduction

A *quasigroup* is a set G together with a binary operation $*$ such that the maps $L(a) : G \to G$ and $R(a) : G \to G$ defined by $[L(a)](x) = a*x$ and $[R(a)](x) = x*a$ are bijective for all $a \in G$. As such, there are operations $\backslash : G \to G$ and $/ : G \to G$ defined by $a\backslash c = b$ and $c/b = a$ if only if $a * b = c$. We often refer to $*$ as the *principal operation* in the quasigroup. A quasigroup is called a *loop* if it has a two-sided neutral element, i.e., an element $e \in G$ such that $e * x = x = x * e$ for all $x \in G$. From the viewpoint of universal algebra, one may view the *variety of quasigroups* as consisting of universal algebras $(G, *, \backslash, /)$ satisfying the four identities:

$$a * (a\backslash b) = b,\ \ (b/a) * a = b,\ \ a\backslash(a * b) = b,\ \ (b * a)/a = b.$$

In this article, we classify varieties of quasigroups satisfying an additional identity, an identity of so-called *Bol-Moufang type.* Such identities involve three variables, two of which appear once on both sides of the equation and one of which appears twice on both sides. We also require that the variables appear in the same

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order on both sides, and that only one operation (either $*$, $\backslash$, or $/$) appears in the identity. For example, $x*((y*x)*z) = (x*y)*(x*z)$ is an identity of Bol-Moufang type.

The equational perspective is useful in that it lends itself particularly well to automated theorem proving. Indeed, we made considerable use of the automated theorem prover `Prover9` [3] to deduce which implications among identities were valid; virtually all counterexamples were found using the finite model builder `Mace4` [3]. In hindsight, we realized that all the proofs could be written out by hand, only one of them being somewhat long. Therefore, all proofs that appear in this paper are "human" proofs, although some of them would have been difficult to find without the assistance of `Prover9`.

Our work builds upon that of Phillips and Vojtěchovský [5] who carried out this classification for varieties of quasigroups defined by identities of Bol-Moufang type involving only the operation $*$. Using the action of the group S_3 on the conjugates of a quasigroup, we argue that an analogous classification holds for varieties defined solely by $\backslash$ and for varieties defined by solely by $/$; hence, the problem is reduced to an understanding of how a variety defined by an identity involving one of the three operations is related (if at all) to a variety defined by an identity involving another operation. By using the Phillips-Vojtěchovský classification and the S_3-action, we reduce the problem to checking a much smaller number of implications. We then provide necessary counterexamples to complete our classification.

2. Notation and background

For simplicity of reference, we adopt and extend notation introduced by Phillips and Vojtěchovský in [4] and [5] for labeling identities of Bol-Moufang type.

A	$xxyz$
B	$xyxz$
C	$xyyz$
D	$xyzx$
E	$xyzy$
F	$xyzz$

1	0(0(00))
2	0((00)0)
3	(00)(00)
4	(0(00))0
5	((00)0)0

In labeling an identity, the first letter (S, L, or R) refers to the operation used (**s**tar ($*$), **l**eft division ($\backslash$) or **r**ight division ($/$)); the next letter, selected from A through F, refers to the variable ordering as labeled in the above chart, and the two numbers at the end refer to the parenthesization patterns on the two sides of the identity. For example, $LA25$ is the identity $x\backslash((x\backslash y)\backslash z) = ((x\backslash x)\backslash y)\backslash z$, while $SD34$ is the identity $(x*y)*(z*x) = (x*(y*z))*x$. Note also that an identity employing a variable ordering in which x, y, and z are not revealed in alphabetical order (e.g. $zxyz$) is equivalent to one described by the above notation by appropriate permutation of x, y, and z. Thus, there are 180 identities of Bol-Moufang type to consider, 60 for each operation.

