Quasigroups and Related Systems **19** (2011), 23 – 68

Steiner triple systems and their close relatives

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Abstract. This paper is intended to be a gentle and self-contained introduction to Steiner triple systems and other important designs of triples. Topics covered include existence proofs, isomorphism testing, and important techniques which have wide application. Links to the algebraic theory of quasigroups and loops are also discussed.

1. Introduction

This paper accompanies talks given at the Loops11 workshop in Třešť, Czech Republic, from 21st to 23rd July (Červenec), 2011. Knowledge of Steiner triple systems and other designs of triples is a vast field as the reference work, "Triple Systems" by C.J. Colbourn and A. Rosa [17] shows. This was published in 1999 and has over 450 pages of text and nearly 100 pages of bibliography. Although in the 10+ years since its publication the subject has inevitably moved on it is still the reference work to consult and I will refer to it at various points throughout this paper referenced as just C&R. Another indispensable tool is the "Handbook of Combinatorial Designs" edited by C.J. Colbourn and J.H. Dinitz [12]. Now in its second edition I will also refer to this as HB. So within the time and space available it is possible only to give a very brief glimpse of this interesting and fascinating area. I have to be selective; indeed very selective. What has guided my choice are three basic principles. The first of these is to present basic existence results and questions of isomorphism testing. The second is to explore certain techniques which seem to have a wide application. Last, but not least, I want to select topics which I hope will be of most interest or use to an algebraic audience. In this way perhaps I will achieve the aim of at least giving a flavour of the subject. So let us begin.

2010 Mathematics Subject Classification: 05B07
Keywords: triple system, construction, automorphism, configuration, isomorphism, group divisible design, quasigroup, loop.

A *Steiner triple system* of order v, usually denoted by STS(v), is an ordered pair $(V, \mathcal{B})$ where V is a *base set* of *elements* or *points* of cardinality v and $\mathcal{B}$ is a collection of *triples* also called *blocks*, which collectively have the property that every pair of distinct elements of V is contained in precisely one triple. The most well-known examples come from geometry. Let $\mathbb{F}_2$ be the finite field of two elements and $V = (\mathbb{F}_2)^n \setminus \{\mathbf{0}\}$. The set of triples of vectors $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ where $\mathbf{x} + \mathbf{y} + \mathbf{z} = \mathbf{0}$, $\mathbf{x} \neq \mathbf{y} \neq \mathbf{z} \neq \mathbf{x}$, form the blocks of an STS$(2^n - 1)$. Thus V and $\mathcal{B}$ are respectively the points and lines of the projective geometry PG$(n-1, 2)$ and the systems are known as *projective Steiner triple systems*. For $n = 3$, and interpreting the vectors as binary numbers, this gives the following triples 123, 145, 167, 246, 257, 347, 356 as the blocks of an STS(7). Here, and throughout the rest of the paper we will for clarity omit set brackets and commas from triples when there is no danger of confusion.

Further examples are the *affine triple systems*. Let $\mathbb{F}_3$ be the field of three elements and let $V = (\mathbb{F}_3)^n$. Again $\mathcal{B}$ is the set of triples of vectors $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ where $\mathbf{x} + \mathbf{y} + \mathbf{z} = \mathbf{0}$, $\mathbf{x} \neq \mathbf{y} \neq \mathbf{z} \neq \mathbf{x}$, and V and $\mathcal{B}$ are respectively the points and lines of the affine geometry AG$(n, 3)$. For $n = 2$, and interpreting the vectors as ternary numbers, this gives the following triples 012, 345, 678, 036, 147, 258, 048, 372, 615, 057, 138, 246 as the blocks of an STS(9). A wider class which contains the affine Steiner triple systems is the *Hall triple systems*. These were introduces by Hall [51] as Steiner triple systems in which for each $x \in V$, the automorphism group contains an automorphism which is an involution with just x as a fixed point. They can be characterized as Steiner triple systems in which every three points which do not form a triple generate the affine Steiner triple system AG(2, 3) of order 9. Hall triple systems have order $3^m, m \geq 2$, and the smallest such system which is not affine has order 81. More information is contained on pages 496 to 499 of HB.

Less well-known are the so-called *Netto triple systems*. These appear to have been incorrectly attributed to Netto and are not the systems introduced in his paper of 1893 [74]. Perhaps their most elegant description is the following taken from [22]. Let p be prime with $p \equiv 7 \pmod{12}$. Let n be odd and $q = p^n$. Consider the finite field $\mathbb{F}_q = V$ and let ϵ_1 and ϵ_2 be the two primitive sixth roots of unity. Then ϵ_1 and ϵ_2 satisfy the equation $x^2 - x + 1 = 0$. So $\epsilon_1 \epsilon_2 = \epsilon_1 + \epsilon_2 = 1$. Both ϵ_1 and ϵ_2 are quadratic non-residues. The collection $\mathcal{B}$ is determined by specifying the unique triple which contains the pair $\{a, b\}$. Define $x < y$ if $y - x$

is a quadratic residue. Either $a < b$ or $b < a$ but not both. Without loss of generality assume the former. Then the triple containing the pair is $\{a, b, f(a, b)\}$ where $f(x, y) = x\epsilon_1 + y\epsilon_2$. The construction works because both $b < f(a, b)$ and $f(a, b) < a$ and both $f(b, f(a, b)) = a$ and $f(f(a, b), a) = b$.

The above are very special types of Steiner triple system. It was Plűcker in 1835 [80] who first asked the question for which v Steiner triple systems STS(v) exist and stated that a necessary condition is $v \equiv 3 \pmod 6$, later [81] corrected to $v \equiv 1, 3 \pmod 6$. Such values are called *admissible* and are easily derived by counting. Each point $x \in V$ occurs in $r = (v-1)/2$ triples. This is the *replication number*. Hence v must be odd. The total number of triples is $b = v(v-1)/6$ which disallows $v \equiv 5 \pmod 6$. The name comes from the fact that Steiner in 1853 [89] asked a series of questions, the first of which was the existence of what became to be called Steiner triple systems.

> Welche Zahl, N, von Elementen hat die Eigenschaft, dass sich die Elemente so zu dreien ordnen lassen, dass je zwei in *einer*, aber *nur in einer* Verbindung vorkommen?

Six years later a solution was given by Reiss [83], but both Steiner and Reiss had been anticipated by Kirkman [60] in a paper dated 23rd December 1846 and published the next year. There is a remarkable similarity between the papers of Kirkman and Reiss!

Kirkman's paper was the first of any significance in Combinatorial Design Theory and was followed by other important contributions. To quote Biggs [5]

> In this series of papers Kirkman has established an incontestable claim to be regarded as the founding father of the theory of designs. Among his contemporaries, only Sylvester attempted anything comparable, and his papers on Tactic seem to be more concerned with advancing his claims to have discovered the subject than with advancing the subject itself. Not until the *Tactical Memoranda* of E.H. Moore in 1896 is there another contribution to rival Kirkman's.

In one note [61], Kirkman posed the following problem.

> Fifteen young ladies in a school walk out three abreast for seven days in succession: it is required to arrange them daily, so that no two shall walk twice abreast.

What is required here is an STS(15), but one which has an additional property, that of resolvability. In an STS(v), $(V, \mathcal{B})$, a *parallel class* or a *resolution class* is a set of blocks which contain every element precisely once. If the blocks of $\mathcal{B}$ can be partitioned into parallel classes, then the STS(v) is said to be *resolvable*. Such an STS(v), together with its parallel classes, is called a *Kirkman triple system* and denoted by KTS(v). The STS(9) given above has this property of resolvability. Although Kirkman himself always very properly referred to "young ladies" the problem became known as "Kirkman's 15 schoolgirls problem". A solution was given by Cayley [8] in 1850 and the following year Kirkman [62] gave his own solution. Clearly, necessary condition for the existence of a KTS(v) is $v \equiv 3 \pmod 6$, but a proof of its sufficiency, by Ray-Chaudhuri and Wilson [82], did not appear until 1971, fully 120 years after the proof for Steiner triple systems. The same result was also established by Lu in 1965 but remained unpublished until 1990 [66].

Essentially there are two types of construction for Steiner triple systems; recursive and direct. Kirkman's solution is recursive and is described in the next section as well as further later constructions. Direct constructions are considered in Section 3. However before proceeding it is perhaps appropriate to give some enumeration results.

Two Steiner triple systems $(V, \mathcal{B})$ and $(W, \mathcal{D})$ are said to be *isomorphic* if there exists a one-one mapping $\phi : V \to W$ such that every triple $B \in \mathcal{B}$ maps to a triple $\phi(B) \in \mathcal{D}$. In the case of a Kirkman triple system the mapping must also preserve the resolution classes. To within isomorphism the STS(7) and STS(9) are unique with automorphism groups of order 168 and 432 respectively. There are two non-isomorphic STS(13)s. In 1897, Zulauf [100] showed that the known STS(13)s fall into two isomorphism classes and two years later De Pasquale [24] determined that only two isomorphism classes are possible. White, Cole and Cummings [94] first enumerated STS(15)s in 1919; they found 80 non-isomorphic systems. Unaware of their work, Fisher [31] repeated the enumeration in 1940 but found only 79 systems. However the veracity of White, Cole and Cumming's result was confirmed in 1955 by Hall and Swift [52] in one of the first uses of digital computers in Combinatorial Design Theory. Listings and properties, including details of automorphism groups, of these systems are contained in the paper by Mathon, Phelps and Rosa [67], see also pages 65 to 69 of C&R. The combinatorial explosion now takes over. The number of non-isomorphic STS(19)s is 11,084,874,829 published by Kaski and Őstergård in 2004 [59]. A study of the properties of these system is [13]. So enumeration results have appeared at the rate of one in each of the 19th, 20th and 21st centuries. It is interesting to speculate whether we will have to wait until the next century or perhaps the general availability of quantum computing to know the number of non-isomorphic STS(21)s.

There are seven KTS(15)s but these come from only four STS(15)s; there are two non-isomorphic resolutions of systems #1, #7 and #15 and one of #61. They can be found, very conveniently, on page 67 of HB. The solutions mentioned above by Cayley and Kirkman are the two resolutions of system #1 which is the projective STS(15). In 1860, Peirce [76] also gave both solutions together with the one from system #61 and all seven solutions are listed by Mulder [73] and Cole [20]. An early bibliography of 48 papers on "Kirkman's schoolgirls problem" was published by Eckenstein [29].

2. Recursive constructions

So, how did Kirkman prove the existence of Steiner triple systems, or as he called them *triad systems*? He devised two recursive constructions which are given below. But first we need some further definitions. A *partial Steiner triple system* of order v, denoted by PSTS(v), is defined similarly to an STS(v) except that every pair of distinct elements of V is contained in at most one triple. The set of pairs which are not contained in any triple is called the *leave* of the PSTS(v). The constructions also use the concept of a one-factorization of a complete graph K_{2n}. A *one-factor* is a set of n edges which collectively are incident with every vertex of the graph. A *one-factorization* is a partition of all $n(2n-1)$ edges into $2n-1$ one-factors. Denote by Q_v, an STS(v) and by R_v, a PSTS(v) with a leave which consists of a cycle $\mathbb{C}_{v-1}$. The necessary condition for the existence of the latter is $v \equiv 1, 5 \pmod 6$. Kirkman's two constructions are as follows.

1. $Q_{2n+1} \Longrightarrow Q_{4n+3} \Longrightarrow R_{4n+1}$.

2. $R_{2n+1} \Longrightarrow Q_{4n+1} \Longrightarrow R_{4n-1}$.

