Quasigroups and Related Systems **19** (2011), 359 − 368

A study of anti-fuzzy quasi-ideals in ordered semigroups

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Abstract. In this paper, we introduce the concept of anti-fuzzy quasi-ideals in ordered semigroups and investigate the quasi-ideals of ordered semigroups in terms of anti-fuzzy quasi-ideals. We characterize left (resp. right) regular and completely regular ordered semigroups in terms of anti-fuzzy quasi-ideals and semiprime anti-fuzzy quasi-ideals.

1. Introduction

Biswas introduced the concept of an anti-fuzzy subgroup of a group in [3] and studied the basic properties of groups in terms of anti-fuzzy subgroups. Hong and Jun [5] modified Biswas idea and applied it to BCK-algebras. Akram and Dar defined anti-fuzzy left h-ideals of hemirings [2]. Recently Shabir and Nawaz studied anti fuzzy ideals of semigroups [11]. Ahsan et. al in [1] characterize semigroups in terms of fuzzy quasi-ideals. The monograph given by Mordeson and Malik [10] deals with the applications of fuzzy approach to the concepts of automata and formal languages. Fuzzy sets in ordered semigroups were first introduced by Kehayopulu and Tsingelis in [8].

In this paper, we introduce the concept of anti-fuzzy quasi-ideals in ordered semigroups and investigate the basic properties of quasi-ideals of ordered semigroups in terms of anti-fuzzy quasi-ideals. We characterize left (resp. right) regular and completely regular ordered semigroups in terms of anti-fuzzy quasi-ideals. We define semiprime anti-fuzzy quasi-ideals and characterize completely regular ordered semigroups in terms of semiprime anti-fuzzy quasi-ideals.

2010 MSC: 06F05, 06D72, 08A72
Keywords: Fuzzy subset, anti-fuzzy quasi-ideal, simple semigroup, completely regular ordered semigroup.

2. Some basic definitions and results

By an *ordered semigroup* (*po-semigroup*) we mean a *structure* $(S, \cdot, \leqslant)$ in which

$(OS1)$ $(S, \cdot)$ is a *semigroup*,

$(OS2)$ $(S, \leqslant)$ is a *poset*,

$(OS3)$ $(\forall a, b, x \in S)(a \leqslant b \Longrightarrow ax \leqslant bx$ and $xa \leqslant xb)$.

Throughout this paper S will denote an ordered semigroup unless otherwise specified.

For $A, B \subseteq S$, we denote $(A] := \{t \in S \mid t \leqslant h$ for some $h \in A\}$ and $AB := \{ab \mid a \in A, b \in B\}$. Then $A \subseteq (A]$, $(A](B] \subseteq (AB]$, $((A]] = (A]$ and $((A](B]] \subseteq (AB]$.

A non-empty subset A of S is called a *right* (resp. *left*) *ideal* of S if:

(1) $AS \subseteq A$ (resp. $SA \subseteq A$),

(2) $a \in A$ and $S \ni b \leqslant a$ imply $b \in A$.

If A is both a right and a left ideal of S, then it is called an *ideal* of S.

A non-empty subset Q of S is called a *quasi-ideal* of S if:

(1) $(QS] \cap (SQ] \subseteq Q$,

(2) $a \in Q$ and $S \ni b \leqslant a$ imply $b \in Q$.

A subsemigroup B of S is called a *bi-ideal* of S if:

(1) $BSB \subseteq B$,

(2) $a \in B$ and $S \ni b \leqslant a$ imply $b \in B$.

A fuzzy subset f of S is called a *fuzzy left* (resp. *right*) *ideal* of S if:

(1) $x \leqslant y \Longrightarrow f(x) \geqslant f(y)$,

(2) $f(xy) \geqslant f(y)$ (resp. $f(xy) \geqslant f(x)$) for all $x, y \in S$.

If f is both a fuzzy left and a fuzzy right ideal of S. Then it is called a *fuzzy ideal* of S.

