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Central r-naturally fully ordered groupoids with left identity

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Abstract. In this paper a generalized version (r-naturally fully ordered groupoid) of a naturally fully ordered groupoid with left identity in the sense that only right solvability is permissible is embedded in a concrete groupoid of all non-negative real numbers. First, the introduction of centrality makes the r-naturally fully ordered groupoid with left identity order-isomorphic to the positive cone of a fully ordered central quasigroup. Second, the left Archimedean property enables this ordered groupoid to be embedded in the concrete groupoid.

1. Introduction

In this paper we will generalize the classical result of Hölder [6] with the embedding of a fully ordered (briefly, f.o.) semigroup in the additive semigroup of all non-negative real numbers in the context of groupoids. The embedding will be carried out in a concrete groupoid (Example 3.1) consisting of all non-negative real numbers. This approach is similar to that of Hartman [5], who considered the embedding of a f.o. loop in the additive group of all real numbers. Our concern lies in an r-naturally f.o. groupoid with left identity, which is a generalized version of a naturally f.o. groupoid with left identity in the sense that only right solvability is guaranteed. First, the analogous concept to centrality [9] for quasigroups is introduced so that an r-naturally f.o. groupoid can be the positive cone of a f.o. central quasigroup with left identity. Second, the left Archimedean property makes the ordered groupoid embeddable in the concrete groupoid.

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2. Preliminaries

A quasigroup is an algebra $(Q, \cdot, /, \backslash)$ with three binary operations satisfying the following identities:

$$(a \cdot b)/b = a = (a/b) \cdot b \quad \text{and} \quad a\backslash(a \cdot b) = b = a \cdot (a\backslash b).$$

These identities imply that, given $a,\ b \in Q$, the equations $x \cdot b = a$ and $b \cdot x = a$ have unique solutions $x = a/b$ and $x = b\backslash a$, respectively. A loop is a quasigroup Q with an identity element e ($e \cdot a = a = a \cdot e$ for all $a \in Q$). For any $a \in Q$, we denote by R_a and L_a the mappings of Q onto itself defined by the rules $R_a(x) = xa$ and $L_a(x) = ax$, respectively. Moreover, $R_a^{-1}(x) = x/a$, $L_a^{-1}(x) = a\backslash x$. Multiplications expressed implicitly by juxtaposition are meant to bind more strongly than the divisions so as to reduce the number of brackets in quasigroup equalities. For example, $(a{\cdot}b)/b$ reduces to ab/b.

Every quasigroup $(Q, \cdot, /, \backslash)$ is isotopic to a loop. Indeed, if a binary operation $+$ on Q is defined by

$$a + b = R_{e\backslash e}^{-1}(a) \cdot L_e^{-1}(b) \quad \text{for all } a, b \in Q,$$

then it is seen that $(Q, +, e)$, denoted $B(Q)$, is a loop. Assume here that e is a left identity element for $(Q, \cdot)$ ($ea = a$ for all $a \in Q$). Then $e\backslash a = a$ holds for all $a \in Q$ but $a/e = a$ does not unless $a = e$. Hence

$$a + b = (a/e)b. \tag{1}$$

Using (1),

$$a - b = (a/b)e$$

One outstanding benefit of centering is that it makes the loop $B(Q)$ into an abelian group. According to Corollary 3.7 in [9], a central quasigroup $(Q, \cdot, /, \backslash, e)$ with a left identity element e is characterized by the following identities:

$$((a/e)b/e)c = (a/e)((b/e)c); \tag{2}$$

$$(a/e)b = (b/e)a; \tag{3}$$

$$((a/e)b)e = (ae/e)(be). \tag{4}$$

The first two identities show that $B(Q)$ is an abelian group. Indeed, identities (2) and (3) specify the associativity and commutativity of $B(Q)$, respectively. Identity (4) means that right multiplication by e is an automorphism of $B(Q)$, i.e., $R_e(a + b) = R_e(a) + R_e(b)$.

A quasigroup $(Q, \cdot, /, \backslash)$ with a binary relation $\geqslant$ is called a *f.o. quasigroup* if $(Q, \geqslant)$ is a fully ordered set and the following monotony law holds:

(M) $a \geqslant b \Leftrightarrow ax \geqslant bx \Leftrightarrow xa \geqslant xb$ for all $a, b, x \in Q$.