Kirkman's first construction

Let Q_{2n+1} be defined on base set $V = \{x_0, x_1, x_2, \ldots, x_{2n}\}$. Now take the following one-factorization of the complete graph K_{2n+2} on $\{\infty, 0, 1, \ldots, 2n\}$ and assign all the pairs of each one-factor to points of the base set V of the Q_{2n+1} as shown below.

x_0 :	$(\infty,0)$	$(1,2n)$	$(2,2n-1)$	$\dots$	$(n-1,n+2)$	$(n,n+1)$
x_1 :	$(\infty,1)$	$(2,0)$	$(3,2n)$	$\dots$	$(n,n+3)$	$(n+1,n+2)$
$\cdot$						
$\cdot$						
x_{n-1} :	$(\infty,n-1)$	$(n,n-2)$	$(n+1,n-3)$	$\dots$	$(2n-2,0)$	$(2n-1,2n)$
x_n :	(∞,n)	$(n+1,n-1)$	$(n+2,n-2)$	$\dots$	$(2n-1,1)$	$(2n,0)$
x_{n+1} :	$(\infty,n+1)$	$(n+2,n)$	$(n+3,n-1)$	$\dots$	$(2n,2)$	$(0,1)$
$\cdot$						
$\cdot$						
x_{2n} :	$(\infty,2n)$	$(0,2n-1)$	$(1,2n-2)$	$\dots$	$(n-2,n+1)$	$(n-1,n)$

This gives Q_{4n+3}. To obtain R_{4n+1} remove all triples containing 0 or $2n$. The $\mathbb{C}_{4n}$ leave is

$$\infty, x_0, 1, x_{n+1}, 2, x_1, 3, x_{n+2}, 4, x_2, \dots, 2n-2, x_{n-1}, 2n-1, x_{2n}.$$

Kirkman's second construction

Let R_{2n+1} be defined on base set $V = \{x_0, x_1, x_2, \dots, x_{2n}\}$ with $\mathbb{C}_{2n}$ leave $x_0, x_1, x_2, \dots, x_{2n-1}$. Now take the following one-factorization of the complete graph K_{2n} on set $\{\infty, 0, 1, \dots, 2n-2\}$ and assign all the pairs of each one-factor except the pair in the last column to points of the base set V of the R_{2n+1} as shown below.

x_{n-1} :	$(\infty,0)$	$(1,2n-2)$	$(2,2n-3)$	$\dots$	$(n-2,n+1)$	$(n-1,n)$
x_n :	$(\infty,1)$	$(2,0)$	$(3,2n-2)$	$\dots$	$(n-1,n+2)$	$(n,n+1)$
$\cdot$						
$\cdot$						
x_{2n-3} :	$(\infty,n-2)$	$(n-1,n-3)$	$(n,n-4)$	$\dots$	$(2n-4,0)$	$(2n-3,2n-2)$
x_{2n} :	$(\infty,n-1)$	$(n,n-2)$	$(n+1,n-3)$	$\dots$	$(2n-3,1)$	$(2n-2,0)$
x_0 :	(∞,n)	$(n+1,n-1)$	$(n+2,n-2)$	$\dots$	$(2n-2,2)$	$(0,1)$
$\cdot$						
$\cdot$						
x_{n-2} :	$(\infty,2n-2)$	$(0,2n-3)$	$(1,2n-4)$	$\dots$	$(n-3,n)$	$(n-2,n-1)$

Further, for pairs in the last column, assign the pair $(2n-2,0)$ to x_{2n} and all the other pairs to x_{2n-2} and x_{2n-1} alternately starting with the pair $(0,1)$ assigned to x_{2n-2}. Finally adjoin the triples

$$\{1,x_0,x_1\},\ \{2,x_1,x_2\},\ \dots,\ \{2n-2,x_{2n-3},x_{2n-2}\},\ \{\infty,x_{2n-2},x_{2n-1}\},\ \{0,x_{2n-1},x_0\}$$

This gives Q_{4n+1}. To obtain R_{4n-1} remove all triples containing 0 or $2n-2$. The $\mathbb{C}_{4n-2}$ leave is

$$\infty, x_{n-1}, 1, x_{2n-2}, x_{2n-3}, 2n-4, x_{n-3}, 2n-5, x_{2n-4}, 2n-6, x_{n-4}, \dots,$$
$$x_1, 3, x_n, 2, x_0, x_{2n-1}, 2n-3, x_{n-2}$$

Kirkman's work is quite remarkable, made even more so because repeated application of the two constructions gives STS(v) of all admissible orders beginning with the trivial Steiner triple system on just one point and consisting of no triples! First note that

$$Q_1 \Longrightarrow Q_3 \Longrightarrow Q_7 \Longrightarrow R_5 \Longrightarrow Q_9$$

Then, succesively for all $n \geqslant 1$, use the following schema.

$$\begin{gathered} Q_{6n+1} \Longrightarrow R_{6n-1} \Longrightarrow Q_{12n-3} \\ Q_{6n+3} \Longrightarrow R_{6n+1} \Longrightarrow Q_{12n+1} \\ Q_{6n+1} \Longrightarrow Q_{12n+3} \\ Q_{6n+3} \Longrightarrow Q_{12n+7} \end{gathered}$$

Kirkman's one-factorization

The one-factorization used by Kirkman is the one which is now usually denoted by GK_{2n}. It is easily described. Let K_{2n} be the complete graph on vertex set $\{\infty, 0, 1, 2, \ldots, 2n-2\}$. Denote the set of one-factors by $\{F_0, F_1, \ldots, F_{2n-2}\}$. Let F_0 be the set of edges $\{(\infty, 0), (1, 2n{-}2), (2, 2n{-}3), \ldots, (n-1, n)\}$. The remaining one-factors $F_i, 1 \leqslant i \leqslant 2n-2$, are obtained by applying the mapping $x \mapsto x+i$ to F_0, arithmetic modulo $2n-2$ with ∞ as a fixed point.

But this was not Kirkman's method. He used instead a *greedy algorithm*. Representing the vertices of K_{2n} as above, he considered the pairs in lexicographical order and assigned them to one factors in cyclic order without violating the one-factor criterion. The method is best explained by the following example for K_{10}.

F_0	F_1	F_2	F_3	F_4	F_5	F_6	F_7	F_8
01	02	03	04	05	06	07	08	0∞
		12	13	14	15	16	17	18
	1∞			23	24	25	26	27
28			2∞			34	35	36
37	38				3∞			45
46	47	48					4∞	
	56	57	58					
5∞			67	68				
		6∞			78			
				7∞		8∞		

It is not immediately obvious that this method works, nor that it gives GK_{2n}. It is not well-known but possibly should be. More details are contained in the paper by Anderson [2].

Other recursive constructions

The $Q_{2n+1} \Longrightarrow Q_{4n+3}$ construction is an STS$(v) \Longrightarrow$ STS$(2v+1)$ construction which can use any one-factorization of K_{v+1}. Let $(V, \mathcal{B})$ be an STS(v). For any one-factorization of K_{v+1} with vertex set W, assign all the pairs of each one-factor to one of the points of V to form further triples $\mathcal{T}$. Then $(V \cup W, \mathcal{B} \cup \mathcal{T})$ is an STS$(2v+1)$. In employing this construction we may use the STS(v) itself to determine the one-factorization. For each point $x \in V$, let there be a point $x' \in W$. Further let $\infty \in W$. If $\{x, y, z\} \in \mathcal{B}$ then put $\{x, y', z'\}, \{x', y, z'\}, \{x', y', z\} \in T$. Finally, for all $x \in V$, put $\{x, x', \infty\} \in T$.

A further recursive construction is STS$(v) \Longrightarrow$ STS$(3v-2)$. Take three STS(v)s $(V_0 \cup \{\infty\}, \mathcal{B}_0)$, $(V_1 \cup \{\infty\}, \mathcal{B}_1)$, $(V_2 \cup \{\infty\}, \mathcal{B}_2)$. Now take a Latin square of side $v-1$ with the rows, columns and entries indexed respectively by the points of the sets V_0, V_1, V_2. Let $\mathcal{T}$ be the set of {row, column, entry} triples. Then $(V_0 \cup V_1 \cup V_2 \cup \{\infty\}, \mathcal{B}_0 \cup \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{T})$ is an STS$(3v-2)$. The three initial STS(v)s need not be isomorphic.

An exact tripling construction i.e. STS$(v) \Longrightarrow$ STS$(3v)$ is the following. Let $(V, \mathcal{B})$ be an STS(v). Let $W = V \times \{0, 1, 2\}$. If $\{x, y, z\} \in V$, then put $\{(x, i), (y, j), (z, k)\} \in \mathcal{D}$ for $i = j = k$ and $i \neq j \neq k \neq i$ and also $\{(x, 0), (x, 1), (x, 2)\} \in \mathcal{D}$ for all $x \in V$. Then $(W, \mathcal{D})$ is an STS$(3v)$.

The last construction can be generalized into a direct product construction, i.e., STS(u) & STS$(v) \Longrightarrow$ STS(uv). Let $(U, \mathcal{A})$ be an STS(u) and $(V, \mathcal{B})$ be an STS(v). Let $(a, x), (b, y) \in U \times V$. If $a \neq b$ define c by $\{a, b, c\} \in \mathcal{A}$ and similarly if $x \neq y$ define z by $\{x, y, z\} \in \mathcal{B}$. The block set $\mathcal{D}$ is defined by specifying for each distinct pair $\{(a, x), (b, y)\}$, the third element of the block. The following is easily seen to be consistent. If $a \neq b$ and $x \neq y$, then put $\{(a, x), (b, y), (c, z)\} \in \mathcal{D}$. If $a \neq b$ and $x = y$, put $\{(a, x), (b, x), (c, x)\} \in \mathcal{D}$ and if $a = b$ and $x \neq y$, put $\{(a, x), (a, y), (a, z)\} \in \mathcal{D}$. Then $(V \times W, \mathcal{D})$ is an STS(uv).

All of the above constructions can be obtained as special cases of a recursive construction due to Moore [72]. Let $(U, \mathcal{A})$ be an STS(u) and $(V \cup W, \mathcal{B} \cup \mathcal{C})$ be an STS(v) which contains as a subsystem an STS(w), $(W, \mathcal{C})$. Take u copies of the STS(v) on base sets $V_i \cup W, 1 \leqslant i \leqslant u$. Index the u systems by the points of the set U and take a Latin square of side $v - w$. Across each set of three systems of the u STS(v)s as determined by the blocks of the STS(u) adjoin new triples determined by the set of {row, column, entry} triples as in the $3v-2$ construction above. What results is an STS$(w + u(v-w))$. The $3v-2$ construction corresponds to when $w = 1$ and $u = 3$. A $3v$ construction is obtained by choosing $w = 0$ and $u = 3$ and a direct product construction by choosing $w = 0$. Finally, the $2v+1$ construction in which the STS(v) is used to produce the one-factorization is the case where $v = 3$ and $w = 1$ (and u renamed as v).

3. Direct constructions

In 1939, Bose [6] published a landmark paper on Design Theory in which he gave a direct construction for Steiner triple systems of order $v \equiv 3 \pmod 6$ based on a cyclic group of odd order. The method can be extended and it is in a more generalized form that we now present it.

Bose construction
Let $(Q, \circ)$ be a commutative idempotent quasigroup of order $2s+1$ and let $V = Q \times \{0,1,2\}$. The blocks of an STS$(6s+3)$, $(V, \mathcal{B})$, are defined as follows.

$$(A) \;\; \{(x,0),(x,1),(x,2)\},\; x \in Q$$

$$(B1) \;\; \{(x,0),(y,0),(z,1)\},\; x,y \in Q, x \neq y, z = x \circ y$$

$$(B2) \;\; \{(x,1),(y,1),(z,2)\},\; x,y \in Q, x \neq y, z = x \circ y$$

$$(B3) \;\; \{(x,2),(y,2),(z,0)\},\; x,y \in Q, x \neq y, z = x \circ y$$

Such quasigroups are easy to construct. Abelian groups of odd order possess unique square roots, so if G is an Abelian group of order $2s+1$ and we write $x \circ y = z$ if $xy = z^2$ then $(G, \circ)$ is a commutative idempotent quasigroup. Also non-isomorphic quasigroups defined on Q may be used to construct the blocks (B1), (B2), (B3).

A further generalization [48] is the following. Let $(W, \mathcal{D})$ be an STS$(6m+3)$ which contains a parallel class $\mathcal{P}$. For each block of the system, assign an arbitrary but fixed order to the points. Call a typical block $\{a,b,c\}$ and denote the ordering by $a < b < c$. The blocks of an STS$((2s+1)(6m+3))$ on base set $Q \times W$ are defined as follows.

$$(A) \;\; \{(x,a),(x,b),(x,c)\},\; x \in Q,\; \{a,b,c\} \in \mathcal{D}$$

$$(B1) \;\; \{(x,a),(y,a),(z,b)\},\; x,y \in G, x \neq y, z = x \circ y,\; \{a,b,c\} \in \mathcal{P}$$

$$(B2) \;\; \{(x,b),(y,b),(z,c)\},\; x,y \in G, x \neq y, z = x \circ y,\; \{a,b,c\} \in \mathcal{P}$$

$$(B3) \;\; \{(x,c),(y,c),(z,a)\},\; x,y \in G, x \neq y, z = x \circ y,\; \{a,b,c\} \in \mathcal{P}$$

$$(C) \;\; \{(x,a),(y,b),(z,c)\},\; x,y \in G, x \neq y, z = x \circ y,\; \{a,b,c\} \in \mathcal{D} \setminus \mathcal{P}$$

When $m = 0$ there are no blocks of type (C) and the construction reverts to the basic Bose construction.

The Bose construction and its variants seem to be a particularly useful tool in constructing Steiner triple systems having prescribed properties. We will meet them again in Section 5 on configurations. The construction also appears in the work of Ducrocq and Sterboul [28] and Grannell, Griggs and Širáň [45] on biembedding pairs of Steiner triple systems in non-orientable and orientable surfaces respectively. Further discussion of this falls well outside the scope of this paper and would take us towards Topological Graph Theory but the interested reader can consult the recent survey paper [40]. Again the subject has moved on since it was written but it still serves as a good introduction and overview of the subject.

A parallel construction for STS($6s+1$) uses a half-idempotent commutative quasigroup. A Latin square is *half-idempotent* if every element appears either twice or zero times on the diagonal. Clearly such squares can only exist for even orders and an easy example is given by any cyclic group of even order. Any half-idempotent Latin square can have its rows and columns relabelled in such a way that the equation $x \circ x = x$ is satisfied by precisely half of the elements. We then have a *half-idempotent quasigroup*. Note that the relabelling can be done in such a way that retains commutativity. In particular, for addition modulo $2s$ the relabelling can be done so that $2x \circ 2y = (2x+1) \circ (2y+1) = x+y$ and $2x \circ (2y+1) = (2x+1) \circ 2y = x+y+s$, $0 \leqslant x, y \leqslant s-1$.