A fuzzy subset f of S is called a *fuzzy subsemigroup* of S if for all $x, y \in S$ $f(xy) \geqslant \min\{f(x), f(y)\}$. A fuzzy subsemigroup f of S is called a *fuzzy bi-ideal* of S if:

(1) $x \leqslant y \Longrightarrow f(x) \geqslant f(y)$,

(2) $f(xyz) \geqslant \min\{f(x), f(z)\}$ for all $x, y \in S$.

For a non-empty *family* of fuzzy subsets $\{f_i\}_{i \in I}$ of S, the fuzzy subsets $\bigwedge_{i \in I} f_i$ and $\bigvee_{i \in I} f_i$ of S are defined as follows:

$$\Big(\bigwedge_{i \in I} f_i\Big)(x) := \inf_{i \in I}\{f_i(x)\}, \qquad \Big(\bigvee_{i \in I} f_i\Big)(x) := \sup_{i \in I}\{f_i(x)\}.$$

For any two fuzzy subsets f and g of S we put

$$(f \circ g)(x) := \begin{cases} \bigvee\limits_{(y,z)\in A_x} \max\{f(y), g(z)\} & \text{if } A_x \neq \emptyset, \\ 0 & \text{if } A_x = \emptyset, \end{cases}$$

where $A_x := \{(y,z) \in S \times S \mid x \leqslant yz\}$.

A fuzzy subset f of S is called a *fuzzy quasi-ideal* of S if:

(1) $x \leqslant y \Longrightarrow f(x) \geqslant f(y)$,

(2) $(f \circ 1) \wedge (1 \circ f) \preceq f$,

where $f \preceq g$ means that $f(x) \leqslant g(x)$ for all $x \in S$.

A fuzzy subset f of S is called an *anti-fuzzy subsemigroup* of S if

$$f(xy) \leqslant \max\{f(x), f(y)\}$$

for all $x, y \in S$.

An anti-fuzzy subsemigroup f of S is called an *anti-fuzzy bi-ideal* of S if:

(1) $x \leqslant y$ implies $f(x) \leqslant f(y)$,

(2) $f(xay) \leqslant \max\{f(x), f(y)\}$

for all $x, a, y \in S$.

For fuzzy subsets f and g of S the product $f * g$ is defined as follows:

$$(f * g)(a) = \begin{cases} \bigwedge\limits_{(y,z)\in A_x} \max\{f(y), g(z)\} & \text{if } A_x \neq \emptyset \\ 1 & \text{if } A_x = \emptyset \end{cases}$$

The fuzzy subsets "$\mathcal{S}$" and "$\mathcal{O}$" of S are defined as

$$\mathcal{S}(x) = 1, \qquad \mathcal{O}(x) = 0$$

for all $x \in S$.

Proposition 2.1. *Let $A, B \subseteq S$. Then*

(*i*) $A \subseteq B$ *if and only if* $f_{B^c} \preceq f_{A^c}$.

(*ii*) $f_{A^c} \vee f_{B^c} = f_{A^c \cup B^c} = f_{(A\cap B)^c}$.

(*iii*) $f_{A^c} * f_{B^c} = f_{(AB]^c}$. □

An ordered semigroup S is called *regular* (see [6]) if for every $a \in S$ there exists $x \in S$ such that $a \leqslant axa$ or equivalently, (1)$(\forall a \in S)(a \in (aSa])$ and (2)$(\forall A \subseteq S)(A \subseteq (ASA])$, and S is called *left* (resp. *right*) *simple* (see [7]) if it has no proper left (resp. right) ideals.

Lemma 2.2. (cf. [7]). *S is left (resp. right) simple if and only if $(Sa] = S$ (resp. $(aS] = S$) for every $a \in S$.* □

An ordered semigroup S is called *left* (resp. *right*) *regular* (see [7]) if for every $a \in S$, there exists $x \in S$ such that $a \leqslant xa^2$ (resp. $a \leqslant a^2x$) or equivalently, (1)$(\forall a \in S)(a \in (Sa^2])$ and (2)$(\forall A \subseteq S)(A \subseteq (SA^2])$. S is called *completely regular* if it is regular, left regular and right regular [7].