If I is an identity of Bol-Moufang type, its *dual* is the identity $I^\vee$ obtained from I by reading from right to left; for example, $(SD34)^\vee$ is $x*((z*y)*x) = (x*z)*(y*x)$; after switching y and z, we identify this as $SD24$. Thus the variable orders A and F are duals of each other, as are B and E, while C and D are self-dual. Similarly, patterns 1 and 5 are dual to each other, as are 2 and 4, whereas 3 is self-dual. Since the other three operations defined on G ($\circ$, $//$, and $\backslash\backslash$) are defined by

$$x \circ y = y * x, \quad x//y = y\backslash x, \quad \text{and} \quad x\backslash\backslash y = y/x$$

an identity of Bol-Moufang type involving any one of these operations is equivalent to an identity involving one of $*$, $\backslash$, or $/$. This explains our restriction to identities of the latter sort.

We say that an identity I *implies* another identity J and write $I \Rightarrow J$ if J holds in every quasigroup satisfying I – in other words, if the variety of quasigroups defined by I is contained in the variety of quasigroups defined by J. We say that I and J are *equivalent* if $I \Rightarrow J$ and $J \Rightarrow I$, or equivalently if I and J define the same variety of quasigroups.

Let G be a quasigroup with principal operation $*$. We refer to the operations in $\mathcal{O} = \{*, \backslash, /\circ, \backslash\backslash, //\}$ as *conjugates* of the principal operation $*$. If $\Box \in \mathcal{O}$ is any operation, we may consider the quasigroup $(G, \Box)$ whose underlying set is G and whose principal operation $*^\Box$ is defined by $a *^\Box b = a\Box b$. We call these quasigroups *conjugates* of the original quasigroup $(G, *)$. There is a natural action of the symmetric group S_3 on $\mathcal{O}$, summarized in Table 1; this extends to an action of S_3 on the conjugates of $(G, *)$ by setting $\sigma \cdot (G, \Box) = (G, \sigma \cdot \Box)$. The table also tells one how to interpret each of the conjugate operations in the various conjugate quasigroups. In particular, given $\sigma \in S_3$, let $\Box$ be the operation in the first column and in the row corresponding to σ. The entries of this row identify each of the six operations $*^\Box$, $\backslash^\Box$, $/^\Box$, $\circ^\Box$, $\backslash\backslash^\Box$, and $//^\Box$ with a corresponding operation in $\mathcal{O}$. For example, if $\sigma = (13)$, we have $\sigma \cdot (G, \cdot) = (G, \backslash)$. The entry in the third row and third column of the table tells us $/^\backslash = \backslash\backslash$; that is, for any $a, b \in G$, $a/^\backslash b = a\backslash\backslash b$.

	$*$	$\backslash$	$/$	$\circ$	$\backslash\backslash$	$//$
1	$*$	$\backslash$	$/$	$\circ$	$\backslash\backslash$	$//$
(1 2)	$\circ$	$\backslash\backslash$	$//$	$*$	$\backslash$	$/$
(1 3)	$\backslash$	$*$	$\backslash\backslash$	$//$	$/$	$\circ$
(2 3)	$/$	$//$	$*$	$\backslash\backslash$	$\circ$	$\backslash$
(1 2 3)	$//$	$/$	$\circ$	$\backslash$	$*$	$\backslash\backslash$
(1 3 2)	$\backslash\backslash$	$\circ$	$\backslash$	$/$	$//$	$*$

Table 1. Action of S_3 on $\mathcal{O}$

Conjugacy is particularly important in that it allows us to reduce further the number of implications among Bol-Moufang identities we need to consider. Extending the action of S_3 on $\mathcal{O}$ to an action on the set of all Bol-Moufang identities involving a single operation, we have the following:

Lemma 2.1. *Let I be an identity involving (only) one operation and J an identity involving a single (potentially different) operation. Then*

$$(I \Rightarrow J) \iff (\sigma \cdot I \Rightarrow \sigma \cdot J) \text{ for any } \sigma \in S_3.$$

Proof. Suppose $I \Rightarrow J$. If $\sigma \cdot I$ holds in some quasigroup $(G, *)$, then I holds in $\sigma^{-1}(G, *)$. Thus, J holds in $\sigma^{-1}(G, *)$, so $\sigma \cdot J$ holds in $(G, *)$. The proof of the reverse implication is similar. □