So let $(Q, \circ)$ be a half-idempotent quasigroup of order $2s$ and let $V = Q \times \{0,1,2\} \cup \{\infty\}$. The blocks of an STS($6s+1$), $(V, \mathcal{B})$, are defined as follows.

$$(A)\ \ \{(x,0),(x,1),(x,2)\},\ x \in Q,\ x \circ x = x$$

$$(B1)\ \ \{(x,0),(y,0),(z,1)\},\ x,y \in Q, x \neq y, z = x \circ y$$

$$(B2)\ \ \{(x,1),(y,1),(z,2)\},\ x,y \in Q, x \neq y, z = x \circ y$$

$$(B3)\ \ \{(x,2),(y,2),(z,0)\},\ x,y \in Q, x \neq y, z = x \circ y$$

$$(C1)\ \ \{\infty,(x,0),(x \circ x,1)\},\ x \in Q,\ x \circ x \neq x$$

$$(C2)\ \ \{\infty,(x,1),(x \circ x,2)\},\ x \in Q,\ x \circ x \neq x$$

$$(C3)\ \ \{\infty,(x,2),(x \circ x,0)\},\ x \in Q,\ x \circ x \neq x$$

A completely different direct construction of Steiner triple systems was given by Schreiber [86] and Wilson [96], see also [43].

Schreiber-Wilson construction

Let G be an Abelian group of order $n \equiv -1, 1 \pmod 6$ with the operation written additively and $v = n + 2$. First list all sets of triples $\{x, y, z\}$ such that $x + y + z = 0$. These fall into three types.

1. $\{x, y, z\},\ x, y, z \in G, x \neq y \neq z \neq x$
2. $\{x, x, -2x\},\ x \in G \setminus \{0\}$
3. $\{0, 0, 0\}$

The total number of triples is $(n+2)(n+1)/6 = v(v-1)/6$, the exact number of blocks contained in an STS(v). The idea is to leave type 1 triples as constructed and to replace repeated elements by two new elements, X and Y. Clearly the type 3 triple becomes XY0. So the efficacy of the construction depends on the type 2 triples. These fall into orbits under the mapping $i \mapsto -2i \pmod{n}$ and for the replacement to be done must all have even length. The condition for this is number theoretic; for every prime divisor p of n, the order of $-2 \pmod{p}$ must be even. The following example for $v = 15$ illustrates the construction well.

Example 3.1. Let G be the cyclic group of order 13, $\mathbb{Z}_{13}$, with the elements 10, 11, 12 being denoted by A, B, C respectively. The triples are as follows.

Type 1: 01C, 02B, 03A, 049, 058, 067, 12A, 139, 148, 157, 238, 247,

256, 346, 3BC, 4AC, 59C, 5AB, 68C, 69B, 78B, 79A.

Type 2: 11B, BB4, 445, 553, 337, 77C, CC2, 229, 998, 88A, AA6, 661

Type 3: 000

Here the type 2 triples form a single orbit and so replacing the repeated elements by X and Y respectively (and the type 3 block by XY0) gives the triples

X1B,YB4,X45,Y53,X37,Y7C,XC2,Y29,X98,Y8A,XA6,Y61,XY0

The STS(15) constructed is #37 in the standard listing on pages 65 to 69 of C&R. □

However all is not lost when there are odd length orbits under the mapping $i \mapsto -2i \pmod{n}$. In that case these orbits occur in pairs, mapped to one another by $i \mapsto -i \pmod{n}$. Proceed as before as far as possible with the replacement of repeated elements but with the extra proviso that if the triple $gg(-2g)$ in one orbit becomes the block X$g(-2g)$ then also $(-g)(-g)2g$ in the "negative" orbit becomes X$(-g)2g$. There remain two triples, $xx(-2x)$ in one orbit and $(-x)(-x)2x$ in the other orbit, in which the repeated element cannot be replaced by either X or Y without introducing a repeated pair. To solve this problem discard the triples $0x(-x)$ and $0(2x)(-2x)$ and include four new triples. The already defined blocks include $\mathrm{X}(-2x)(4x)$ or $\mathrm{Y}(-2x)(4x)$. In the former case the four new blocks are $0x(-2x)$, $0(-x)(2x)$, $\mathrm{X}x(-x)$, $\mathrm{Y}(2x)(-2x)$. For the latter interchange X and Y. The procedure is illustrated well by the following example for $v = 13$.

Example 3.2. Let G be the cyclic group of order 11, $\mathbb{Z}_{11}$. Denote the element 10 by A. The type 2 triples fall into two orbits

119, 994, 443, 335, 551 and AA2, 227, 778, 886, 66A

which under replacement become

X19, Y94, X43, Y35, 051 and XA2, Y27, X78, Y86, 06A

The type 1 triples 01A and 056 become Y1A and X56 respectively. The other (unchanged) type 1 triples are

029, 038, 047, 128, 137, 146, 236, 245, 39A, 48A, 57A, 589, 679

which together with XY0 give the 26 blocks of an STS(13). □

4. Automorphisms

Further constructions are based on assumed automorphisms. For a Steiner triple system of order v the obvious candidate is the cyclic group of the same order. So let $(V, \mathcal{B})$ be an STS(v) where $V = \mathbb{Z}_v$ and the automorphism is generated by the mapping $i \mapsto i + 1 \pmod{v}$. Considering the case $v = 6s + 1$, the STS(v) will comprise $(v-1)/6$ orbits of triples under the mapping. Suppose that the set $\{0, a, a+b\}$ is a block of such an orbit. Then the other blocks in the same orbit which contain the point 0 are $\{v-a, 0, b\}$ and $\{v-(a+b), v-b, 0\}$. Since the group acts transitively on the points, a necessary and sufficient condition for the existence of an STS(v) with a cyclic automorphism, denoted by CSTS(v), is that there exists a partition of $\mathbb{Z}_v^*$ into $(v-1)/6$ subsets each of the form $\{a, b, a+b, v-a, v-b, v-(a+b)\}$. Equivalently we seek a partition of the integers $\{1, 2, \ldots, 3s\}$ into s triples $\{a, b, c\}$ where either $a+b = c$ or $a+b+c \equiv 0 \pmod{v}$. Thus as an example for $s = 3$ such a partition is given by the equations $1 + 4 = 5$, $2 + 6 = 8$, $3+7+9 \equiv 0 \pmod{19}$ and starter blocks for a CSTS(19) under the action of the mapping $i \mapsto i+1 \pmod{19}$ are $\{0, 1, 5\}$ or $\{0, 4, 5\}$, $\{0, 2, 8\}$ or $\{0, 6, 8\}$, $\{0, 3, 10\}$ or $\{0, 7, 10\}$. Alternative choices for the starter blocks can, and indeed often do, give non-isomorphic systems. The problem of partitioning the set $\{1, 2, \ldots, 3s\}$ into s triples $\{a, b, c\}$ with $a + b = c$ or $a + b + c \equiv 0 \pmod{v}$ is known as *Heffter's first difference problem*, $\mathrm{HDP}_1(s)$ [54].

For $v = 6s + 3$, a cyclic system must contain the short orbit generated from the starter block $\{0, v/3, 2v/3\}$. By the same argument as in the previous paragraph, starter blocks for the other orbits can be obtained from a similar partition of the integers $\{1, 2, \ldots, 3s + 1\} \setminus \{2s + 1\}$. For example for $v = 15$ we have $1 + 3 = 4$, $2 + 6 + 7 \equiv 0 \pmod{15}$ giving starter blocks $\{0, 1, 4\}$ or $\{0, 3, 4\}$, $\{0, 2, 8\}$ or $\{0, 6, 8\}$, $\{0, 5, 10\}$. This is *Heffter's second difference problem*, $\mathrm{HDP}_2(s)$.

Solutions to both of Heffter's difference problems, except for $\mathrm{HDP}_2(1)$ for which no solution exists, were first given by Peltesohn [77], and are reproduced below in condensed form.

$v = 18s + 1, s \geqslant 2$
$(3i + 1, 4s - i + 1, 4s + 2i + 2)$ $\quad 0 \leqslant i \leqslant s - 1$
$(3i + 2, 8s - i, 8s + 2i + 2)$ $\quad 0 \leqslant i \leqslant s - 1$
$(3i + 3, 6s - 2i - 1, 6s + i + 2)$ $\quad 0 \leqslant i \leqslant s - 2$
$(3s, 3s + 1, 6s + 1)$

$v = 18s + 7, s \geqslant 1$
$(3i + 1, 8s - i + 3, 8s + 2i + 4)$ $\quad 0 \leqslant i \leqslant s - 1$
$(3i + 2, 6s - 2i + i, 6s + i + 3)$ $\quad 0 \leqslant i \leqslant s - 1$
$(3i + 3, 4s - i + 1, 4s + 2i + 4)$ $\quad 0 \leqslant i \leqslant s - 1$
$(3s + 1, 4s + 2, 7s + 3)$

$v = 18s + 13, s \geqslant 1$
$(3i + 1, 4s - i + 3, 4s + 2i + 4)$ $\quad 0 \leqslant i \leqslant s$
$(3i + 2, 6s - 2i + 3, 6s + i + 5)$ $\quad 0 \leqslant i \leqslant s - 1$
$(3i + 3, 8s - i + 5, 8s + 2i + 8)$ $\quad 0 \leqslant i \leqslant s - 1$
$(3s + 2, 7s + 5, 8s + 6)$

$v = 18s + 3, s \geqslant 1$
$(3i + 1, 8s - i + 1, 8s + 2i + 2)$ $\quad 0 \leqslant i \leqslant s - 1$
$(3i + 2, 4s - i, 4s + 2i + 2)$ $\quad 0 \leqslant i \leqslant s - 1$
$(3i + 3, 6s - 2i - 1, 6s + i + 2)$ $\quad 0 \leqslant i \leqslant s - 1$

$v = 18s + 9, s \geqslant 4$
$(3i + 1, 4s - i + 3, 4s + 2i + 4) \quad 0 \leqslant i \leqslant s$
$(3i + 2, 8s - i + 2, 8s + 2i + 4) \quad 2 \leqslant i \leqslant s - 2$
$(3i + 3, 6s - 2i + 1, 6s + i + 4) \quad 1 \leqslant i \leqslant s - 2$
$(2, 8s + 3, 8s + 5)$
$(3, 8s + 1, 8s + 4)$
$(5, 8s + 2, 8s + 7)$
$(3s - 1, 3s + 2, 6s + 1)$
$(3s, 7s + 3, 8s + 6)$

$v = 18s + 15, s \geqslant 1$
$(3i + 1, 4s - i + 3, 4s + 2i + 4) \quad 0 \leqslant i \leqslant s$
$(3i + 2, 8s - i + 6, 8s + 2i + 8) \quad 0 \leqslant i \leqslant s$
$(3i + 3, 6s - 2i + 3, 6s + i + 6) \quad 0 \leqslant i \leqslant s - 1$

The above leaves the values $v = 7, 13, 15, 19, 27, 45, 63$ still to be done but we leave these as exercises for the reader. In case of difficulty see pages 31 and 32 of C&R.

We can therefore state the following theorem.

Theorem 4.3. *There exists a cyclic STS*(v) *for all* $v \equiv 1, 3 \pmod 6$ *except* $v = 9$. □

A restricted form of Heffter's first difference problem was considered by Skolem. In [87] he introduced the problem of partitioning the set $\{1, 2, \ldots, 2s\}$ into ordered pairs $(a_i, b_i), i = 1, 2, \ldots s$, such that $b_i - a_i = i$. An example for s=4 is $(6, 7)$, $(1, 3)$, $(2, 5)$, $(4, 8)$ which is usually more succinctly represented as 23243114 and called a *Skolem sequence*. Given a Skolem sequence then the set of triples $\{(i, s + a_i, s + b_i) : 1 \leqslant i \leqslant s\}$ is a solution of $\mathrm{HDP}_1(s)$. So the above example yields the solution $1 + 10 = 11$, $2 + 5 = 7$, $3 + 6 = 9$, $4 + 8 = 12$.

Skolem proved that the sequences, which he called $1, +1$ systems, exist if and only if $s \equiv 0, 1 \pmod 4$. In a second paper [88] he pointed out that for $s \equiv 2, 3 \pmod 4$, if it could be proved that the set $\{1, 2, \ldots, 2s - 1, 2s + 1\}$ could be similarly partitioned then this too would yield a solution to $\mathrm{HDP}_1(s)$. An example for $s = 6$ is 11345364252*6 and these are known as *hooked Skolem sequences*. Their existence was determined by O'Keefe [75]. Details of the construction of both Skolem and hooked Skolem sequences are given below.