If $\emptyset \neq A \subseteq S$, then the set $(A \cup (AS \cap SA)]$ is the quasi-ideal of S generated by A.

Lemma 2.3. (cf. [6]). *S is completely regular if and only if $A \subseteq (A^2SA^2]$ for every $A \subseteq S$. Equivalently, if $a \in (a^2Sa^2]$ for every $a \in S$.* □

3. Anti-fuzzy quasi-ideals

Definition 3.1. A fuzzy subset f of S is called an *anti-fuzzy quasi-ideal*if

(1) $(f * \mathcal{O}) \vee (\mathcal{O} * f) \succeq f$,

(2) $x \leqslant y$ implies $f(x) \leqslant f(y)$ for all $x, y \in S$.

As a consequence of the transfer principle for fuzzy sets (cf. [9] we obtain the following two theorems.

Theorem 3.2. *Let $\emptyset \neq A \subseteq S$. Then A is a quasi-ideal of S if and only if the characteristic function f_{A^c} of the complement of A is an anti-fuzzy quasi-ideal of S.*

Theorem 3.3. *Let f be a fuzzy subset of S. Then each non-empty level $L(f;t)$ is a quasi-ideal if and only if f is an anti-fuzzy quasi-ideal.*

Example 3.4. The set $S = \{a, b, c, d, f\}$ with the multiplication

$\cdot$	a	b	c	d	f
a	a	a	a	a	a
b	a	b	a	d	a
c	a	f	c	c	f
d	a	b	d	d	b
f	a	f	a	c	a

and the order $\leqslant := \{(a,a), (a,b), (a,c), (a,d), (a,f), (b,b), (c,c), (d,d), (f,f)\}$ is an ordered semigroup with the following quasi-ideals:

$\{a\}, \{a,b\}, \{a,c\}, \{a,d\}, \{a,f\}, \{a,b,d\}, \{a,c,d\}, \{a,b,f\}, \{a,c,f\}, S$.

For a fuzzy set f defined by $f(a) = 0.3$, $f(b) = 0.5$, $f(c) = f(f) = 0.8$, $f(d) = 0.6$ we have

$$L(f;t) := \begin{cases} S & \text{if} \quad t \in [0.8, 1), \\ \{a,b,d\} & \text{if} \quad t \in [0.6, 0.8), \\ \{a,b\} & \text{if} \quad t \in [0.5, 0.6), \\ \{a\} & \text{if} \quad t \in [0.3, 0.5), \\ \emptyset & \text{if} \quad t \in [0, 0.3). \end{cases}$$

$L(f;t)$ is a quasi-ideal. By Theorem 3.3, f is an anti-fuzzy quasi-ideal. □

Lemma 3.5. *Every anti-fuzzy quasi-ideal of S is its anti-fuzzy bi-ideal.*

Proof. Let $x, y, z \in S$. Then $xyz = x(yz) = (xy)z$. Hence $(x, yz) \in A_{xyz}$ and $(xy, z) \in A_{xyz}$. Since $A_{xyz} \neq \emptyset$, we have

$$\begin{aligned} f(xyz) &\leqslant [(f * \mathcal{O}) \vee (\mathcal{O} * f)]\,(xyz) \\ &= \max\left[\bigwedge_{(p,q)\in A_{xyz}} \max\{f(p), \mathcal{O}(q)\}, \bigwedge_{(p_1,q_1)\in A_{xyz}} \max\{\mathcal{O}(p_1), f(q_1)\}\right] \\ &\leqslant \max[\max\{f(x), \mathcal{O}(yz)\}, \max\{\mathcal{O}(xy), f(z)\}] \\ &= \max[\max\{f(x), 0\}, \max\{0, f(z)\}] = \max[f(x), f(z)]. \end{aligned}$$