Corollary 2.2. *Any implication among identities of Bol-Moufang type is equivalent to one of the form $SUvw \Rightarrow LXab$.*

Proof. By Lemma 2.1, any implication whose premise $LUvw$ is equivalent, by application of the permutation $\sigma = (1\ 3)$, to an implication with premise $SUvw$. Similarly, any implication whose premise is $RUvw$ is equivalent, by application of (2 3), to an implication with premise $SUvw$. Now all implications of the form $SUvw \Rightarrow SXab$ have been determined by Phillips and Vojtěchovský [5], so it remains only to consider implications of the form $SUvw \Rightarrow LXab$ or $SUvw \Rightarrow RXab$. However, by applying (1 2), we see that the latter is equivalent to $S(Uvw)^{\vee} \Rightarrow L(Xab)^{\vee}$. □

A convenient summary of rules for converting implications is given in Table 2.

Before	After
$LUvw \Rightarrow SXab$	$SUvw \Rightarrow LXab$
$LUvw \Rightarrow LXab$	$SUvw \Rightarrow SXab$
$LUvw \Rightarrow RXab$	$SUvw \Rightarrow R(Xab)^{\vee}$
$RUvw \Rightarrow SXab$	$SUvw \Rightarrow RXab$
$RUvw \Rightarrow RXab$	$SUvw \Rightarrow SXab$
$RUvw \Rightarrow LXab$	$SUvw \Rightarrow L(Xab)^{\vee}$
$SUvw \Rightarrow RXab$	$S(Uvw)^{\vee} \Rightarrow L(Xab)^{\vee}$

Table 2. Conversion of implications

3. The main result

In this section we classify all valid implications among identities of Bol-Moufang type. By Corollary 2.2, we may restrict attention to implications of the form $SUvw \Rightarrow LXab$.

We will make heavy use of the Hasse diagram in Figure 1 which summarizes the results of [5]. Each node corresponds to a distinct variety of quasigroups defined by a single Bol-Moufang identity involving (only) the operation $*$. Inside the node is the abbreviated name of the variety, together with one identity which defines it. The full name of the variety corresponding to each abbreviation, together with the complete statement of the defining identity and what type of neutral element (2-sided, left, right, or none) exists, may be found in Table 5. The Hasse diagram is to be interpreted as follows: if there is a path from some variety to another variety on a lower level, then the upper variety is contained in the lower variety; that is, the identity defining the upper variety implies the one defining the lower variety. Note that by Proposition 2.1, there is a corresponding Hasse diagram for each of the other operations $\backslash$ and $/$.

For convenience, we say that an implication $SUvw \Rightarrow LXab$ is *irreducible* if whenever Vxy is an identity such that $SUvw \Rightarrow SVxy \Rightarrow LXab$, we must have $SUvw \Leftrightarrow SVxy$, and whenever Vxy is an identity such that $SUvw \Rightarrow LVxy \Rightarrow LXab$, we must have $LVxy \Leftrightarrow LXab$. It is clear that all valid implications may be constructed from a list of valid irreducible implications and the relevant Hasse diagram.

Theorem 3.1. *The only valid irreducible implications of the form* $SUvw \Rightarrow LXab$ *are* $SA25 \Rightarrow LB25$, $SB15 \Rightarrow LA35$, *and* $SC24 \Rightarrow LA35$.

Proof. We begin by arguing that all the implications described above are valid. Note first that in a loop both sides of the identity $LA35$: $(x\backslash x)\backslash(y\backslash z) = ((x\backslash x)\backslash y)\backslash z$ are equal to $y\backslash z$. Since $SB15$ and $SC24$ define varieties of loops, each of these implies $LA35$. From Table 2, $SA25 \Rightarrow LB25$ is equivalent to $SF14 \Rightarrow RE14$. The proof of the latter is rather lengthy and is deferred to Section 4.

