A note on M-hypersystems and N-hypersystems in Γ-semihypergroups

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Abstract. In this paper, we have introduced the notions of M-hypersystem and N-hypersystem in Γ-semihypergroups, and some related properties are investigated. We have also proved that left Γ-hyperideal $P$ of a Γ-semihypergroup $S$ is quasi-prime if and only if $S\setminus P$ is an M-hypersystem.

1. Introduction

In 1986, Sen and Saha [3] defined the notion of a $Γ$-semigroup as a generalization of a semigroup. Recently, Davvaz, Hila and et. al. [1, 2] introduced the notion of $Γ$-semihypergroup as a generalization of a semigroup, a generalization of a semihypergroup and a generalization of a $Γ$-semigroup. The notion of a $Γ$-hyperideal of a $Γ$-semihypergroup was introduced in [1].

Let $S$ and $Γ$ be two non-empty sets. Then $S$ is called a $Γ$-semihypergroup if every $γ \in Γ$ is a hyperoperation on $S$, i.e., $xγy \subseteq S$ for every $x, y \in S$, and for every $α, β \in Γ$ and $x, y, z \in S$ we have $xα(yβz) = (xαy)βz$. Let $S$ be a $Γ$-semihypergroup and $γ \in Γ$. A non-empty subset $A$ of $S$ is called a sub-$Γ$-semihypergroup of $S$ if $xγy \subseteq A$ for every $x, y \in A$. A $Γ$-semihypergroup $S$ is called commutative if for all $x, y \in S$ and $γ \in Γ$, we have $xγy = yγx$.

Example 1.1. Let $S = [0, 1]$ and $Γ = \mathbb{N}$. For every $x, y \in S$ and $γ \in Γ$, we define $γ : S^2 \rightarrow P^\ast(S)$ by $xγy = \left[0, \frac{xy}{γ}\right]$. Then $γ$ is a hyperoperation.

For every $x, y, z \in S$ and $α, β \in Γ$, we have $(xαy)βz = \left[0, \frac{xyz}{αβ}\right] = xα(yβz)$. Thus $S$ is a $Γ$-semihypergroup.

Example 1.2. Let $(S, ◦)$ be a semihypergroup and $Γ$ be a non-empty subset of $S$. We define $xγy = x ◦ y$ for every $x, y \in S$ and $γ \in Γ$. Thus $S$ is a $Γ$-semihypergroup.

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Example 1.3. Let $S = (0, 1)$, $\Gamma = \{\gamma_n \mid n \in \mathbb{N}\}$ and for every $n \in \mathbb{N}$ we define hyperoperation $\gamma_n$ on $S$ as follows

$$x\gamma_n y = \left\{ \frac{xy}{2^k} \mid 0 \leq k \leq n \right\}.$$ 

Then $x\gamma_n y \subset S$ and for every $m, n \in \mathbb{N}$ and $x, y, z \in S$

$$ (x\gamma_n y)\gamma_m z = \left\{ \frac{xyz}{2^k} \mid 0 \leq k \leq n + m \right\} = x\gamma_n (y\gamma_m z). $$

So, $S$ is a $\Gamma$-semihypergroup.

A $\Gamma$-semihypergroup $S$ is called regular if for all $a \in S$ and $\alpha, \beta \in \Gamma$ there exists $x \in S$ such that $a \in ax\beta a$.

A non-empty subset $A$ of $S$ is a left (right) $\Gamma$-hyperideal of $S$ if $A\Gamma S \subseteq A$ ($S\Gamma A \subseteq A$). A $\Gamma$-hyperideal is both a left and right $\Gamma$-hyperideal.

A left $\Gamma$-hyperideal $P$ is quasi-prime if for any left $\Gamma$-hyperideals $A$ and $B$ such that $A\Gamma B \subseteq P$ it follows $A \subseteq P$ or $B \subseteq P$.

A left $\Gamma$-hyperideal $P$ is quasi-semiprime if any left $\Gamma$-hyperideal $A$ from $A\Gamma A \subseteq P$ it follows $A \subseteq P$.

2. M-hypersystem and N-hypersystem

A $\Gamma$-semihypergroup $S$ is called fully $\Gamma$-hyperidempotent if every $\Gamma$-hyperideal is idempotent.