Let $x, y \in S$, then $xy = x(y)$ and hence $(x, y) \in A_{xy}$. Since $A_{xy} \neq \emptyset$, we have

$$\begin{aligned} f(xy) &\leqslant [(f * \mathcal{O}) \vee (\mathcal{O} * f)](xy) \\ &= \max\left[\bigwedge_{(p,q)\in A_{xy}} \max\{f(p), \mathcal{O}(q)\}, \bigwedge_{(p,q)\in A_{xy}} \max\{\mathcal{O}(p), f(q)\}\right] \\ &\leqslant \max[\max\{f(x), \mathcal{O}(y)\}, \max\{\mathcal{O}(x), f(y)\}] \\ &= \max[\max\{f(x), 0\}, \max\{0, f(y)\}] = \max[f(x), f(y)]. \end{aligned}$$

Let $x, y \in S$ be such that $x \leqslant y$. Then $f(x) \leqslant f(y)$, because f is an anti-fuzzy quasi-ideal of S. Thus f is an anti-fuzzy bi-ideal of S. □

The converse of above Lemma is not true, in general.

Example 3.6. The set $S = \{a, b, c, d\}$ with the multiplication table

$\cdot$	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	b	a
d	a	a	b	b

and the order $\leqslant := \{(a,a), (b,b), (c,c), (d,d), (a,b)\}$ is an ordered semigroup.

$\{a, d\}$ is its bi-ideal but not a quasi-ideal.

For a fuzzy set $f(a) = f(d) = 0.7$, $f(b) = f(c) = 0.3$ we have

$$L(f;t) := \begin{cases} S & \text{if} \quad t \in [0.7, 1), \\ \{a, d\} & \text{if} \quad t \in [0.3, 0.7), \\ \emptyset & \text{if} \quad t \in [0, 0.3). \end{cases}$$

$L(f;t)$ is a bi-ideal for every t, but for $t \in [0.3, 0.7)$ it is not a quasi-ideal of S. By Theorem 3.3, f is an anti-fuzzy bi-ideal of S but not an anti-fuzzy quasi-ideal of S. □

4. Completely regular ordered semigroups

Theorem 4.1. *The the following are equivalent:*

(*i*) *S is regular, left and right simple,*

(*ii*) *every anti-fuzzy quasi-ideal of S is a constant function.*

Proof. (*i*) $\Longrightarrow$ (*ii*). Let S be a fixed regular, left and right simple ordered semigroup. Let f be an anti-fuzzy quasi-ideal of S. We consider the set $E_\Omega = \{e \in S \mid e^2 \geqslant e\}$. E_Ω is non-empty, because for $a \in S$ there exists $x \in S$ such that $a \leqslant axa$, hence $(ax)^2 = (axa)x \geqslant ax$, which means that $ax \in E_\Omega$.

(A) We first prove that f is a constant function on E_Ω. That is, $f(e) = f(t)$ for every $t \in E_\Omega$. In fact: since S is left and right simple, we have $(St] = S$ and $(tS] = S$. But $e \in S$. Then $e \in (St]$ and $e \in (tS]$. Thus $e \leqslant xt$ and $e \leqslant ty$ for some $x, y \in S$. If $e \leqslant xt$ then $e^2 = ee \leqslant (xt)(xt) = (xtx)t$ and $(xtx, t) \in A_{e^2}$. If $e \leqslant ty$ then $e^2 = ee \leqslant (ty)(ty) = t(yty)$ and $(t, yty) \in A_{e^2}$.

Since $A_{e^2} \neq \emptyset$ we have

$$\begin{aligned} f(e^2) &\leqslant ((f * \mathcal{O}) \vee (\mathcal{O} * f))(e^2) = \max[(f * \mathcal{O})(e^2), (\mathcal{O} * f)(e^2)] \\ &= \max\left[\bigwedge_{(y_1,z_1)\in A_{e^2}} \max\{f(y_1), \mathcal{O}(z_1)\}, \bigwedge_{(y_2,z_2)\in A_{e^2}} \max\{\mathcal{O}(y_2), f(z_2)\}\right] \\ &\leqslant \max[\max\{f(t), \mathcal{O}(yty)\}, \max\{\mathcal{O}(xtx), f(t)\}] \\ &= \max[\max\{f(t), 0\}, \max\{0, f(t)\}] = \max[f(t), f(t)] = f(t). \end{aligned}$$