We now show that no other irreducible implications hold. We begin by giving examples showing that the maximal identity $SA12$ in the Hasse diagram does not imply any minimal identity $LUvw$ when Uvw is equivalent to neither $A35$ nor $B25$. Observe that a quasigroup satisfying $SA12$ is necessarily a group. If $G = \mathbb{Z}_3 = \{e, a, b\}$ is a cyclic group of order 3 in which e denotes the neutral element and some identity $LUvw$ holds in G, then both sides of $LUvw$ must be equal when the element a is substituted for each of the variables x, y, and z. Now if $v = 1$, the left hand side of $LUvw$ is $a\backslash(a\backslash(a\backslash a)) = a\backslash(a\backslash e) = a\backslash b = a$. Similar computations show that if $v = 2, 3$, or 5 we obtain e and if $v = 4$ we obtain b. All this implies that the only identities $LUvw$ which could possibly hold in G are of form $LU23$, $LU25$ or $LU35$. Referencing Figure **??**, we are reduced to showing $SA12 \not\Rightarrow LUvw$ where $Uvw \in \{A23, E25, F25\}$. In fact, none of these three identities holds in S_3, the symmetric group on three letters: to show that $LA23$ does not hold, we take $x = z = (1\ 2)$, $y = (1\ 2\ 3)$, and to show that $LE25$ and $LF25$ do not hold we take $x = y = (1\ 2)$, $z = (1\ 2\ 3)$.

To show that $SB23$ does not imply $LB25$, we consider a nonassociative extra loop (i.e., a loop satisfying $SB23$) defined by Goodaire et. al. in [2]. We describe here a construction of this loop due to Chein [1]: given a group G, define $M(G, 2) = G\times\{0, 1\}$, where $(g, 0)(h, 0) = (gh, 0)$, $(g, 0)(h, 1) = (hg, 1)$, $(g, 1)(h, 0) = (gh^{-1}, 1)$

and $(g,1)(h,1) = (h^{-1}g, 0)$. For our counterexample, we consider $M(D_4, 2)$, where D_4 is the dihedral group of order 8 defined by generators R and F satsifying $R^4 = F^2 = 1$ and $RF = FR^{-1}$. Now define elements of $M(D_4, 2)$ by $x = (R, 1)$, $y = (R, 0)$ and $z = (F, 1)$; direct computation then shows that $LB25$ does not hold. The counterexamples associated to each of the remaining (potential) implications are described in Table 3. The entries in every third column correspond to quasigroups whose multiplication tables are catalogued in Section ; in each case below the counterexample is obtained by taking $x = y = z = 0$. □

Uvw	Xab	No.	Uvw	Xab	No.	Uvw	Xab	No.	Uvw	Xab	No.
A13	A35	3	F13	A35	1	A35	A35	10	A23	B25	6
A15	A35	5	F14	A35	1	B45	A35	2	B25	B25	9
A23	A35	6	F15	A35	8	C15	A35	2	F14	B25	1
A25	A35	7	F34	A35	1	C45	A35	4	F34	B25	1

Table 3. Table of counterexamples

By converting the implications of Theorem 3.1 using Table 2, one obtains a complete list of valid irreducible implications. The results are summarized below in Table 4; each box consists of logically equivalent implications.

$SA25 \Rightarrow LB25$	$LA25 \Rightarrow SB25$	$RA25 \Rightarrow LE14$
$SF14 \Rightarrow RE14$	$LF14 \Rightarrow RB25$	$RF14 \Rightarrow SE14$
$SB15 \Rightarrow LA35$	$LB15 \Rightarrow SA35$	$RB15 \Rightarrow LF13$
$SB15 \Rightarrow RF13$	$LB15 \Rightarrow RA35$	$RB15 \Rightarrow SF13$
$SC24 \Rightarrow LA35$	$LC24 \Rightarrow SA35$	$RC24 \Rightarrow LF13$
$SC24 \Rightarrow RF13$	$LC24 \Rightarrow RA35$	$RC24 \Rightarrow SF13$

Table 4. Valid irreducible implications

4. Proof of $SF14 \Rightarrow RE14$

In this section we give a proof that $SF14$ implies $RE14$, based on output from `Prover9`. Since $SF14$ has been shown to be equivalent to $SD14$ [5], we prove instead $SD14 \Rightarrow RE14$, as the output from `Prover9` is easier to parse. Although the proof is not particularly intuitive, it is short enough to be written out, and doing so ensures that all proofs in this article are "human" proofs.