The law (M) implies that (see [3, Lemma 3.1])

(D) $a \geqslant b \Leftrightarrow a/x \geqslant b/x,\ x\backslash a \geqslant x\backslash b,\ x/b \geqslant x/a,\ b\backslash x \geqslant a\backslash x$.

A *f.o. central quasigroup with left identity* is a f.o. quasigroup with left identity satisfying (2) to (4). In a f.o. central quasigroup Q with left identity, it is clear from (M) and (D) that $(B(Q), \geqslant)$ is a f.o. abelian group. Therefore the positive cone of Q is defined by $Q^+ = \{a \in Q \,|\, a \geqslant e\}$. A groupoid $(Q, \cdot)$ with a full order $\geqslant$ that satisfies (M) is a *f.o. groupoid.*

3. r-naturally f.o. groupoid and centrality

We follow the terminology of [4] for ordering. An element a of a f. o. groupoid P is *r-positive* or *l-positive* according as $xa \geqslant x$ or $ax \geqslant x$ for all $x \in P$. A f.o. groupoid is called *r-positively ordered* or *l-positively ordered* if all of its elements are r-positive or l-positive, respectively. If a f. o. groupoid contains a left identity element e, then a is l-positive if and only if $a \geqslant e$, whereas a strictly positive element $a > e$ is not always r-positive. However, it will be shown at the end of this section that the introduction of centrality makes an r-positively ordered groupoid equivalent to the positive cone of a f.o. central quasigroup. For this the concept of a naturally ordered groupoid is generalized in such a way that only the existence of a right solution is permissible. A f.o. groupoid P is said to be *r-naturally ordered* if it is r-positively ordered and

$$a > b \text{ implies that } xb = a \text{ for some } x \in P.$$

Note that by (M) the solution x is unique. This condition implies that P has a right division that is “partially” defined on P in the sense that its domain is a subset of P: Set $x = a/b$ with $a > b$. Then $(a/b)b = a$ and $ab/b = a$ are satisfied. Also, P has a partial left division that is always definable on a f.o. groupoid. Since $x = b$ is a unique solution to $ax = ab$ by (M), we can define $x = a\backslash ab$ so that $a\backslash ab = b$ and $a(a\backslash ab) = ab$. Specifically, $e\backslash b = b$. Consequently, P is regarded as a set equipped with three binary operations: the groupoid multiplication and the partial right and left divisions.

Example 3.1. We define a binary operation $\oplus$ on the set $\mathbb{R}^+$ of all non-negative real numbers by

$$a \oplus b = \alpha a + b \quad \text{for some } \alpha \geqslant 1.$$

The set $\mathbb{R}^+$ with this operation and the usual order is an r-naturally f.o. groupoid with a left identity element 0. □

Example 3.2. Let Q be a f.o. central quasigroup with a left identity element e. Assume that $xe \geqslant x$ for all $x \in Q$. Then Q^+ is an r-naturally f.o. groupoid with a left identity element e. □

Throughout the paper, unless otherwise specified, we will use the symbol e to denote a left identity element, and let P be an r-naturally f.o. groupoid with left identity. The trivial case where P has just a single element e will always be excluded. Centrality of P is defined in a similar way to centrality of quasigroups (see [9] for the specific definition of central quasigroups). We now consider the Cartesian product P^2 as a partial algebra $(P^2, \cdot, /, \backslash)$ with componentwise groupoid multiplication and componentwise partial right and left divisions. An equivalence relation W on P is a *congruence* if it is a subalgebra of P^2. The diagonal $\widehat{P} = \{(a,a) \mid a \in P\}$ is a subalgebra of P^2. An r-naturally f.o. groupoid P is defined to be *central* if there exists a congruence W on P^2 having $\widehat{P}$ as a congruence class. In addition we will call this W a *centering congruence*. The equivalence class of $(a,b) \in P^2$ under W is denoted by $(a,b)^W$, i.e., $(a,b)^W = \big\{(x,y) \in P^2 \mid (x,y)W(a,b)\big\}$, and the set of equivalence classes by P^2/W.