Since $e \in E_\Omega$, we have $e^2 \geqslant e$ and $f(e^2) \geqslant f(e)$. Thus $f(e) \leqslant f(t)$. On the other hand since S is left and right simple and $e \in S$, we have $S = (Se]$ and $S = (eS]$. Since $t \in S$ we have $t \in (Se]$ and $t \in (eS]$. Then $t \leqslant ze$ and $t \leqslant es$ for some $z, s \in S$. If $t \leqslant ze$ then $t^2 = tt \leqslant (ze)(ze) = (zez)e$ and $(zez, e) \in A_{t^2}$. If $t \leqslant es$ then $t^2 = tt \leqslant (es)(es) = e(ses)$ and $(e, ses) \in A_{t^2}$. Since $A_{t^2} \neq \emptyset$ we have

$$\begin{aligned} f(t^2) &\leqslant ((f * \mathcal{O}) \vee (\mathcal{O} * f))(t^2) = \max[(f * \mathcal{O})(t^2), (\mathcal{O} * f)(t^2)] \\ &= \max\left[\bigwedge_{(y_1,z_1)\in A_{t^2}} \max\{f(y_1), \mathcal{O}(z_1)\}, \bigwedge_{(y_2,z_2)\in A_{t^2}} \max\{\mathcal{O}(y_2), f(z_2)\}\right] \\ &\leqslant \max[\max\{f(e), \mathcal{O}(ses)\}, \max\{\mathcal{O}(zez), f(e)\}] \\ &= \max[\max\{f(e), 0\}, \max\{0, f(e)\}] = \max[\max\{f(e), f(e)\}] = f(e). \end{aligned}$$

Since $t \in E_\Omega$ then $t^2 \geqslant t$ and $f(t^2) \geqslant f(t)$. Thus $f(t) \leqslant f(e)$. Consequently, $f(t) = f(e)$.

(B) Now we prove that f is a constant function on S. That is, $f(t) = f(a)$ for every $a \in S$. In fact: since S is regular and $a \in S$, there exists $x \in S$ such that $a \leqslant axa$. We consider the elements ax and xa of S. Then by $(OS3)$, we have $(ax)^2 = (axa)x \geqslant ax$ and $(xa)^2 = x(axa) \geqslant xa$, then $ax, xa \in E_\Omega$ and by (A) we have $f(ax) = f(t)$ and $f(xa) = f(t)$. Since $(ax)(axa) \geqslant axa \geqslant a$, then $(ax, axa) \in A_a$ and $(axa)(xa) \geqslant axa \geqslant a$, then $(axa, xa) \in A_a$ and hence $A_a \neq \emptyset$. Since f is an anti-fuzzy quasi-ideal of S, we have

$$\begin{aligned} f(a) &\leqslant ((f * \mathcal{O}) \vee (\mathcal{O} * f))(a) = \max[(f * \mathcal{O})(a), (\mathcal{O} * f)(a)] \\ &= \max\left[\bigwedge_{(y_1,z_1)\in A_a} \max\{f(y_1), \mathcal{O}(z_1)\}, \bigwedge_{(y_2,z_2)\in A_a} \max\{\mathcal{O}(y_2), f(z_2)\}\right] \\ &\leqslant \max[\max\{f(ax), \mathcal{O}(axa)\}, \max\{\mathcal{O}(axa), f(xa)\}] \\ &= \max\left[\max\{f(ax), 0\}, \max\{0, f(xa)\}\right] = \max\left[f(ax), f(xa)\right] = f(t). \end{aligned}$$