For convenience, we write xy in place of $x * y$ and use juxtaposition notation to save parentheses. The notation $a \mapsto b$ (where a and b are formal expressions involving quasigroup elements and operations) means "substitute b for a".

We begin with the identity $SD14$:

$$(x \cdot yz)x = x(y \cdot zx).$$

This readily implies

$$(x \cdot yz)\backslash(x(y \cdot zx)) = x \tag{1}$$

and

$$[x(y \cdot zx)]/x = x \cdot yz. \tag{2}$$

On the other hand, substituting $y \mapsto y/z$ in SD14 gives

$$xy \cdot x = x(y/z \cdot zx). \tag{3}$$

By replacing $y \mapsto y/(zx)$ in (2), we have $(xy)/x = x[y/(zx) \cdot z]$. Substituting $y \mapsto x$ and $z \mapsto y$, we obtain

$$x = x \cdot (x/(yx))y \tag{4}$$

and dividing by x on the left yields

$$x\backslash x = (x/yx)y. \tag{5}$$

Returning to (1) and replacing $z \mapsto z/x$ we have $x = [x \cdot y(z/x)]\backslash[x \cdot y(z/x \cdot x)]$, which simplifies to

$$x = [x \cdot y(z/x)]\backslash[x \cdot yz]. \tag{6}$$

Replacing $y \mapsto x\backslash y$ in (3), we have

$$yx = x[(x\backslash y)/z \cdot zx]. \tag{7}$$

Putting $x \mapsto y/zy \cdot z$, $y \mapsto x$, and $z \mapsto y$ in (7), we have

$$x(y/zy \cdot z) = (y/zy \cdot z)[((y/zy \cdot z)\backslash x)/y \cdot y(y/zy \cdot z)]$$

which by (4) simplifies to $(y/zy \cdot z)[((y/zy \cdot z)\backslash x)/y \cdot y]] = x$. Thus $x = x(y/zy \cdot z) = x(y\backslash y)$ by (5), which establishes the existence of a right neutral element.
Using this we argue

$$[x/(y/z \cdot x)]y = z\backslash[z \cdot [x/(y/z \cdot x)]y] = [z \cdot (x\backslash x)]\backslash[z \cdot [x/(y/z \cdot x)]y].$$

Now using (5), the above may be written as

$$[z \cdot [x/(y/z \cdot x)](y/z)]\backslash[z \cdot [x/(y/z \cdot x)]y]$$

which by (6) reduces to z. Summarizing, we have

$$[x/(y/z \cdot x)]y = z. \tag{8}$$

Dividing this equation on the right by y on the right yields

$$x/(y/z \cdot x) = z/y, \tag{9}$$

and if instead we substitute $y \mapsto yz$, we obtain

$$x/yx \cdot yz = z. \tag{10}$$

Returning to (3) and substituting $z \mapsto z/(xz)$, we have $xy \cdot x = x(y/(z/xz) \cdot (z/xz)x)$. By (5), the right hand side reduces to $x(y/(z/xz) \cdot z\backslash z) = x(y/(z/xz))$. Thus, we have