A partial ternary operation on P is defined by

$$p(a,b,c) = (a/b)c \quad \text{provided that } a \geqslant b.$$

This definition does not entails the identity $p(a,b,b) = a$ in case of $a < b$. Therefore useful methods cannot be used to obtain the following properties of centering congruences. To solve this problem, we provide a new ternary operation p_s on P defined by

$$p_s(a,b,c) = (sa/b)c \quad \text{provided that } sa \geqslant b$$

Indeed, even for $a < b$ by right solvability and (M) we can take $s \in P$ such that $sa \geqslant b$. Then since $p_s(a,b,b) = sa$, it follows that $s\backslash p_s(a,b,b) = a$. Also, $s\backslash p_s(a,a,b) = b$. Using the operation p_s, we obtain similar results to

Propositions 3.1, 3.3, and 3.4 in [9]. The result similar to Proposition 3.1 guarantees the existence of a centering congruence on P^2. The other results are listed in the form in which they will be used in what follows.

Proposition 3.3. *Let P be a central r-naturally f.o. groupoid with left identity and let W be a centering congruence on P^2. Then*

(RR) $(a,b) \in P^2 \Rightarrow (a,a)W(b,b)$;

(RS) $(a,b)W(a',b') \Rightarrow (b,a)W(b',a')$;

(RT) $(a,b)W(a',b')$ and $(b,c)W(b',c') \Rightarrow (a,c)W(a',c')$.

Proof. The proof is much the same as that of [9]. Therefore we prove only (RT) because the operation p_s is necessary in case of $a < b$, $b > c$, $a' < b'$, and $b' > c'$. Assume that $(a,b)W(a',b')$ and $(b,c)W(b',c')$. Take $s \in P$ such that $sa \geqslant b$ and $sa' \geqslant b'$ (which is possible by setting $s = \max(u,v)$ such that $ua \geqslant b$, $va' \geqslant b'$). Then

$$\begin{array}{ll} (s,s)W(s,s) & \text{by (RR)}, \\ (a,b)W(a',b') & \text{is given}, \\ (b,b)W(b',b') & \text{by (RR)}, \\ (b,c)W(b',c') & \text{is given}, \\ (s,s)W(s,s) & \text{by (RR)} \end{array}$$

$$\Rightarrow (s\backslash p_s(a,b,b), s\backslash p_s(b,b,c))W(s\backslash p_s(a',b',b'), s\backslash p_s(b',b',c')).$$

Hence we obtain $(a,c)W(a',c')$, as required for (RT). □

Proposition 3.4. *Let P be a central r-naturally f.o. groupoid with left identity and let W be a centering congruence on P^2. Then W is uniquely specified by*

$$\text{if } c \geqslant a, \text{ then } (a,b)W(c,d) \Leftrightarrow d = p(c,a,b). \tag{5}$$

$$\text{if } a \geqslant b, \text{ then } (a,b)W(c,d) \Leftrightarrow c = p(a,b,d). \tag{6}$$

Applying (RS) of W and with the use of (6), we have

$$\text{if } a < b, \text{ then } (a,b)W(c,d) \Leftrightarrow d = p(b,a,c). \tag{7}$$

Lemma 3.5. *If P is an r-naturally f. o. groupoid with left identity, then $a \geqslant e$ for all $a \in P$.*

Proof. By r-positivity we have $aa \geqslant a$ for all $a \in P$, i.e., $aa \geqslant ea$. Hence by (M) we obtain $a \geqslant e$ for all $a \in P$. □

Lemma 3.6. *Let P be a central r-naturally f.o. groupoid with left identity. Then identity* (3) *is satisfied for all elements of P, and if $a \geqslant be$ then*

$$b\backslash a = (a/be)e.$$

Proof. Let $a, b \in P$ be arbitrary positive elements. Both (5) and (7) guarantee the existence of $c \in P$ such that $(e, a)W(b, c)$. Note here that c is uniquely determined. Hence $p(b, e, a) = p(a, e, b)$, or $(b/e)a = (a/e)b$. To prove the latter part, assume that $a \geqslant be$ and let $x \in P$ be such that $bx = a$. Then since $a/be \in P$ by right solvability, it follows from (3) that $(be/e)((a/be)e) = (a/be)(be) = a$. Hence by (M) $x = (a/be)e$. □

Theorem 3.7. *Let P be a central r-naturally f.o. groupoid with left identity and let W be a centering congruence on P^2. Then the quotient P^2/W is a f.o. central quasigroup with left identity, and P is o-isomorphic* (*order-isomorphic*) *to the positive cone of P^2/W.*

Proof. Let $P_W = \{(a, b)^W \mid a \geqslant b\}$ and $N_W = \{(a, b)^W \mid a \leqslant b\}$ be the sets of positive and negative elements in P^2/W, respectively. The ordering on P_W and N_W is determined by:

(a) $\geqslant$ on P_W: $(a, b)^W \geqslant (c, d)^W$ if and only if $p(a, b, e) \geqslant p(c, d, e)$,

(b) $\geqslant$ on N_W: $(a, b)^W \geqslant (c, d)^W$ if and only if $p(b, a, e) \leqslant p(d, c, e)$.