Since S is left and right simple we have $(Sa] = S$, and $(aS] = S$. Since $t \in S$, we have $t \in (Sa]$ and $t \in (aS]$. Then $t \leqslant pa$ and $t \leqslant aq$ for some $p, q \in S$. Then $(p, a) \in A_t$ and $(a, q) \in A_t$. Since $A_t \neq \emptyset$, and f is an anti-fuzzy quasi-ideal of S, we have

$$\begin{aligned} f(t) &\leqslant ((f * \mathcal{O}) \vee (\mathcal{O} * f))(t) = \max[(f * \mathcal{O})(t), (\mathcal{O} * f)(t)] \\ &= \max \left[\bigwedge_{(y_1,z_1)\in A_t} \max\{f(y_1), \mathcal{O}(z_1)\}, \bigwedge_{(y_2,z_2)\in A_t} \max\{\mathcal{O}(y_2), f(z_2)\} \right] \\ &\leqslant \max\left[\max\{f(a), \mathcal{O}(q)\}, \max\{\mathcal{O}(p), f(a)\}\right] \\ &= \max\left[\max\{f(a), 0\}, \max\{0, f(a)\}\right] = f(a). \end{aligned}$$

Thus $f(t) \leqslant f(a)$ and $f(t) = f(a)$.

$(ii) \Longrightarrow (i)$. Let $a \in S$. Then the set $(aS]$ is a quasi-ideal of S. Indeed: $(aS] \cap (Sa] \subseteq (aS]$, and $x \in (aS]$ and $S \ni y \leqslant x \in (aS]$ imply $y \in ((aS]] = (aS]$. Since $(aS]$ is quasi-ideal of S, by Theorem 3.2, the characteristic function $f_{(aS]^c}$ of $(aS]$ is an anti-fuzzy quasi-ideal of S. By hypothesis, $f_{(aS]^c}$ is a constant function, that is, there exists $t \in \{0, 1\}$ such that $f_{(aS]^c}(x) = t$ for every $x \in S$. Let $(aS] \subset S$ and a be an element of S such that $a \notin (aS]$, then $f_{(aS]^c}(a) = 1$. On the other hand, since $a^2 \in (aS]$, then $f_{(aS]^c}(a^2) = 0$, a contradiction to the fact that $f_{(aS]^c}$ is a constant function. Hence $(aS] = S$. By symmetry we can prove that $(Sa] = S$.

Since $a \in S$ and $S = (aS] = (Sa]$, we have $a \in (aS] = (a(Sa]] = (aSa]$, consequently S is regular. □

Theorem 4.2. *S is completely regular if and only if for every anti-fuzzy quasi-ideal f of S we have $f(a) = f(a^2)$ for every $a \in S$.*

Proof. Let S be completely regular and f be an anti-fuzzy quasi-ideal of S. Since S is left and right regular we have $a \in (Sa^2]$ and $a \in (a^2S]$ for every $a \in S$. Then there exists $x, y \in S$ such that $a \leqslant xa^2$ and $a \leqslant a^2y$. Hence $(x, a^2), (a^2, y) \in A_a$. Since $A_a \neq \emptyset$, we have

$$\begin{aligned} f(a) &\leqslant ((f * \mathcal{O}) \vee (\mathcal{O} * f))(a) = \max[(f * \mathcal{O})(a), (\mathcal{O} * f)(a)] \\ &= \max \left[\bigwedge_{(y_1,z_1)\in A_a} \max\{f(y_1), \mathcal{O}(z_1)\}, \bigwedge_{(y_2,z_2)\in A_a} \max\{\mathcal{O}(y_2), f(z_2)\} \right] \\ &\leqslant \max\left[\max\{f(a^2), \mathcal{O}(y)\}, \max\{\mathcal{O}(x), f(a^2)\}\right] \\ &= \max\left[\max\{f(a^2), 0\}, \max\{0, f(a^2)\}\right] \\ &= \max\left[f(a^2), f(a^2)\right] = f(a^2) = f(aa) \leqslant \max\{f(a), f(a)\} = f(a). \end{aligned}$$

Hence $f(a) = f(a^2)$.