$v = 4s$

$(i, 4s-1-i)$	$1 \leqslant i \leqslant s-1$
$(s+2+i, 3s-1-i)$	$0 \leqslant i \leqslant s-3$
$(4s+i, 8s-i)$	$0 \leqslant i \leqslant 2s-1$
$(s, s+1), (2s, 4s-1), (2s+1, 6s)$	

$v = 4s+1$

$(i, 4s+1-i)$	$1 \leqslant i \leqslant s$
$(s+2+i, 3s+1-i)$	$1 \leqslant i \leqslant s-2$
$(4s+2+i, 8s+2-i)$	$0 \leqslant i \leqslant 2s-1$
$(s+1, s+2), (2s+1, 6s+2)$	
$(2s+2, 4s+1)$	

$v = 4s+2$

$(i, 4s+2-i)$	$1 \leqslant i \leqslant 2s$
$(4s+3+i, 8s+4-i)$	$1 \leqslant i \leqslant s-1$
$(5s+2-i, 7s+3-i)$	$1 \leqslant i \leqslant s-1$
$(2s+1, 6s+2), (4s+2, 6s+3)$	
$(4s+3, 8s+5), (7s+3, 7s+4)$	

$v = 4s-1$

$(i, 4s-1-i)$	$1 \leqslant i \leqslant s-1$
$(s+1+i, 3s-i)$	$1 \leqslant i \leqslant s-2$
$(4s+i, 8s-2-i)$	$1 \leqslant i \leqslant 2s-2$
$(s, s+1), (2s, 4s-1)$	
$(2s+1, 6s-1), (4s, 8s-1)$	

Another type of automorphism is 1-rotational. This is an automorphism which consists of a $(v-1)$-cycle together with a fixed point. Usually systems having such an automorphism are represented on a base set $V = \mathbb{Z}_{v-1} \cup \{\infty\}$ with the automorphism generated by the mapping $i \mapsto i+1 \pmod{v-1}$ and fixing the point ∞. In [79] Phelps and Rosa proved the following.

Theorem 4.4. *A 1-rotational $STS(v)$ exists if and only if $v \equiv 3, 9 (\mathrm{mod}\, 24)$.*

Proof. We first prove necessity. Consider orbits of pairs of elements under the automorphism. There is one half-orbit generated from the starter block $\{0, (v-1)/2\}$ and $(v-1)/2$ full orbits. Now consider the orbit of triples generated from the starter block $\{\infty, 0, \alpha\}$. It also contains the block $\{\infty, \alpha, 2\alpha\}$. Thus $\alpha = (v-1)/2$ and this is a half-orbit which contains the half-orbit of pairs and full orbit containing the point ∞. There are $(v-3)/2$ orbits of pairs remaining. If $v \equiv 1 \pmod 6$, there is a third-orbit of triples generated from the starter block $\{0, (v-1)/3, 2(v-1)/3\}$. This contains the orbit of pairs generated from $\{0,(v-1)/3\}$ with the other $(v-5)/2$ orbits of pairs appearing in full orbits of triples. But this is impossible since $(v-5)/2$ is not divisible by 3. If $v \equiv 3 \pmod 6$, then $(v-3)/6$ full orbits of triples are required to complete the system. Thus $v \equiv 3, 9, 15, 21 \pmod{24}$.

Now consider the set of pairs $S = \{\{x,y\} : 0 \leqslant x < (v-1)/2, (v-1)/2 \leqslant y < v-1\}$. The cardinality of S is $(v-1)^2/4$ and $(v-1)/2$ of the pairs occur in the orbit generated from $\{\infty, 0, (v-1)/2\}$. This leaves $(v-1)(v-3)/4$ pairs. Now every block in the rest of the system contains either none or two pairs from S. Moreover the blocks occur in pairs: if $\{a,b,c\}$ is a block then so is $\{a+(v-1)/2, b+(v-1)/2, c+(v-1)/2\}$. Hence $(v-1)(v-3)/4$ must be divisible by 4 which eliminates the cases $v \equiv 15, 21 \pmod{24}$.

To prove sufficiency put $v = 6t+3$ where $t = 4s$ or $4s+1$. Then there exists a Skolem sequence of order t, $(a_i, b_i), i = 1, 2, \ldots t$. The following are then the starter blocks for a 1-rotational STS(v).

$$\{\infty, 0, (v-1)/2\} \cup \{\{0, i, t+b_i\} : 1 \leqslant i \leqslant t\}. \qquad \square$$

The concept of 1-rotational can be generalized. A Steiner triple system, STS(v), is k-rotational if it admits an automorphism consisting of k cycles of length $(v-1)/k$ together with a fixed point. In the same paper [79] in which they determined the spectrum of 1-rotational Steiner triple systems, Phelps and Rosa also proved the following.

Theorem 4.5. *A 2-rotational STS(v) exists if and only if* $v \equiv 1, 3, 7, 9, 15, 19 \pmod{24}$. $\square$

Cho [10] then determined the spectrum of 3-rotational and 4-rotational systems.

Theorem 4.6. *A 3-rotational STS(v) exists if and only if* $v \equiv 1, 19 \pmod{24}$.

Theorem 4.7. *A 4-rotational STS(v) exists if and only if* $v \equiv 1, 9, 13, 21 \pmod{24}$. $\square$

A particularly interesting case is when $k = (v-1)/2$, i.e., the automorphism consists of an involution fixing one element. Such systems are called *reverse* Steiner triple systems and in fact were studied before general rotational systems. The combined work of Doyen [25], Rosa [85], and Teirlinck [91] gives the following result.

Theorem 4.8. *A reverse STS(v) exists if and only if* $v \equiv 1, 3, 9, 19 \pmod{24}$.

However the ultimate result in this area is due to Colbourn & Jiang [15] who determined the spectrum of k-rotational STS(v) for all k with $1 \leqslant k \leqslant (v-1)/2$. Their result is given in the next theorem.

Theorem 4.9. *A k-rotational STS(v) exists if and only if*

1. $v \equiv 3 \pmod 6$ *if* $k = 1$, *and*

2. $v \equiv 1 \pmod k$, *and*

3. $v \not\equiv 7, 13, 15, 21 \pmod{24}$ *if* $(v-1)/k$ *is even.* □

Various other automorphism types have also been considered which space does not allow to be discussed here. But particular mention should be made of the work of Calahan and Gardner, further details of which are in Section 7.4 on pages 134 to 140 of C&R and the relevant papers in the Bibliography. Mendelsohn [69], [70] proved that every abstract group is the automorphism group of some Steiner triple system.

Finally in this section it is probably appropriate to ask about Steiner triple systems which have only the identity automorphism, so-called *automorphism-free* systems. There are none of orders 7, 9, and 13 but 36 of the 80 STS(15)s and all but 164,758 of the 11,084,874,829 STS(19)s are automorphism-free. The question was considered by Lindner & Rosa [63] who constructed automorphism-free systems for $v = 15, 19, 21, 25, 27, 33$ and then used various "doubling" constructions, including the STS(v) $\Longrightarrow$ STS($2v+1$) construction with the one-factorization GK_{v+1} described in Section 2, to complete the spectrum.

Theorem 4.10. *An automorphism-free STS(v) exists if and only if* $v \equiv 1, 3 \pmod 6$ *and* $v \geqslant 15$. □

Babai [3] in fact proved that almost all Steiner triple systems are automorphism-free.

5. Configurations

In the context of a Steiner triple system, a *configuration* is simply a small number of blocks which may appear in the system. Perhaps the first question to ask therefore is for given n, the number of blocks, how many non-isomorphic configurations are there? Trivially when $n = 1$ there is just one, a single block, and when $n = 2$ there are two, a pair of parallel blocks (denoted by A_1) and a pair of blocks intersecting in a common point (denoted by A_2). Denoting the number of non-isomorphic configurations with n blocks by $\mathrm{C}(n)$, it is also easy to work out that $\mathrm{C}(3) = 5$ and these are shown in Figure 5.1.

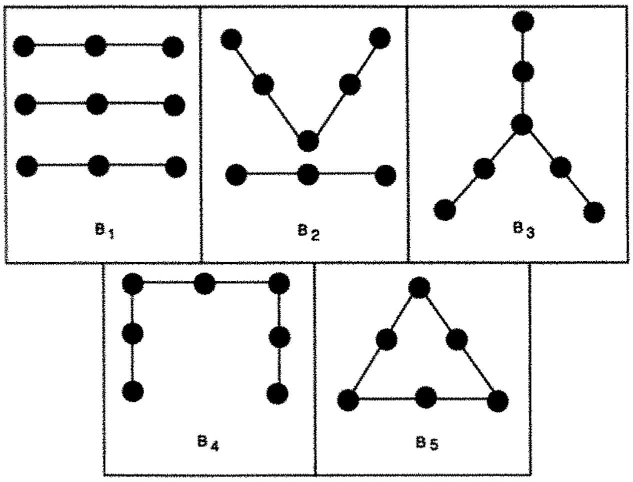

Fig. 5.1. 3-block configurations.

A nice exercise for a student is to determine the value of C(4). It is 16 and these are illustrated in Figure 5.2. Beyond this a computer is needed and the values for $5 \leqslant n \leqslant 10$ are given below. There seems to be no known formula to determine these values.

n	5	6	7	8	9	10
$\mathrm{C}(n)$	56	282	1865	17100	207697	3180571

Of more interest is counting the number of occurrences of each configuration in an STS(v), a study of which was initiated in [41]. For a single block this is $v(v-1)/6$ and for A_2, a pair of intersecting blocks, is $v \times \binom{r}{2} = v(v-1)(v-3)/8$ where $r = (v-1)/2$ is the replication number, i.e. number of blocks through any given point. For A_1, first note that given any block of an STS(v), there are $v(v-1)/6 - 3(v-3)/2 - 1 = (v-3)(v-7)/6$ disjoint blocks. The number of occurrences of A_1 therefore is $(v(v-1)/6 \times (v-3)(v-7)/6)/2 = v(v-1)(v-3)(v-7)/72$. Without too much difficulty, by reasoning along the same lines, formulae for the five 3-block configurations can be obtained. These are given below where b_1 is the number of occurrences of B_i.

$$b_1 = v(v-1)(v-3)(v-7)(v^2-19v+96)/1296$$
$$b_2 = v(v-1)(v-3)(v-7)(v-9)/48$$
$$b_3 = v(v-1)(v-3)(v-4)/48$$
$$b_4 = v(v-1)(v-3)(v-7)/8$$
$$b_5 = v(v-1)(v-3)/6$$

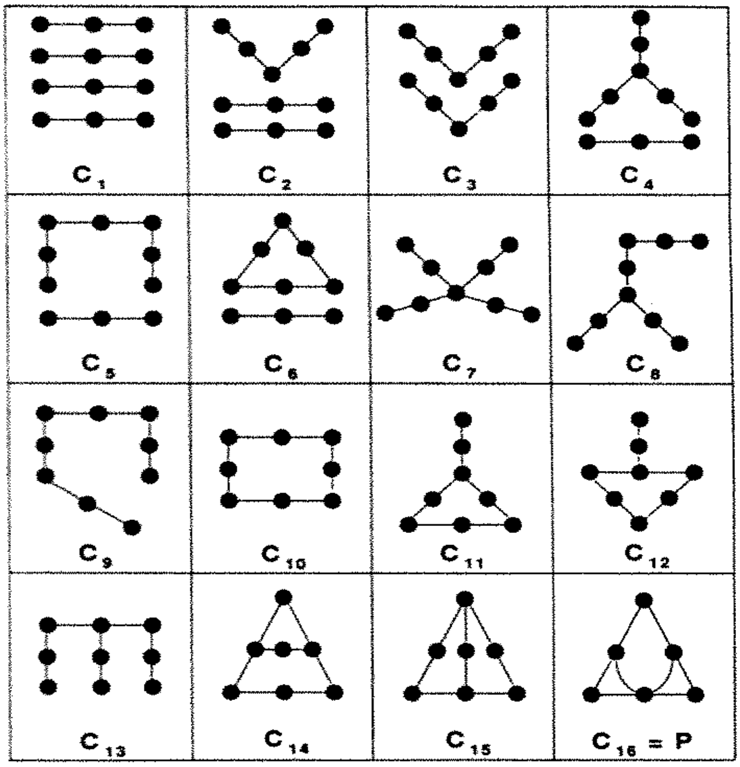

Fig. 5.2. 4-block configurations.

So, for any given $v \equiv 1, 3 \pmod 6$, the number of occurrences of every 1, 2, and 3-block configuration is the same in all Steiner triple systems of that order. But at 4-block configurations the situation changes. The configuration C_{16} now plays a key role. It is the "tightest" of the 4-block configurations having only 6 points and is more usually known as a *quadrilateral* or *Pasch configuration*, P. Recall that there are two non-isomorphic STS(13)s. One of these contains 13 Pasch configurations and the other contains 8. So the formulae for the number of occurrences of 4-block configurations in an

STS(v) cannot all be functions of v. This leads to the following definitions. A configuration will be called *constant* if the formula for its number of occurrences in an STS(v) is a function of v; otherwise it is called *variable.* In fact only 5 of the 4-block configurations are constants and 11 are variable. Formulae for these configurations were first given in [41]. They are repeated below, where again we adopt the convention that c_i is the formula for C_i. All can be expressed in terms of the order v of the Steiner triple system and the number of Pasch configurations p in the particular STS(v). We write n_v for $v(v-1)(v-3)$.

$$
\begin{aligned}
c_1 &= n_v(v-9)(v-10)(v-13)(v^2-22v+141)/31104 + p\\
c_2 &= n_v(v-9)(v-10)(v^2-22v+129)/576 - 6p\\
c_3 &= n_v(v-9)^2(v-11)/128 + 3p\\
c_4 &= n_v(v-7)(v-9)(v-11)/288\\
c_5 &= n_v(v-9)(v^2-20v+103)/48 + 12p\\
c_6 &= n_v(v-9)(v-10)/36 - 4p\\
c_7 &= n_v(v-5)(v-7)/384\\
c_8 &= n_v(v-7)(v-9)/16\\
c_9 &= n_v(v-9)^2/8 - 12p\\
c_{10} &= n_v(v-8)/8 + 3p\\
c_{11} &= n_v(v-7)/4\\
c_{12} &= n_v(v-9)/4 + 12p\\
c_{13} &= n_v(v^2-18v+85)/48 - 4p\\
c_{14} &= n_v/4 - 6p\\
c_{15} &= n_v/6\\
c_{16} &= p
\end{aligned}
$$

Of course the number of occurrences of all of the variable configurations can be expressed in terms of the order v and the number of occurrences of any one of them. However the Pasch configuration is the most natural for a number of reasons which will become clearer later. As well as having the least number of points of all the 4-block configurations, observe that it is also the only n-block configuration, $1 \leqslant n \leqslant 4$, in which every point has degree at least 2. These formulae immediately raise two interesting and significant areas of investigation.