Rules (a) and (b) are based on (6) and (7), respectively. Since it is the case that $(a, b)^W > (c, d)^W$ whenever $(a, b)^W \in P_W$ with $a > b$, $(c, d)^W \in N_W$ with $c < d$, (a) and (b) provide a full order on P^2/W. Further, multiplication on P^2/W is defined by

(c) multiplication: $(a, b)^W (c, d)^W = (ac, bd)^W$.

For (c) we may use an element of the form $(s, e)^W$ with $s = p(a, b, e) \geqslant e$ or $(e, t)^W$ with $t = p(b, a, e) \geqslant e$ based on whether each $(a, b)^W$ is positive or negative. We show that P^2/W is right and left solvable. First, the following is clear from (a) and (b): if $a \geqslant b$, then $x = a/b$ is a solution to

$(x,e)^W(b,e)^W = (a,e)^W$ and $(e,x)^W(e,b)^W = (e,a)^W$. Second, if $a \geqslant be$, then by Lemma 3.6 $x = (a/be)e$ is a solution to $(b,e)^W(x,e)^W = (a,e)^W$ and $(e,b)^W(e,x)^W = (e,a)^W$. We consider solutions to the other cases of equations.

Case 1R. $(e,x)^W(b,e)^W = (a,e)^W$ with $a < b$: Since in this case $(a,e)^W = (b, p(b,a,e))^W$ by (5), we have $xe = p(b,a,e)$. Hence $x = b/a$.

Case 1L. $(b,e)^W(e,x)^W = (a,e)^W$ with $a < be$: Since in this case $(a,e)^W = (be, p(be,a,e))^W$ by (5), we have $x = p(be,a,e)$. Hence $x = (be/a)e$.

Case 2R. $(x,e)^W(e,b)^W = (a,e)^W$: Since $(a,e)^W = (p(a,e,b), b)^W$ by (6), we have $xe = p(a,e,b)$. Hence $x = (a/e)b/e$.

Case 2L. $(e,b)^W(x,e)^W = (a,e)^W$: Since $(a,e)^W = (p(a,e,be), be)^W$ by (6), we have $x = p(a,e,be)$. Hence $x = (a/e)(be)$.

A similar method guarantees the existence of a solution to each of the above equations into which $(e,a)^W$ is substituted for $(a,e)^W$. Obviously, $(s,s)^W (= (e,e)^W)$ is a left identity element for multiplication. Therefore the right division on P^2/W is defined as follows:

$$(a_1,a_2)^W/(b_1,b_2)^W = (sa_1/b_1, sa_2/b_2)^W, \tag{8}$$

where $sa_1 \geqslant b_1$, $sa_2 \geqslant b_2$. Indeed, setting $s = b/a$ and $s = b/e$ give the solutions in Cases 1R and 2R, respectively. By making use of Lemma 3.6, the left division is also defined as

$$(b_1,b_2)^W \backslash (a_1,a_2)^W = ((sa_1/b_1e)e, (sa_2/b_2e)e)^W, \tag{9}$$

where $sa_1 \geqslant b_1e$, $sa_2 \geqslant b_2e$. Setting $s = be/a$ and $s = b$ give the solutions in Cases 1L and 2L, respectively. (Note here that by (3) $ba = (a/e)(be)$.)

To show that (M) is satisfied for P^2/W, assume that $(a,e)^W \geqslant (b,e)^W$, so that $a \geqslant b$. Since $ac \geqslant bc$ by (M), $(a,e)^W(c,e)^W \geqslant (b,e)^W(c,e)^W$. The converse is also valid. We next provide three cases to prove $(a,e)^W(e,c)^W \geqslant (b,e)^W(e,c)^W$.

Case 1. ae, $be \geqslant c$: Since $(ae/c)e \geqslant (be/c)e$ by (M) and (D), the required inequality follows from (a).

Case 2. ae, $be < c$: Since $(c/ae)e \leqslant (c/be)e$ by (M) and (D), we obtain from (b) the required inequality.