$$x(y/(z/xz)) = xy \cdot x. \tag{11}$$

Using (11), (2), and (10) we reason

$$(y/zy)(zx \cdot z) = (y/zy)(z(x/(y/zy)) = (y/zy \cdot zx)/(y/zy) = x/(y/zy).$$

Thus we have

$$x/(y/zy) = (y/zy)(zx \cdot z). \tag{12}$$

We are finally ready to prove $RE14$. Applying (9), we have $(x/(y/z))/y = (x/[x/((z/y)x)])/y$, which by (12) equals $[(x/((z/y)x)) \cdot ((z/y)x) \cdot (z/y)]/y$. Using (3) we may rewrite this as $[(x/((z/y)x)) \cdot ((z/y) \cdot (x/w)(w \cdot (z/y)))]/y$, where for convenience we write $w = y/(z/y)$. By (10), the above expression reduces to $[(x/w) \cdot (w \cdot (z/y))]/y = [x/(y/(z/y)) \cdot y]/y = x/(y/(z/y)$, which establishes $RE14$. □

5. Counterexamples

1.

*	0	1	2
0	1	0	2
1	2	1	0
2	0	2	1

2.

*	0	1	2
0	1	0	2
1	0	2	1
2	2	1	0

3.

*	0	1	2	3	4	5	6	7	8
0	1	2	4	0	6	3	8	5	7
1	2	4	6	1	8	0	7	3	5
2	0	1	2	3	4	5	6	7	8
3	7	5	3	8	0	6	1	4	2
4	6	8	7	4	5	2	3	1	0
5	3	0	1	5	2	7	4	8	6
6	8	7	5	6	3	4	0	2	1
7	5	3	0	7	1	8	2	6	4
8	4	6	8	2	7	1	5	0	3

4.

*	0	1	2	3	4	5
0	1	2	4	0	5	3
1	2	0	5	1	3	4
2	0	1	3	2	4	5
3	4	5	2	3	0	1
4	5	3	0	4	1	2
5	3	4	1	5	2	0

5.

*	0	1	2	3	4
0	1	4	3	0	2
1	3	0	4	2	1
2	0	1	2	3	4
3	2	3	1	4	0
4	4	2	0	1	3

6.

*	0	1	2	3
0	1	0	3	2
1	2	3	0	1
2	0	1	2	3
3	3	2	1	0

7.

*	0	1	2	3	4	5
0	1	0	4	5	2	3
1	3	2	5	4	0	1
2	0	1	2	3	4	5
3	5	4	3	2	1	0
4	2	3	0	1	5	4
5	4	5	1	0	3	2

8.

*	0	1	2	3	4
0	1	2	4	3	0
1	3	0	2	4	1
2	0	4	3	1	2
3	4	1	0	2	3
4	2	3	1	0	4

9.

*	0	1	2	3	4	5
0	1	3	0	5	2	4
1	0	1	2	3	4	5
2	4	2	5	0	3	1
3	5	4	3	2	1	0
4	2	0	4	1	5	3
5	3	5	1	4	0	2

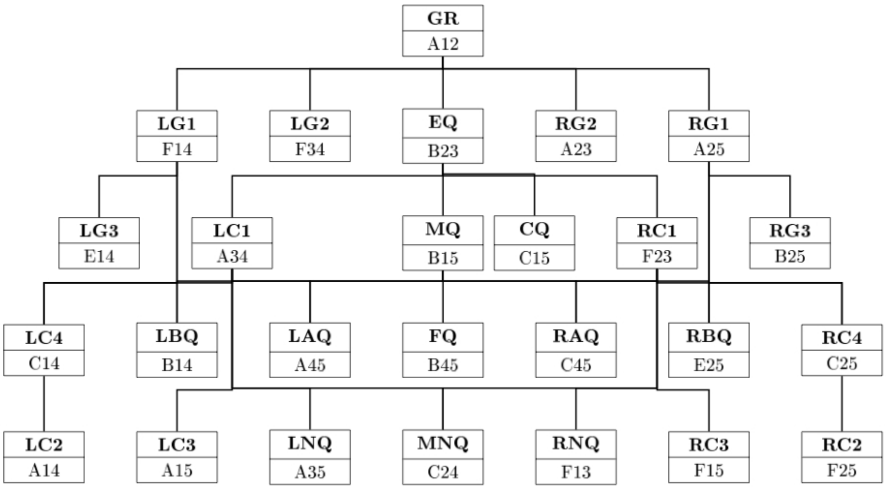
10.