The first is to identify, for each n, an easily described subset of configurations such that for admissible v the number of occurrences of any n-block configuration in an STS(v) can be expressed in terms of v and the number of occurrences of each member of the subset. This idea was considered by Horák, Phillips, Wallis and Yucas [55]. They make the following definitions.

Definitions. A *generating set* M for n-block configurations is a set of m-block configurations, $1 \leqslant m \leqslant n$, such that the number of occurrences of any n-block configuration can be expressed as a linear combination of the number of occurrences of the configurations in M, where the coefficients are polynomials in v. A *basis* is a minimal generating set.

So using this terminology, the single block is a basis for 1-, 2-, and 3-block configurations and the single block and the Pasch configuration form a basis for 4-block configurations. The main result in [55] is the following important theorem.

Theorem 5.1. *The single block, together with all m-line configurations, $1 \leqslant m \leqslant n$, having all points of degree at least 2, form a generating set for the n-line configurations in a Steiner triple system.* □

The only 5-block configuration having all points of degree at least 2 is the so-called *mitre*, shown in Figure 5.3. Formulae for the number of occurrences of 5-block configurations are given in [21], with minor corrections in [39]. Already these are becoming complex. For example, that for 5 non-intersecting (parallel) blocks, where m is the number of mitres is

$$v(v-1)(v-3) \times (v^7 - 91v^6 + 3588v^5 - 79510v^4 + 1069873v^3 - 8742231v^2 + 40167162v - 80101224)/933120 + (v-16)(v-21)p/6 + 2m$$

There are five 6-block, nineteen 7-block, and 153 8-block configurations having all points of degree at least 2 and formulae for the number of occurrences of the 6-block, 7-block, and 8-block configurations are given on the website [32]. In all of these cases it is known that the generating set is also a basis but in general this is not proved. Indeed, Horák, Phillips, Wallis and Yucas make the following conjecture.

Conjecture 5.2. *The single block, together with all m-line configurations, $1 \leqslant m \leqslant n$, having all points of degree at least 2, form a basis for the n-line configurations in a Steiner triple system.*

The second area is to answer the question: what are the constant configurations? There seems to be little doubt what the answer to this is, though proving it certainly doesn't appear easy and may in fact be quite difficult. Define an *n-star* to be an n-block configuration in which all n blocks intersect at a common point called the *centre*. The following conjecture is also made in [55].

Conjecture 5.3. *For $n \geqslant 4$, an n-block configuration in a Steiner triple system is constant if and only if it can be obtained from the $(n-1)$-star by adjoining a block.*

In general this can be done in precisely five ways. The "adjoined block" can be disjoint from the $(n-1)$-star, intersect at the centre or intersect at one, two, or three points. The proof that these configurations are constant is straightforward, and formulae are given in [55]. Note that the conjecture is not true for $n < 4$. The configuration B_1, three non-intersecting blocks, is the sole exception.

A third conjecture was also made in [39]. It is easily verified that the four 3-block configurations obtained by removing each of the four blocks in turn from a 4-block configuration uniquely determine the 4-block configuration, and the same is true for the five 4-block configurations obtained from a 5-block configuration.

Conjecture 5.4. *Every n-block configuration, $n \geqslant 4$, is uniquely characterized by the n configurations on $n-1$ blocks, each of which is obtained by removing a single block from the given n-block configuration.*

Again note that the conjecture is not true for the 2-block or 3-block configurations (both B_3 and B_5 give three pairs of intersecting blocks). Given that this conjecture is analogous to the graph reconstruction conjecture, this too may be difficult to prove.

Another important topic is that of avoidance. In 1973, Erdős [30] conjectured that for every integer $k \geqslant 4$, there exists $v_0(k)$ such that if $v > v_0(k)$ and if v is admissible, then there exists an STS(v) with the property that it contains no configuration having n blocks and $n+2$ points for any n satisfying $4 \leqslant n \leqslant k$. Such an STS$(v)$ is said to be k-*sparse*. Clearly, a k-sparse system is also k'-sparse for every k' satisfying $4 \leqslant k' \leqslant k$. The reason why configurations having two more points than blocks form the focus of the conjecture lies in the following theorem and its corollary which are formally proved in [33].

Theorem 5.5. *Suppose that $n \geqslant 2$ and that v is admissible with $v \geqslant n+3$. Then any STS(v) contains a configuration having n blocks and $n+3$ points.* □

Corollary 5.6. *For every integer $d \geqslant 3$ and for every integer n satisfying $n \geqslant \lceil d/2 \rceil$ there exists $v_0(n,d)$ such that for all admissible $v \geqslant v_0(n,d)$, every STS(v) contains a configuration having n blocks and $n+d$ points.* □

So a 4-sparse STS(v) is just one which contains no Pasch configurations. Such systems are more commonly known as *anti-Pasch*. But constructing these systems is not straightforward. The Bose construction gives a good start. As was observed by Doyen [26], when G is the cyclic group of order $2s+1$, the construction yields an anti-Pasch STS(v), whenever $v = 6s+3$ is not divisible by 7. The case when v is divisible by 7 was resolved by Brouwer [7]. The case where $v \equiv 1 \pmod 6$ seems to be much harder and is based on work contained in two papers [65] and [46]. The definitive result is as follows.

Theorem 5.7. *There exists an anti-Pasch STS(v) for all $v \equiv 1, 3 (\mathrm{mod}\, 6)$ except $v = 7, 13$.* □

There are two configurations with 5 blocks and 7 points. One is the *mia* (Fano arrow or Farrow), shown in Figure 5.3, obtained by extending the Pasch configuration with an extra block through any of the three pairs of uncovered points. So systems avoiding the mia are the same as anti-Pasch systems. The other configuration is the mitre. So 5-sparse systems are those which are both anti-Pasch and anti-mitre. But first, Colbourn, Mendelsohn, Rosa and Širáň [16] considered systems which were just anti-mitre. They showed that these exist for all $v \equiv 3, 7, 9, 19, 21, 27 \pmod{36}$. The proof uses both the Bose construction and the standard "doubling" construction STS(v) $\Longrightarrow$ STS($2v+1$) with the one-factorization based on the STS(v). They also pointed out that the Netto systems are anti-mitre. Combined with the result of Robinson [84] that Netto systems STS(p^n) are also anti-Pasch if and only if $p \equiv 19 \pmod{24}$, this gives an infinite class of 5-sparse Steiner triple systems. The spectrum was extended by Ling [64] who proved that if there exists a transitive anti-mitre (resp. 5-sparse) STS(v), $v \equiv 1 \pmod 6$, (and the Netto systems are transitive), and an anti-mitre (resp. 5-sparse) STS(w), (including $w = 3$), then there exists an anti-mitre (resp. 5-sparse) STS(vw). Further work by Fujiwara [35],[36] and Wolfe [98] finally established the definitive result for anti-mitre systems.

Theorem 5.8. *There exists an anti-mitre STS(v) for all $v \equiv 1, 3 \,(\mathrm{mod}\, 6)$ except $v = 9$.* □

With regard to 5-sparse systems Wolfe has proved that these exist for "almost all" admissible v (meaning arithmetic set density 1 in the set of all admissible orders) [97] and for all $v \equiv 3 \pmod 6$ with $v \geqslant 21$ [99]. For 6-sparse STS(v), as well as the Pasch configuration and the mitre, the systems also have to avoid two further configurations, the *crown* and the *6-cycle*, also shown in Figure 5.3. In two papers [33], [34], Forbes, Grannell and Griggs gave constructions for infinite classes of these. In particular they are are known to exist for all sufficiently large v of the form $3p$, p prime and $p \equiv 3 \pmod 4$. There is no known 7-sparse STS(v).

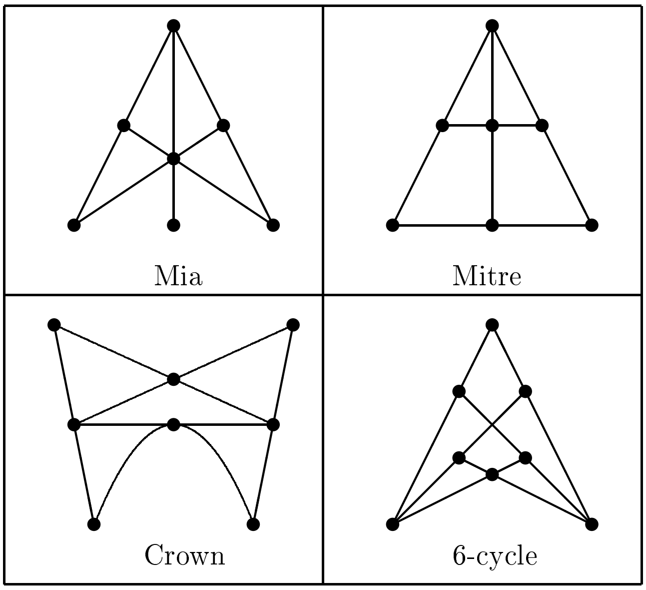

Fig. 5.3 Some 5 and 6-block configurations.

6. Isomorphism testing

Given two Steiner triple systems, $(V, \mathcal{B})$ and $(W, \mathcal{D})$, of the same order v, a fundamental question is whether they are isomorphic and how to determine this? Clearly to examine all $v!$ bijections from V to W is not possible. A different approach is needed. We therefore define an *invariant* to be any property of an STS(v), $(V, \mathcal{B})$, which remains fixed under all possible $v!$ permutations of the base set V. Then if the invariants of two Steiner triple systems differ they are non-isomorphic though of course if they are the same no conclusion can be drawn. Trivially, the number of blocks $b = v(v-1)/6$ is such an invariant but this would be totally useless in determining isomorphism as would any constant configuration. However a variable configuration will be of use. The first candidate is the Pasch configuration P and we already noted in the last section that the number of these in the two STS(13)s differ.

For the 80 STS(15)s, the number of Pasch configurations in each of the systems is given on pages 65 to 69 of C&R. It varies from 105 for the projective STS(15) to 0 for the unique anti-Pasch system of this order. In total there are 27 different values but only 8 of these, $105, 73, 57, 32, 15, 11, 2, 0$, occur as the number of Pasch configurations of just one STS(15). At the other extreme there are seven systems with both 7 and 6 Pasch configurations and six systems with 10, 9, and 8 Pasch configurations. Nevertheless some progress has been made and further tests, such as counting the numbers of other variable configurations such as the mitre and the crown, can be applied to try to distinguish the systems further. However, in addition to simply just counting the number of Pasch configurations, other statistics can also be compiled. For any STS(v), $(V, \mathcal{B})$, and any variable configuration C, let $n(C)$ be the number of occurrences of the configuration in the STS(v). Further for each point $x \in V$ and block $B \in \mathcal{B}$, let $n(C, x)$ and $n(C, B)$ be the number of configurations C in which the point x and the block B respectively, appear. The *point-configuration vector* is then defined as the vector $(x_0, x_1, \ldots, x_{n(C)})$ where x_i is the cardinality of the set $\{x \in V : n(C, x) = i\}$, i.e. the number of points in the system which occur in precisely i configurations. The *block-configuration vector* is defined analogously. These two vectors give much more information and in fact are sufficient to identify individual STS(15)s.

So we have a general strategy. First compute the point-configuration and block-configuration vectors of variable configurations for the two Steiner triple systems under consideration. Any difference implies that the systems are non-isomorphic. If not, so that one suspects that the two systems may be isomorphic, then the information obtained can be used to determine the isomorphism. As a simple example there exist STS(19)s containing just one Pasch configuration, in fact 35,758 of them [13]. Therefore 6 points occur in one Pasch configuration and 13 points in no Pasch configuration. So if we wish to test whether two such systems are isomorphic this simple observation immediately reduces the number of possible bijections from 19! to $6! \times 13!$, a saving in the computational effort by a factor of over 25,000. (In fact, since the automorphism group of the Pasch configuration has order 24, this can be reduced further to $24 \times 13!$). Further tests can then be applied to reduce this number further until all the remaining possibilities can be tested individually.