Proposition 2.1. If $S$ is $\Gamma$-semihypergroup and $A, B$ are $\Gamma$-hyperideal of $S$, then the following are equivalent:

(a) $S$ is fully $\Gamma$-hyperidempotent,
(b) $A \cap B = \langle A\Gamma B \rangle$,
(c) the set of all $\Gamma$-hyperideals of $S$ form a semilattice $(L_S, \land)$, where $A \land B = \langle A\Gamma B \rangle$.

Proof. (a) $\Rightarrow$ (b) Always hold $A\Gamma B \subseteq A \cap B$, for any $\Gamma$-hyperideals $A$ and $B$ of $S$. Hence $(A\Gamma B) \subseteq A \cap B$.

Converse let $x \in A \cap B$. If $\langle x \rangle$ denote the principle left $\Gamma$-hyperideal generated by $x$, then $x \in \langle x \rangle \Gamma \langle x \rangle \subseteq \langle A\Gamma B \rangle$. Thus $x \in \langle A\Gamma B \rangle$. Therefore $A \cap B \subseteq \langle A\Gamma B \rangle$, which proves (b).

(b) $\Rightarrow$ (c) $A \land B = \langle A\Gamma B \rangle = A \cap B = B \cap A = \langle B\Gamma A \rangle = B \land A$.

(c) $\Rightarrow$ (b) Let $(L_S, \land)$ be a semilattice. Then $A = A \land A = \langle A\Gamma A \rangle = A\Gamma A$. Hence $S$ is fully $\Gamma$-hyperidempotent.
Corollary 2.2. If $\Gamma$-semihypergroup $S$ is regular, then $S = STS$.  

A subset $M$ of $\Gamma$-semihypergroup $S$ is called an $M$-hypersystem if for all $a, b \in M$, there exist $x \in S$ and $\alpha, \beta \in \Gamma$ such that $a \alpha x \beta b \subseteq M$.

A subset $N$ of $\Gamma$-semihypergroup $S$ is called an $N$-hypersystem if for all $a \in N$, there exist $x \in S$ and $\alpha, \beta \in \Gamma$ such that $a \alpha x \beta a \subseteq N$.

Obviously, each $M$-hypersystem is an $N$-hypersystem.

Example 2.3. The set $S_i = (0, 2^{-i})$, where $i \in \mathbb{N}$, is an $M$-hypersystem of a $\Gamma$-semihypergroup $S$ defined in Example 1.3. The set $T_i = (0, 4^{-i})$, where $i \in \mathbb{N}$, is its an $N$-hypersystem of $S$.  

Example 2.4. The set $T = [0, t]$, where $t \in [0, 1]$, is an $M$-hypersystem and an $N$-hypersystem of a $\Gamma$-semihypergroup defined in Example 1.1.  

Theorem 2.5. Let $P$ be a left $\Gamma$-hyperideal of $\Gamma$-semihypergroup $S$. Then the following are equivalent:

1. $P$ is a quasi-prime,
2. $A \Gamma B = (A \Gamma B) \subseteq P \Rightarrow A \subseteq P$ or $B \subseteq P$ for all left $\Gamma$-hyperideals,
3. $A \notin P$ or $B \notin P \Rightarrow A \Gamma B \notin P$ for all left $\Gamma$-hyperideals,
4. $a \notin P$ or $b \notin P \Rightarrow a \Gamma b \notin P$ for all $a, b \in S$,
5. $a \Gamma b \subseteq P \Rightarrow a \in P$ or $b \in P$ for all $a, b \in S$.

Proof. (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) is straightforward.

(1) $\Rightarrow$ (4) Let $\langle a \rangle \Gamma \langle b \rangle \subseteq P$. Then by (1) either $\langle a \rangle \subseteq P$ or $\langle b \rangle \subseteq P$, which implies that either $a \in P$ or $b \in P$.

(4) $\Rightarrow$ (2) Let $A \Gamma B \subseteq P$. If $a \in A$ and $b \in B$, then $\langle a \rangle \Gamma \langle b \rangle \subseteq P$, now by (4) either $a \in P$ or $b \in P$, which implies that either $A \subseteq P$ or $B \subseteq P$.