*	0	1	2	3	4
0	1	3	0	4	2
1	0	1	2	3	4
2	4	0	1	2	3
3	2	4	3	1	0
4	3	2	4	0	1

Variety	Abbrev.	Defining identity	Name	Neutral elt.
Groups	GR	$x(yz) = (xy)z$	$A12$	2
RG1-quasigroups	RG1	$x((xy)z) = ((xx)y)z$	$A25$	L
LG1-quasigroups	LG1	$x(y(zz)) = (x(yz))z$	$F14$	R
RG2-quasigroups	RG2	$x(x(yz)) = (xx)(yz)$	$A23$	L
LG2-quasigroups	LG2	$(xy)(zz) = (x(yz))z$	$F34$	R
RG3-quasigroups	RG3	$x((yx)z) = ((xy)x)z$	$B25$	L
LG3-quasigroups	LG3	$x(y(zy)) = (x(yz))y$	$E14$	R
Extra q.	EQ	$x((yx)z) = (xy)(xz)$	$B23$	2
Moufang q.	MQ	$x(y(xz)) = ((xy)x)z$	$B15$	2
Left Bol q.	LBQ	$x(y(xz)) = (x(yx))z$	$B14$	R
Right Bol q.	RBQ	$x((yz)y) = ((xy)z)y$	$E25$	L
C-quasigroups	CQ	$x(y(yz)) = ((xy)y)z$	$C15$	0
LC1-quasigroups	LC1	$(xx)(yz) = (x(xy))z$	$A34$	2
LC2-quasigroups	LC2	$x(x(yz)) = (x(xy))z$	$A14$	0
LC3-quasigroups	LC3	$x(x(yz)) = ((xx)y)z$	$A15$	L
LC4-quasigroups	LC4	$x(y(yz)) = (x(yy))z$	$C14$	R
RC1-quasigroups	RC1	$x((yz)z) = (xy)(zz)$	$F23$	2
RC2-quasigroups	RC2	$x((yz)z) = ((xy)z)z$	$F25$	0
RC3-quasigroups	RC3	$x(y(zz)) = ((xy)z)z$	$F15$	R
RC4-quasigroups	RC4	$x((yy)z) = ((xy)y)z$	$C25$	L
Left alternative q.	LAQ	$x(xy) = (xx)y$	$A45$	L
Right alternative q.	RAQ	$x(yy) = (xy)y$	$C45$	R
Flexible q.	FQ	$x(yx) = (xy)x$	$B45$	0
Left nuclear q.	LNQ	$(xx)(yz) = ((xx)y)z$	$A35$	L
Middle nuclear q.	MNQ	$x((yy)z) = (x(yy))z$	$C24$	2
Right nuclear q.	RNQ	$x(y(zz)) = (xy)(zz)$	$F13$	R

Table 5. Definitions of varieties of quasigroups

Figure 1. Varieties of Bol-Moufang type under $*$



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References

[1] **O. Chein**, *Moufang loops of small order*, Mem. AMS **13** (1978), no. 197.

[2] **E. G. Goodaire, S. May, M. Raman**, *The Moufang loops of order less than 64.* Nova Science Publishers, 1999.

[3] **W. McCune**, *Prover9, equational reasoning tool and Mace4, finite model builder.* Available at `http://www.cs.unm.edu/~mccune/mace4/`.

[4] **J. D. Phillips and P. Vojtěchovský**, *The varieties of loops of Bol-Moufang type.* Algebra Universalis **54** (2005), no. 3, 259 – 271.

[5] **J. D. Phillips and P. Vojtěchovský**, *The varieties of quasigroups of Bol-Moufang type: An equational reasoning approach.* J. Algebra **293** (2005), 17 – 33.

[6] **J. D. H. Smith**, *An Introduction to quasigroups and their representations*, Studies in Advanced Mathematics, Chapman and Hall, 2007.

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