Another very useful invariant is cycle structure. Let $(V, \mathcal{B})$ be an STS(v). For each pair $x, y \in V$, define a graph $G_{x,y}$ with vertex set $V \setminus \{x, y, z\}$ where $\{x, y, z\} \in \mathcal{B}$ with two vertices u, v being joined by an edge if either $\{x, u, v\}$ or $\{y, u, v\} \in \mathcal{B}$. The graph $G_{x,y}$ is a union of cycles of even length greater than 2 and these can be recorded as a list of cycle lengths in ascending order. The *cycle structure* is the collection of all such *cycle lists*. The idea for this invariant goes back to the work of White, Cole and Cummings [94] on the enumeration of STS(15)s where a cycle list is called a *type of interlacing*. It completely distinguishes non-isomorphic STS(15)s. It is worth noting that the number of Pasch configurations can also be computed from cycle structure. A Pasch configuration, say with blocks $\{a, b, c\}, \{a, y, z\}, \{x, b, z\}, \{x, y, c\}$, will occur as a 4-cycle in the graphs $G_{a,x}, G_{b,y}, G_{c,z}$. So adding the total number of 4-cycles in the cycle structure and dividing by 3 will give the number of Pasch configurations.

Before leaving cycle structure it is perhaps appropriate to take a little detour. Steiner triple systems in which all cycle lists are the same are called *uniform* and are of some interest. The projective, Hall, and Netto systems are uniform but apart from these little is known. In [42], Grannell, Griggs and Murphy constructed uniform STS(v) for $v = 43, 13063$, and 34303 with all cycle lists $4, v - 7$. A further uniform STS(v) with $v = 180907$ is given in [33] with cycle lists $4, 12, 180888$. Uniform Steiner triple systems in which each cycle list is $v - 3$ are called *perfect* and here only 14 systems are known. The smallest four are the unique STS(7), the unique STS(9), one of the three STS(25)s with automorphism group $\mathcal{Z}_5 \times \mathcal{Z}_5$ [92], and a cyclic STS(33). Then in [42], perfect systems of order, 79, $139, 367, 811, 1531, 25771, 50923, 61339, 69991$ were constructed and a further system of order 135859 was given in [33]. Unfortunately it is now known that the method used cannot yield an infinite class. Having no 4-cycles, perfect systems are anti-Pasch and those of order 79, 367, 811 are also 5-sparse whilst that of order 139 is 6-sparse, a very interesting Steiner triple system indeed.

Another invariant of a Steiner triple system is a directed graph known as the train. Let $(V, \mathcal{B})$ be an STS(v). Define a mapping f from the set of all 3-subsets of V to itself by $f(\{x, y, z\}) = \{a, b, c\}$ where $\{a, y, z\}, \{x, b, z\}$, $\{x, y, c\} \in \mathcal{B}$. The digraph which represents this mapping is the *train* of the STS(v). It comprises a number of components, all of which consist of a single directed cycle with pendant directed trees that are directed towards the cycle. If $\{a, b, c\} \in \mathcal{B}$ then the directed cycle will be a directed loop on the vertex; the only place where loops will occur. Further if $f(\{x,y,z\}) = \{a,b,c\}$ where $\{a,b,c\} \in \mathcal{B}$ then the four blocks $\{a,y,z\}, \{x,b,z\}, \{x,y,c\}, \{a,b,c\}$ are a Pasch configuration. So by computing the sum of the indegrees of all the vertices which represent blocks of the STS(v) and dividing by 4, the number of Pasch configurations can again be obtained. The idea of the train was developed by White [93] but is rather cumbersome to represent since the digraph has $v(v-1)(v-2)/6$ vertices. Accordingly, Colbourn, Colbourn and Rosenbaum [19] suggested using a summary of the information contained in the digraph. This is called the *compact train* and is defined as a set of ordered triples (m,n,p) where such a triple means that the train contains p components with m vertices, n of which have indegree zero (after discounting the directed cycle from each component). Trains also completely distinguish non-isomorphic STS(15)s and compact trains nearly do except that systems #6 and #7 both have compact train $(13,12,1)(13,10,18)(13,9,16)$. However the former has 37 Pasch configurations and the latter has 33.

The information in the train can also be summarized by the tricolour vector. This was introduced in [50] primarily as an invariant for one-factorizations of the complete graph but is applicable to Steiner triple systems. In the train, define v_i to be the number of vertices having indegree equal to i. The *tricolour vector* is then $(v_0, v_1, v_2, \ldots, v_m)$ where m is the maximum indegree, and the *tricolour number* is the value of v_0. The tricolour number varies from 420 for the projective STS(15) to 60 for the anti-Pasch STS(15). There are 62 different values occurring with 47 appearing once, 12 appearing twice and 3 appearing thrice. It is therefore a more discriminating invariant than counting Pasch configurations. The tricolour vectors do distinguish the STS(15)s completely; in fact the first three components are sufficient.

7. Group divisible designs

A natural generalization of a Steiner triple system is a group divisible design. Let S be a set of positive integers. A 3-*group divisible design*, usually denoted by 3-GDD, is an ordered triple $(V, \mathcal{G}, \mathcal{B})$ where V is a base set of cardinality v, $\mathcal{G}$ is a partition of V into parts, called *groups*, whose cardinality belongs to S, and $\mathcal{B}$ is a collection of triples or blocks which collectively have the property that every pair of elements from different groups is contained in precisely one triple and no pair of distinct elements from the same group occur in any triple. Alternatively, every pair of distinct elements occur in either a group or a block but not both. If the partition of V is into t_i groups of cardinality g_i, $i = 1, 2, \dots, n$ so that $v = \Sigma_{i=1}^{n} t_i g_i$, the 3-GDD is said to be of type $g_1^{t_1} g_2^{t_2} \dots g_n^{t_n}$. The use of the word "group" in this definition is perhaps misleading; it has nothing to do with Group Theory.

Example 7.1. Let $V = \{1, 2, 3, 4, A, B, C, D, E, F\}$ and $\mathcal{G}$ be the partition $\{1, 2, 3, 4\}, \{A, B\}, \{C, D\}, \{E, F\}$. Take the triples $\mathcal{B}$ to be 1AC, 1BE, 1CF, 2AD, 2BF, 2CE, 3AE, 3BD, 3CF, 4AF, 4BC, 4DE. Then $(V, \mathcal{G}, \mathcal{B})$ is a 3-GDD of type $4^1 2^3$. □

A Steiner triple system of order v is a 3-GDD of type 1^v. Further, by defining the sets of pairs through any chosen point as the groups and then deleting that point from the design, gives a 3-GDD of type $2^{(v-1)/2}$. For an STS$(6s+3)$ with a parallel class, by defining each block of the parallel class as a group, a 3-GDD of type 3^{2s+1} is obtained. A Latin square of side v is a 3-GDD of type v^3.

More generally, necessary and sufficient conditions for the existence of 3-GDDs in which every group has the same cardinality, i.e. of type g^t, are (1) $t \geqslant 3$, (2) $(t-1)g \equiv 0 \pmod 2$, (3) $t(t-1)g^2 \equiv 0 \pmod 3$, [53] or in tabular form as below.

Value of g	Value of t
1 or 5 (mod 6)	1 or 3 (mod 6)
2 or 4 (mod 6)	0 or 1 (mod 3)
3 (mod 6)	1 (mod 2)
0 (mod 6)	no constraint

Also of particular note are 3-GDDs in which all groups except one are of the same cardinality, i.e. of type $g^t u^1$. Necessary and sufficient conditions are the following [14].

1. if $g > 0$ then $t \geqslant 3$, or $t = 2$ and $u = g$, or $t = 1$ and $u = 0$, or $t = 0$,
2. $u \leqslant g(t-1)$ or $gt = 0$,
3. $g(t-1) + u \equiv 0 \pmod 2$ or $gt = 0$,
4. $gt \equiv 0 \pmod 2$ or $u = 0$,
5. $g^2 t(t-1)/2 + gtu \equiv 0 \pmod 3$.

The importance of group divisible designs is their use in a construction of Wilson [95]. The construction is applicable to group divisible designs having any block size but is presented here just in the context of 3-GDDs.

Wilson's fundamental construction

Let $(V, \mathcal{G}, \mathcal{B})$ be a 3-GDD (called the *master* GDD), and the partition $\mathcal{G}$ of V be $G_1, G_2, \ldots, G_t$. Further let w be a function (called a *weight function*) from the base set V to the set $\mathbb{Z}_0^+$ of non-negative integers which has the property that if $\{x, y, z\} \in \mathcal{B}$ then there exists a 3-GDD of type $w(x)w(y)w(z)$ (called a *slave* GDD). Then there exists a 3-GDD of type $\Sigma_{x\in G_1} w(x)\ \Sigma_{x\in G_2} w(x) \ldots \Sigma_{x\in G_t} w(x)$.

Wilson's construction has wide application throughout Design Theory and has been used in many creative ways. Below is just one example which, although quite straightforward, will hopefully give some idea of the power of the technique. In Section we introduced the Pasch configuration and discussed the existence of anti-Pasch STS(v); systems which contain no Pasch configurations. At another extreme it is perhaps appropriate to ask whether there exist STS(v) in which the blocks of the system can be partitioned into Pasch configurations. This is one of the questions considered in [49] and the relevant material, together with explanatory comments, are reproduced in the proof of the following theorem.

Theorem 7.2. *There exists an STS(v) which is decomposable into copies of the Pasch configuration if and only if $v \equiv 1, 9 \pmod{24}$.*

Proof. We first prove necessity. In order for an STS(v) to be decomposable into Pasch configurations, the number of blocks $b = v(v-1)/6$ must be divisible by 4. Hence $v \equiv 1, 9 \pmod{24}$.

The first possible value of v is therefore 25. If we choose $V = \mathbb{Z}_{25}$ then a cyclic STS(25) will consist of 4 orbits under the mapping $i \mapsto i + 1 \pmod{25}$. We seek such a system which contains a Pasch configuration with one block from each orbit. Then the action of the mapping will guarantee that the system decomposes into Pasch configurations. It is not too difficult to construct a system by hand. The one given in [49] has starter blocks $\{0, 1, 6\}$, $\{0, 2, 16\}$, $\{0, 3, 10\}$, $\{0, 4, 12\}$ and a Pasch configuration with one block from each orbit is $\{0, 1, 6\}$, $\{1, 3, 17\}$, $\{3, 6, 13\}$, $\{13, 17, 0\}$.

The second possible value of v is 33. We use the same approach as for $v = 25$, seeking a system with $V = \mathbb{Z}_{11} \times \{1, 2, 3\}$ with automorphism $i \mapsto i + 1 \pmod{11}$ acting on the first co-ordinate and leaving the second co-ordinate fixed. There are 16 orbits in all and starter blocks both for the

system and the decomposition into Pasch configurations are
$0_1 0_2 0_3$, $0_1 6_2 10_3$, $3_1 0_2 10_3$, $3_1 6_2 0_3$; $0_1 1_1 3_1$, $8_1 1_1 10_2$, $8_1 3_1 6_3$, $0_1 10_2 6_3$;
$0_2 1_2 4_2$, $6_2 1_2 3_3$, $6_2 4_2 10_1$, $0_2 3_3 10_1$; $0_3 7_3 5_3$, $8_3 7_3 6_1$, $8_3 5_3 10_2$, $0_3 6_1 10_2$.
This system was found by computer.

The above are two of the ingredients needed in applying Wilson's construction. The third is a 3-GDD of type 4^3 which is also decomposable into Pasch configurations. This will be the slave GDD. Let $\mathbb{K}_4$ be the Klein 4-group with elements $\{e, x, y, z\}$ where e is the identity. Let $V = \mathbb{K}_4 \times \{1, 2, 3\}$, with GDD partition $G_i = \mathbb{K}_4 \times \{i\}$, $i = 1, 2, 3$. The 3-GDD has 16 blocks which are generated from the Pasch configuration $e_1 e_2 e_3$, $e_1 y_2 z_3$, $x_1 e_2 z_3$, $x_1 y_2 e_3$ under the action of $\mathbb{K}_4$.

We can now use the construction. Take as the master GDD, a 3-GDD of type 6^t, $t \geqslant 3$, and weight every point with 4. Replace every block of the master GDD with the slave 3-GDD of type 4^3 above which can be decomposed into Pasch configurations. Each group of the master GDD is now expanded and has cardinality 24. Adjoin a further point ∞ and on every expanded group together with this point place the Pasch decomposable STS(25) constructed above. The result is a Pasch decomposable STS$(24t + 1)$.

For the case where $v = 24t + 9$ a different master GDD is needed, one in which one of the groups has different cardinality from the others. Take as the master GDD, a 3-GDD of type $6^t 8^1$, $t \geqslant 3$. Then just proceed as in the former case but on the expanded block of 32 points, together with the point ∞, place the Pasch decomposable STS(33). □

The reader will have noticed that in fact we have not quite proved the theorem. The master GDDs used exist only for $t \geqslant 3$. We already have Pasch decomposable STS(25) and STS(33) but this still leaves the two values $v = 49$ and $v = 81$. This is a common feature of GDD constructions; often small values "fall through the net" and have to be considered individually. Nevertheless we have proved the existence of Pasch decomposable STS(v) for all $v \equiv 1, 9 \pmod{24}$ with only two exceptions and we have done this by building the systems from just three basic ingredients. Using Wilson's construction has enabled us to assemble these ingredients to obtain what we require. A Pasch decomposable STS(49) was also given in [49] but no such system for $v = 81$. So the theorem was proved for all $v \equiv 1, 9$ (mod 24) except possibly $v = 81$. Of course the authors of the paper did not believe that this was a genuine exception; just that the method used was unable to deal with this value. Sometimes filling in the "missing" values to complete the spectrum can be the most difficult part of the proof. Often it is necessary to embark on a lengthy computer search which can present a significant challenge. To complete this story, a Pasch decomposable STS(81) does exist; it was found later by Adams, Billington and Rodger [1].