(1) $\Rightarrow$ (5) Let $P$ be a left $\Gamma$-hyperideal of $\Gamma$-semihypergroup $S$ and $a \Gamma S \Gamma b \subseteq P$. Then, by (2), (3) and (1), we get $STa \Gamma (a \Gamma S \Gamma b) \subseteq STP \subseteq P$, that is, $STa \Gamma (a \Gamma S \Gamma b) = (STa \Gamma a) \Gamma (STb)$. Thus, $(STa) \Gamma (STb) \subseteq P$ implies either $STa \Gamma a \subseteq P$ or $STb \subseteq P$.

Since $STa$ and $STb$ are left $\Gamma$-hyperideals, for $L(a) = (a \cup STa)$ we have

\[
L(a) \Gamma L(a) \Gamma L(a) = (a \cup STa) \Gamma (a \cup STa) \Gamma (a \cup STa) \\
\subseteq a \Gamma a \cup a \Gamma STa \cup STa \Gamma a \cup STa \Gamma STa \Gamma a \cup STa \\
\subseteq STa \subseteq P.
\]

Hence $L(a) \Gamma L(a) \Gamma L(a) = (L(a) \Gamma L(a)) \Gamma L(a) \subseteq P$. Since $P$ is quasi-prime and $L(a) \Gamma L(a)$ is a left $\Gamma$-hyperideal of $S$ we have $L(a) \Gamma L(a) \subseteq P$. 

□
or \( L(a) \subseteq P \). If \( L(a) \subseteq P \), then \( a \in L(a) \subseteq P \). Let \( L(a) \Gamma L(a) \subseteq P \). Since \( P \) is quasi-prime, \( L(a) \subseteq P \). Thus, \( a \in L(a) \subseteq P \), i.e., \( a \in P \).

(5) \( \Rightarrow \) (1) Assume that \( AB \subseteq P \), where \( A \) and \( B \) are left \( \Gamma \)-hyperideals of \( S \) such that \( A \not\subseteq P \). Then there exist \( x \in A \) such that \( x \notin P \). Hence \( x \Gamma y \subseteq AB \subseteq \Gamma B \subseteq P \) for all \( y \in B \). Then, by (5), \( y \in P \).

**Proposition 2.6.** A left \( \Gamma \)-hyperideal \( P \) of \( \Gamma \)-semihypergroup \( S \) is quasi-prime if and only if \( S \setminus P \) is an \( M \)-hypersystem.

**Proof.** Let \( S \setminus P \) be an \( M \)-hypersystem and \( a \Gamma ST \subseteq P \) for some \( a, b \in S \setminus P \). Then there exist \( x \in S \) and \( \alpha, \beta \in \Gamma \) such that \( aax \beta b \subseteq S \setminus P \). This implies that \( aax \beta b \not\subseteq P \), which is a contradiction. Hence either \( a \in P \) or \( b \in P \).

Conversely, if \( P \) is quasi-prime and \( x, y \in S \setminus P \), then for \( z \in S \) and \( \alpha, \beta \in \Gamma \) such that \( x \alpha z \beta y \not\subseteq P \) we have \( x \alpha z \beta y \subseteq P \), i.e., either \( x \in P \) or \( y \in P \). So, \( S \setminus P \) is an \( M \)-hypersystem.

**Proposition 2.7.** A left \( \Gamma \)-hyperideal \( P \) of \( \Gamma \)-semihypergroup \( S \) is quasi-semiprime if and only if \( S \setminus P \) is an \( N \)-hypersystem.

**Proof.** Let \( S \setminus P \) be an \( N \)-hypersystem and \( a \Gamma ST \subseteq P \) with \( a \notin P \). Then \( aax \beta b \subseteq S \setminus P \) for some \( x \in S \) and \( \alpha, \beta \in \Gamma \). Thus \( aax \beta a \not\subseteq P \), which is a contradiction. Hence \( a \in P \). The converse statement is obvious.

**Theorem 2.8.** Let \( S \) be \( \Gamma \)-semihypergroup and \( P \) a proper left \( \Gamma \)-hyperideal of \( S \). Then the following are equivalent:

1. \( P \) is quasi-prime,
2. \( a \Gamma M \Gamma b \subseteq P \) implies \( a \in P \) or \( b \in P \),
3. \( S \setminus P \) is an \( M \)-system,
4. \( S \setminus P \) is an \( N \)-system.

\( \square \)

**References**