Conversely, let $a \in S$ and let $Q(a^2)$ be the quasi-ideal generated by a^2. Then $Q(a^2) = (a^2 \cup (a^2S \cap Sa^2)]$. By Theorem 3.2, the characteristic function $f_{Q(a^2)^c}$ is an anti-fuzzy quai-ideal of S. By hypothesis $f_{Q(a^2)^c}(a) = f_{Q(a^2)^c}(a^2)$. Since $a^2 \in Q(a^2)$, we have $f_{Q(a^2)^c}(a^2) = 0$, then $f_{Q(a^2)^c}(a) = 0$ and $a \in Q(a^2) = (a^2 \cup (a^2S \cap Sa^2)]$. Then $a \leqslant a^2$ or $a \leqslant a^2x$ and $a \leqslant ya^2$ for some $x, y \in S$. If $a \leqslant a^2$ then $a \leqslant a^2 = aa \leqslant a^2a^2 = aaa^2 \leqslant a^2aa^2 \in a^2Sa^2$ and so $a \in (a^2Sa^2]$. If $a \leqslant a^2x$ and $a \leqslant ya^2$ then $a \leqslant (a^2x)(ya^2) = a^2(xy)a^2 \in a^2Sa^2$ and so $a \in (a^2Sa^2]$. □

A subset T of S is called *semiprime* if for every $a \in S$ such that $a^2 \in T$ we have $a \in T$. An anti-fuzzy quasi-ideal f of S is called *semiprime* if $f(a) \leqslant f(a^2)$ all $a \in S$.

Theorem 4.3. *S is completely regular if and only if every its anti-fuzzy quasi-ideal is semiprime.*

Proof. Let S be completely regular and f be its anti-fuzzy quasi-ideal. Then $f(a) \leqslant f(a^2)$ for $a \in S$. Indeed: since S is left and right regular, there exist $x, y \in S$ such that $a \leqslant xa^2$ and $a \leqslant a^2y$ then $(x, a^2) \in A_a$ and $(a^2, y) \in A_a$. Since $A_a \neq \emptyset$, and f is an anti-fuzzy quasi-ideal of S, we have

$$\begin{aligned}
f(a) &\leqslant ((f * \mathcal{O}) \vee (\mathcal{O} * f))(a) = \max[(f * \mathcal{O})(a), (\mathcal{O} * f)(a)] \\
&= \max\left[\bigwedge_{(y_1,z_1)\in A_a} \max\{f(y_1), \mathcal{O}(z_1)\}, \bigwedge_{(y_2,z_2)\in A_a} \max\{\mathcal{O}(y_2), f(z_2)\}\right] \\
&= \max\left[\max\{f(a^2), \mathcal{O}(y)\}, \max\{\mathcal{O}(x), f(a^2)\}\right] \\
&\leqslant \max\left[\max\{f(a^2), 0\}, \max\{0, f(a^2)\}\right] = \max\left[f(a^2), f(a^2)\right] = f(a^2).
\end{aligned}$$

Conversely. Let f be an anti-fuzzy quasi-ideal of S such that $f(a) \leqslant f(a^2)$ for all $a \in S$. By Theorem 3.2, the characteristic function $f_{Q(a^2)^c}$ of the quasi-ideal $Q(a^2)$ is an anti-fuzzy quai-ideal of S. By hypothesis $f_{Q(a^2)^c}(a) \leqslant f_{Q(a^2)^c}(a^2)$. Since $a^2 \in Q(a^2)$, we have $f_{Q(a^2)^c}(a^2) = 0$, then $f_{Q(a^2)^c}(a) = 0$ and $a \in Q(a^2) = (a^2 \cup (a^2S \cap Sa^2)]$. Thus $a \leqslant a^2$ or $a \leqslant a^2p$ and $a \leqslant qa^2$ for some $p, q \in S$. If $a \leqslant a^2$ then $a \leqslant a^2 = aa \leqslant a^2a^2 = aaa^2 \leqslant a^2aa^2 \in a^2Sa^2$ and so $a \in (a^2Sa^2]$. If $a \leqslant a^2p$ and $a \leqslant qa^2$ then $a \leqslant (a^2p)(qa^2) = a^2(pq)a^2 \in a^2Sa^2$ and so $a \in (a^2Sa^2]$. Consequently, S is completely regular. □

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Received September 02, 2010
Revised February 8, 2011

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