Before leaving this topic, it is worth noting that with little extra work we can prove that the systems constructed in the above theorem not only partition into Pasch configurations, they also partition into sets of four parallel blocks, i.e. configurations C_1 in Figure 5.2. All that is needed to do is to partition the three ingredients used, STS(25), STS(33), and 3-GDD of type 4^3 into configurations C_1. It is completely straightforward and is left as an exercise for the reader. More information about decomposing Steiner triple systems into configurations can be found in the papers [1], [47], [49], [56].

8. Mendelsohn and directed triple systems

The blocks of a Steiner triple system are unordered. In this section we consider the situation where order is introduced. There are two possibilities. A *cyclic* triple, which will be denoted by (x, y, z), contains the ordered pairs (x, y), (y, z), (z, x) and a *transitive* triple, denoted by $[x, y, z]$ contains the ordered pairs (x, y), (y, z), (x, z). Systems of cyclic triples were the first to be considered, by Mendelsohn [71], and very appropriately are named after him. Thus a *Mendelsohn triple system* of order v, usually denoted by MTS(v), is an ordered pair $(V, \mathcal{B})$ where V is a base set of cardinality v and $\mathcal{B}$ is a collection of cyclic triples which collectively have the property that every ordered pair of distinct elements of V is contained in precisely one cyclic triple. An elementary counting argument establishes that a necessary condition for the existence of an MTS(v) is $v \equiv 0, 1 \pmod 3$ and systems do exist for all of these orders except that there is no MTS(6).

An MTS(3) on base set $\{a, b, c\}$ consists of the two triples (a, b, c) and (c, b, a). An MTS(4) on base set $\{a, b, c, d\}$ has triples (a, b, c), (d, b, a), (c, d, a), (d, c, b). They are the unique Mendelsohn triple systems of these orders. There are three non-isomorphic Mendelsohn triple systems of order 7 detailed in the example below.

Example 8.1. All three systems will be defined on base set $V = \mathbb{Z}_7$.
System #1: Develop the triples $(0, 1, 3)$ and $(0, 3, 1)$ under the action of the mapping $i \mapsto i + 1 \pmod 7$.
System #2: Develop the triples $(0, 1, 3)$ and $(0, 3, 2)$ under the action of

the mapping $i \mapsto i+1 \pmod 7$.
SYSTEM #3: The triples are $(0,1,2)$, $(0,2,1)$, $(0,3,4)$, $(0,4,3)$, $(0,5,6)$, $(0,6,5)$, $(1,3,5)$, $(1,4,6)$, $(1,5,4)$, $(1,6,3)$, $(2,3,6)$, $(2,4,5)$, $(2,5,3)$, $(2,6,4)$. □

The numbers of non-isomorphic MTS(v) for $v = 9, 10, 12$ are 18 [68], 143 [37] [38], 4,905,693 [23]. The Mendelsohn triple systems of order 9 are listed on pages 533 and 534 of HB. A further isomorphism invariant is available for Mendelsohn triple systems. For a Steiner triple system, $(V, \mathcal{B})$, the *neighbourhood* of a point $x \in V$ is the set $N(x) = \{\{u,v\} : \{x,u,v\} \in \mathcal{B}\}$. Cycle structure can then be thought of as the graphs obtained from all *double neighbourhoods*, i.e. $N(x) \cup N(y)$, $x, y, \in V$, $x \neq y$. However for a Mendelsohn triple system, the neighbourhood of a point will be a set of ordered pairs, which form a collection of directed cycles. The set of all these single neighbourhoods is an invariant of an MTS(v).

Two recursive constructions for Mendelsohn triple systems are given on pages 442 and 443 of C&R.

Theorem 8.2. *If there exists an MTS*(v) *then there exists an MTS*$(2v+1)$.

Proof. Let $(V, \mathcal{B})$ be an MTS(v) and W be a set of cardinality $v+1$, disjoint from V. Let L be a Latin square of side $v+1$ with rows and columns indexed by W and entries from $V \cup \{\infty\}$, where $L(i,i) = \infty$, $i \in W$. Now put $\mathcal{D} = \{(i, L(i,j), j) : i, j, \in W, i \neq j\}$. Then $(V \cup W, \mathcal{B} \cup \mathcal{D})$ is an MTS($2v+1$). □

Theorem 8.3. *If there exists an MTS*(v) *then there exists an MTS*$(2v+4)$.

Proof. Let $(V, \mathcal{B})$ be an MTS(v) where V is disjoint from $\mathbb{Z}_{v+4}$. Let $\mathcal{T}$ be the set of triples obtained by the action of the mapping $i \mapsto i+1$ (mod $v+4$) on the starter triple $(0,1,3)$. For each $d \in \mathcal{Z}_{v+4} \setminus \{0, 1, 2, v+1\}$, let $\mathcal{D} = \{(x_d, i, i+d) : 0 \leqslant i \leqslant v+3, x_d \in V\}$ where the elements x_d run through all elements of V and addition is modulo $v+4$. Then $(V \cup \mathbb{Z}_{v+4}, \mathcal{B} \cup \mathcal{T} \cup \mathcal{D})$ is an MTS($2v+4$). □

Given Mendelsohn triple systems of orders $3, 4, 13, 16$ the above two theorems are sufficient to give the entire spectrum of MTS(v). Systems for the latter two values are given in the next two examples.

Example 8.4. For an MTS(13), let $V = \mathbb{Z}_{13}$. The blocks are obtained by the action of the mapping $i \mapsto i+1 \pmod{13}$ on the starter triples $(0,1,4)$, $(4,3,0)$, $(0,2,7)$, $(7,5,0)$. □

Example 8.5. For an MTS(16), let $V = \mathbb{Z}_{13} \cup \{\infty_1, \infty_2, \infty_3\}$. The blocks are obtained by the action of the mapping $i \mapsto i + 1 \pmod{13}$ on the starter triples $(\infty_1, 0, 7)$, $(\infty_2, 0, 9)$, $(\infty_3, 0, 10)$, $(0, 1, 5)$, $(0, 2, 7)$, $(0, 3, 1)$, with the points $\infty_1, \infty_2, \infty_3$ as fixed points, together with the two blocks $(\infty_1, \infty_2, \infty_3)$ and $(\infty_3, \infty_2, \infty_1)$. □

Given an MTS(v), if every cyclic triple (x, y, z) is replaced by the corresponding unordered triple $\{x, y, z\}$ a *twofold triple system*, usually denoted by TTS(v), is obtained. This is a collection of triples in which every pair occurs precisely twice. The TTS(v) so obtained is called the *underlying* TTS(v) of the MTS(v) and may contain repeated triples. If it does not then it is called *simple* and the MTS(v) is said to be *pure*. Bennett and Mendelsohn [4] proved the following theorem.

Theorem 8.6. *There exists a pure MTS(v) for all $v \equiv 0, 1 \pmod 3$ except $v = 3, 6$.* □

In fact, the results presented in this section are a good start in proving this. The construction given in the proof of Theorem 8.2 does not introduce repeated triples provided that the Latin square used is anti-symmetric. Nor does that in the proof of Theorem 8.3 if v is odd. If v is even, replace the starter triple $(0, 1, 3)$ with $(0, 1, (v + 4)/2)$ and let $d \in \mathcal{Z}_{v+4} \setminus \{0, 1, (v + 2)/2, (v + 4)/2\}$. Of the initial systems used in these constructions those of orders 4 and 13 are pure. So what is required is a pure MTS(7) (one is given in the example above) and a pure MTS(10) to replace the MTS(3), and a pure MTS(16).

Also given an MTS(v), if every cyclic triple (x, y, z) is replaced by (z, y, x), another MTS(v), called the *converse* of the original MTS(v), is obtained. The converse is not necessarily isomorphic to the original but a system where this is the case is said to be *self-converse*. Chang, Yang and Kang [9] proved the following theorem.

Theorem 8.7. *There exists a self-converse MTS(v) for all $v \equiv 0, 1 \pmod 3$ except $v = 6$.* □

We now turn our attention to directed triple systems. These were introduced by Hung and Mendelsohn [57] and the formal definition is as follows. A *directed triple system* of order v, usually denoted by DTS(v), is an ordered pair $(V, \mathcal{B})$ where V is a base set of cardinality v and $\mathcal{B}$ is a collection of transitive triples which collectively have the property that every ordered pair of distinct elements of V is contained in precisely one transitive triple.

Again a necessary condition for existence is $v \equiv 0, 1 \pmod 3$ and this is also sufficient with no exceptions.

Directed triple systems exist in greater numbers than their Mendelsohn counterparts. Enumeration results for $v \leqslant 7$ were obtained by Colbourn and Colbourn [18]. The DTS(3) is of course unique: on base set $\{a, b, c\}$, it consists of the transitive triples $[a, b, c]$ and $[c, b, a]$. But there are 3 non-isomorphic DTS(4)s. On base set $\{a, b, c, d\}$, they are (1) $[a, b, c]$, $[b, a, d]$, $[c, d, a]$, $[d, c, b]$, (2) $[a, b, c]$, $[b, a, d]$, $[c, d, b]$, $[d, c, a]$, (3) $[a, b, c]$, $[c, a, d]$, $[b, d, a]$, $[d, c, b]$. There are 32 non-isomorphic DTS(6)s and 2,368 non-isomorphic DTS(7)s (compared to no MTS(6) and just 3 MTS(7)s).

In respect of pure directed triple systems, there is a stronger result than for Mendelsohn triple systems. Colbourn and Colbourn [11] proved the following theorem.

Theorem 8.8. *Every twofold triple system is the underlying system of some directed triple system.* □

This is certainly not true for Mendelsohn triple systems. There are 36 non-isomorphic TTS(9)s but only 16 of them are underlying systems of the 18 MTS(9)s.

As with MTS(v), the converse of a DTS(v) is also a DTS(v), not necessarily isomorphic to the original. Kang, Chang and Yang [58] established the spectrum of self-converse DTS(v).

Theorem 8.9. *There exists a self-converse DTS(v) for all* $v \equiv 0, 1 \pmod 3$ *except* $v = 6$. □

An existence proof for directed triple systems can be adapted from and follows closely Theorems 8.2 and 8.3 for Mendelsohn triple systems. But to finish this section an alternative proof is given; one which uses Wilson's fundamental construction. We will need certain ingredients to implement this and we give these first as examples.

Example 8.10. For clarity brackets and commas are omitted from directed triples.

DTS(6): 013, 124, 230, 341, 402, 054, 150, 251, 352, 453.
DTS(9): 012, 345, 678, 036, 147, 258, 048, 156, 237, 057, 138, 246, 310, 872, 654, 520, 761, 843, 740, 851, 632, 860, 421, 753.
DTS(10): 021, 203, 130, 312, 054, 506, 460, 645, 087, 809, 790, 978, 347, 158, 269, 593, 671, 482, 836, 914, 725, 274, 385, 196, 952, 763, 841, 628, 439, 517.

This latter example is taken from [44]. Note that the DTS(10) contains a subsystem DTS(4) on the set $\{0, 1, 2, 3\}$, a feature which will be crucial in the proof below. The triples also have the interesting property that if they are interpreted as cyclic triples instead of transitive triples they form an MTS(10).

3-GDD of type 2^3: Let $V = \{a, b, c, x, y, z\}$ with partition $\{a, x\}$, $\{b, y\}$, $\{c, z\}$. The directed triples are abc, ayz, xbz, xyc, zyx, cbx, cya, zba. □

Theorem 8.11. *There exists a DTS*(v) *for all* $v \equiv 0, 1 \pmod 3$.

Proof. The proof is divided into different residue classes.

(1) $v = 6s + 1,\ s \geqslant 1$. Let $\{\{0, a_i, a_i + b_i\} : 1 \leqslant i \leqslant s\}$ be a set of orbit starters under the mapping $i \mapsto i + 1 \pmod{6s + 1}$. For a DTS$(v)$, choose orbit starters $[0, a_i, a_i + b_i]$ and $[a_i + b_i, a_i, 0]$ under the same mapping or, for a pure system, $[0, a_i, a_i + b_i]$ and $[a_i + b_i, b_i, 0]$.

(2) $v = 6s + 3,\ s \geqslant 0$. As in case (1), for $s \neq 1$ take a set of cyclic orbit starters. It will not be possible in this case to construct a pure DTS(v) because of the short orbit starter $\{0, 2s + 1, 4s + 2\}$. A DTS(9) is given in the above example.

(3) $v = 12s + 6,\ s \geqslant 0$. For $s = 0$, a DTS(6) is given above. Otherwise take a 3-GDD of type 3^{2s+1} and weight every point with 2. Replace every block of the GDD by the slave directed 3-GDD of type 2^3 given in the above example and every expanded group by the DTS(6).

(4) $v = 12s + 4,\ s \geqslant 0$. The three non-isomorphic DTS(4)s are given above. For $s \geqslant 1$, Take a 3-GDD of type 2^{3s+1}, weight every point with 2, and proceed as in case (3), using the slave directed 3-GDD and a DTS(4).

(5) $v = 12s,\ s \geqslant 1$. This is exactly the same as the previous case starting with a 3-GDD of type 2^{3s}.

(6) $v = 12s + 10,\ s \geqslant 0$. This is a slightly more difficult case and illustrates a further extension of the use of Wilson's construction. For $s = 0$, a DTS(10) is given above. Otherwise take a 3-GDD of type 3^{2s+1}, weight every point with 2, and replace every block of the GDD by the slave directed 3-GDD as before. The expanded groups of the master GDD have cardinality 6 so adjoin four further points, say a, b, c, d. On every expanded block, together with a, b, c, d, place a DTS(10) containing a DTS(4) subsystem so that this subsystem is on the four adjoined points. Recall that we remarked that the DTS(10) in the example above had such a subsystem. □

9. Quasigroups and loops

A *Steiner quasigroup* or *squag* is a pair $(Q, \cdot)$ where Q is a set and $\cdot$ is an operation on Q satisfying the identities

$$x \cdot x = x, \qquad y \cdot (x \cdot y) = x, \qquad x \cdot y = y \cdot x.$$

If $(V, \mathcal{B})$ is an STS(v), then a Steiner quasigroup $(Q, \cdot)$ is obtained by letting $Q = V$ and defining $x \cdot y = z$ where $\{x, y, z\} \in \mathcal{B}$. The process is reversible; if Q is a Steiner quasigroup, then a Steiner triple system is obtained by letting $V = Q$ and $\{x, y, z\} \in \mathcal{B}$ where $x \cdot y = z$ for all $x, y \in Q$, $x \neq y$. Thus there is a one-one correspondence between all Steiner triple systems and all Steiner quasigroups, see for example Theorem V.1.11 of [78]. A Steiner quasigroup is also known as an *idempotent totally symmetric quasigroup*, see Remark 2.12 on page 153 of HB. In a similar vein, a *Steiner loop* or *sloop* is a pair $(L, \cdot)$ where L is a set containing an identity element, say e, and $\cdot$ is an operation on L satisfying the identities

$$e \cdot x = x, \qquad x \cdot x = e, \qquad y \cdot (x \cdot y) = x, \qquad x \cdot y = y \cdot x.$$

If $(V, \mathcal{B})$ is an STS(n), then a Steiner loop $(L, \cdot)$ is obtained by letting $L = V \cup \{e\}$ and defining $x \cdot y = z$ where $\{x, y, z\} \in \mathcal{B}$. Again the process is reversible. All of the above is well-known in both the algebraic and the combinatorial communities.

For a Steiner loop, a natural question is whether it can ever be a group and if so to identify both the group and the Steiner triple system from which it comes? The answer, which is also well-known, is in the affirmative and is easy to determine. Let $(L, \cdot)$ be a Steiner loop. Then if it is also a group, since every non-identity element has order 2, it is elementary Abelian of order 2^n, $n \geqslant 2$. The corresponding Steiner triple system thus has order $2^n - 1$ and is the projective Steiner triple system of that order introduced in the Introduction and obtained by suitably identifying elements of the group with vectors in $(\mathbb{F}_2)^n$.

A further question which now arises is whether there are any other algebraic identities which a Steiner loop may satisfy which lead to other interesting classes of Steiner triple system? However, before considering this question, it is instructive to present a different proof of the above result that if a Steiner loop is associative then it comes from a projective Steiner triple system. This alternative proof is not group-theoretic but combinatorial, relying on results from Design Theory. Let $(V, \mathcal{B})$ be an STS(v) and $(L, \cdot)$, where $L = V \cup \{e\}$, the associated Steiner loop. If any of x, y, z are equal to e or to one another then associativity is satisfied. If $\{x, y, z\} \in \mathcal{B}$ then $(x \cdot y) \cdot z = x \cdot (y \cdot z) = e$. Now suppose that $\{x, y, z\} \notin \mathcal{B}$. Then the block, say b_1, containing x, y also contains the element $x \cdot y$. Similarly, the block, say b_2, containing y, z also contains the element $y \cdot z$. Now consider the block b_3 containing $x \cdot y$ and z. The third point is $(x \cdot y) \cdot z$. Similarly the third point in the block b_4 containing x and $y \cdot z$ is $x \cdot (y \cdot z)$. If the associative law holds then these two third points are the same and the four blocks b_1, b_2, b_3, b_4 contain six points, $x, y, z, x \cdot y, y \cdot z, x \cdot y \cdot z$, i.e. they form a Pasch configuration. The number of sets $\{x, y, z\} \notin \mathcal{B}$ is $v(v-1)(v-2)/6 - v(v-1)/6 = v(v-1)(v-3)/6$, so the STS($v$) must contain at least $v(v-1)(v-3)/24$ Pasch configurations. But this is the maximum number that can occur and does so only in the projective systems [90]. In fact the cycle structure of the projective systems contains only 4-cycles.

We now introduce a concept which we call *fractional associativity*. In order to do this we express associativity in a different notation. By introducing left and right translations, $x \cdot y$ can be written as either $L_x(y)$ or $R_y(x)$. The associative law can then be expressed as $L_x R_z = R_z L_x$. Then $1/n$th associativity is defined by $(L_x R_z)^n = (R_z L_x)^n$. Clearly if an operation is $1/n$th associative then it is $1/m$th associative for all $m \geqslant n$ with associativity being the case where $n = 1$. Now consider a Steiner loop $(L, \cdot)$ where the operation is 1/2-associative, in conventional notation, $x \cdot ((x \cdot (y \cdot z)) \cdot z) = (x \cdot ((x \cdot y) \cdot z)) \cdot z$. Then a straightforward, but somewhat tedious, analysis shows that the cycle structure of the corresponding Steiner triple system must contain only 4-cycles and 8-cycles. This class of STS(v) is wider than just the projective systems. It contains the STS(15) #2 in the standard listing in [67] for example. But none of the 11,084,874,829 STS(19)s have this property. The situation merits further investigation.

The Hall triple systems have an elegant characterization in terms of Steiner quasigroups.

Theorem 9.1. *Let $(Q, \cdot)$ be the Steiner quasigroup corresponding to an STS(v), $(V, \mathcal{B})$. Then $(Q, \cdot)$ satisfies the distributive law, i.e., $x \cdot (y \cdot z) = (x \cdot y) \cdot (x \cdot z)$, $x, y, z \in Q$, if and only if $(V, \mathcal{B})$ is a Hall triple system.*

Proof. To prove necessity, we need to show that every three points which do not form a triple generate the unique STS(9).

So let $a, b, p \in V$ where $\{a, b, p\} \notin \mathcal{B}$. Then there exists c, x, such that $\{a, b, c\}, \{a, p, x\} \in \mathcal{B}$. (It is to be understood that when a new letter is introduced it represents a new point.) There also exist z, r, such that $\{b, p, z\}, \{b, x, r\} \in \mathcal{B}$.

Now $a\cdot(b\cdot p) = (a\cdot b)\cdot(a\cdot p)$, i.e. $a\cdot z = c\cdot x = q$. Thus $\{a, z, q\}, \{c, x, q\} \in \mathcal{B}$. Also $a\cdot(b\cdot x) = (a\cdot b)\cdot(a\cdot x)$, i.e., $a\cdot r = c\cdot p = y$. Thus $\{a, r, y\}, \{c, p, y\} \in \mathcal{B}$.

We now have nine points, $a, b, c, p, q, r, x, y, z$ and eight blocks. To complete the STS(9) and also the proof we need to show that $\{b, q, y\}, \{c, r, z\}$, $\{p, q, r\}, \{x, y, z\} \in \mathcal{B}$.

So $b\cdot q = (c\cdot a)\cdot(c\cdot x) = c\cdot(a\cdot x) = c\cdot p = y$. Further $c\cdot r = (b\cdot a)\cdot(b\cdot x) = b\cdot(a\cdot x) = b\cdot p = z$. Next $p\cdot q = (x\cdot a)\cdot(x\cdot c) = x\cdot(a\cdot c) = x\cdot b = r$. Finally $x\cdot y = (p\cdot a)\cdot(p\cdot c) = p\cdot(a\cdot c) = p\cdot b = z$.

To prove sufficiency, suppose first that $\{x, y, z\} \in \mathcal{B}$. Then $x\cdot(y\cdot z) = x\cdot x = x$ and $(x\cdot y)\cdot(x\cdot z) = z\cdot y = x$. If $\{x, y, z\} \notin \mathcal{B}$, then the three points x, y, z generate an STS(9). There exists a, b, c such that $\{a, y, z\}$, $\{x, b, z\}, \{x, y, c\} \in \mathcal{B}$. But $\{a, b, c\} \notin \mathcal{B}$, because the unique STS(9) is anti-Pasch. Therefore there exists l, m, n such that $\{l, b, c\}, \{a, m, c\}, \{a, b, n\} \in \mathcal{B}$, and by considering blocks containing the point a, $\{a, x, l\} \in \mathcal{B}$. Now we obtain $x\cdot(y\cdot z) = x\cdot a = l$ and $(x\cdot y)\cdot(x\cdot z) = c\cdot b = l$. □

Another method of obtaining a loop from a Steiner triple system $(V, \mathcal{B})$ is to choose a point $\alpha \in V$ and define an operation $\circ$ by the rule $x\circ y = (x\cdot y)\cdot\alpha$ where $x\cdot y$ is defined as in the Steiner quasigroup, i.e. $x\cdot x = x$ and $x\cdot y = z$ where $\{x, y, z\} \in \mathcal{B}$. The point α is the identity and every other element has order 3. Different values of α can lead to different loops. If $(V, \mathcal{B})$ is a Hall triple system then the loop obtained is a Moufang loop and different values of α then lead to isomorphic loops. The relationship between Hall triple systems and exponent 3 commutative Moufang loops is one-one.

Less well known seems to be the fact that quasigroups and loops can be obtained from Mendelsohn triple systems by precisely the same procedures as described above for constructing Steiner quasigroups and Steiner loops from Steiner triple systems. The law $y\cdot(x\cdot y) = x$ is usually called semisymmetric and the quasigroups are known as *idempotent semisymmetric quasigroups*, see again Remark 2.12 on page 153 of HB. However the algebraic structures might also appropriately be called *Mendelsohn quasigroups* and *Mendelsohn loops*; they satisfy the same properties as their Steiner counterparts with the exception of commutativity. Similarly there is a one-one correspondence between Mendelsohn triple systems, Mendelsohn quasigroups and Mendelsohn loops.

For a directed triple system, an algebraic structure can also be obtained as above by defining $x\cdot x = x$ and $x\cdot y = z$ for all $x, y \in V$, $x \neq y$ where z is the third element in the transitive triple containing the ordered pair (x, y). However the structure obtained is not necessarily a quasigroup. If $[u, x, y]$ and [y,v,x] are transitive triples then $u \cdot x = v \cdot x = y$. But in fact some DTS($v$)s do yield quasigroups and these are the subject of a recent paper by Drápal, Kozlik and the present author [27]. Such a DTS(v) is called a *Latin directed triple system*, and denoted by LDTS(v), to reflect the fact that in this case the operation table forms a Latin square. The quasigroup so obtained is called a *DTS-quasigroup*. In an analogous way to that described above for Steiner triple systems a loop may also be constructed from an LDTS(n); called a *DTS-loop*.

A necessary and sufficient condition for a directed triple system to be Latin is given in the following theorem, proved in [27].

Theorem 9.2. *Let $D = (V, \mathcal{B})$ be a DTS(v). Then D is an LDTS(v) if and only if $[x, y, z] \in \mathcal{B} \Rightarrow [w, y, x] \in \mathcal{B}$ for some $w \in V$.* □

Latin directed triple systems differ from their Steiner and Mendelsohn counterparts in fundamental ways. One of these is that they are not a variety. Another is that, unlike Steiner and Mendelsohn triple systems, there is not a one-one correspondence between the Latin directed triple systems and the associated quasigroups or loops. A further difference concerns flexibility. The *flexible law* states that $x \cdot (y \cdot x) = (x \cdot y) \cdot x$. As is easily verified, both Steiner quasigroups and loops and Mendelsohn quasigroups and loops all satisfy this law. But this is not the case for DTS-quasigroups and loops. A flexible DTS-quasigroup or loop has an interesting geometric structure and a necessary and sufficient condition is as follows.

Theorem 9.3. *A DTS-quasigroup or DTS-loop obtained from a Latin directed triple system LDTS(v), $D = (V, \mathcal{B})$, satisfies the flexible law if and only if $[x, y, z] \in \mathcal{B} \Rightarrow [x, z \cdot x, y \cdot x] \in \mathcal{B}$.* □

DTS-quasigroups exist for all $v \equiv 0, 1 \pmod 3$ except $v = 4, 6, 10$. More details are in [27].

Finally we remark that the isomorphism invariants, cycle lists and trains, used to distinguish non-isomorphic Steiner and Mendelsohn triple systems, and hence also their associated quasigroups and loops, might also be used more widely. Let $(Q, \cdot)$ be a quasigroup (including a loop). For $x \in Q$ define the *neighbourhood* $N(x)$ as the set of ordered pairs $\{(u, v) : u \cdot v = x\}$. The cycles induced by double neighbourhoods and, if Q is not commutative, the directed cycles induced by single neighbourhoods can be used to help determine isomorphism. Trains can be obtained by defining the mapping f from the set of all cyclic triples of Q to itself by $f((x,y,z)) = (a,b,c)$ where $y \cdot z = a$, $z \cdot x = b$, $x \cdot y = c$. If the quasigroup is commutative then the cyclic triples can be replaced by 3-subsets.

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Received April 15, 2011

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