Case 3. $ae \geqslant c \geqslant be$: By (6) and (7), $(ae,c)^W = ((ae/c)e,e)^W$ and $(be,c)^W = (e,(c/be)e)^W$. Since $ae/c \geqslant e$ and $c/be \geqslant e$ by (D), we obtain $(a,e)^W(e,c)^W \geqslant (b,e)^W(e,c)^W$.

The converse is also seen to be valid. Similarly, we obtain $(e,a)^W \geqslant (e,b)^W \Leftrightarrow (e,a)^W(c,e)^W \geqslant (e,b)^W(c,e)^W \Leftrightarrow (e,a)^W(e,c)^W \geqslant (e,b)^W(e,c)^W$. We further show that $(a,e)^W \geqslant (e,b)^W \Leftrightarrow (a,e)^W(c,e)^W \geqslant (e,b)^W(c,e)^W$. If $c \geqslant be$, then by (6) $(c,be)^W = ((c/be)e,e)^W$. Since $ac \geqslant c$ and $be \geqslant e$ by (M), it follows from (M) and (D) that $ac = (ac/e)e \geqslant (c/be)e$, and hence by (a) $(ac,e)^W \geqslant (c,be)^W$. If $c < be$, then by definition $(c,be)^W < (e,e)^W \leqslant (ac,e)^W$. The converse is trivial because $(a,e)^W$ is always $\geqslant (e,b)^W$. A similar method gives $(a,e)^W \geqslant (e,b)^W \Leftrightarrow (a,e)^W(e,c)^W \geqslant (e,b)^W(e,c)^W$. Thus P^2/W is a f.o. quasigroup with left identity.

We show that P^2/W is central. According to the proof of Lemma 3.2 in [9], which addresses the case where P is a quasigroup, a relation Ω on $P^2/W \times P^2/W$ is defined by

$$((a_1,a_2)^W,(b_1',b_2')^W)\Omega((a_1',a_2')^W,(b_1,b_2)^W) \Leftrightarrow (a_1,a_3)W(a_1',a_3'), \tag{10}$$

where $(a_2,a_3)W(b_1,b_2)$ and $(a_2',a_3')W(b_1',b_2')$. However, since P is a groupoid, a problem arises, i.e., no solution $a_3 \in P$ exists to $(a_2,a_3)W(b_1,b_2)$, for example, when $b_1 > e, b_2 = e$ and $b_1 > a_2$. Therefore the definition of Ω must be revised. The following lemma is provided for this purpose.

Lemma 3.8. *For any (a_1,a_2), $(b_1,b_2) \in P^2$, there exists $(x_1,x_2) \in P^2$ such that $(x_1,x_2)W(a_1,a_2)$ with $x_1 \geqslant b_1$, $x_2 \geqslant b_2$.*

Proof. Let $x_1 \geqslant a_1$. By (5), $x_2 = p(x_1,a_1,a_2)$ satisfies $(x_1,x_2)W(a_1,a_2)$. Take $s \in P$ such that $sa_1 \geqslant b_1$, $sa_2 \geqslant b_2$. Set $x_1 = sa_1$, so that $x_1 \geqslant b_1$. Then $x_2 = ((sa_1)/a_1)a_2 = sa_2 \geqslant b_2$. □

With the aid of Lemma 3.8, a relation Ω is introduced on $P^2/W \times P^2/W$ by setting $((a_1,a_2)^W,(b_1',b_2')^W)\Omega((a_1',a_2')^W,(b_1,b_2)^W)$ if there exist $(x_1,x_2),(x_1',x_2') \in P^2$ such that

$$(x_1,x_2)W(a_1,a_2),\ x_2 \geqslant b_1 \text{ and } (x_1',x_2')W(a_1',a_2'),\ x_2' \geqslant b_1', \tag{11}$$

and such that the relation

$$((x_1,x_2)^W,(b_1',b_2')^W)\Omega((x_1',x_2')^W,(b_1,b_2)^W) \tag{12}$$

is satisfied in the sense of (10). The inequalities $x_2 \geqslant b_1$ and $x_2' \geqslant b_1'$ in (11) are needed only to verify the validity of (12) on the basis of (10). Indeed, in view of these inequalities, it is seen from (5) that $(x_2, x_3)W(b_1, b_2)$ and $(x_2', x_3')W(b_1', b_2')$ have solutions $x_3 = p(x_2, b_1, b_2)$ and $x_3' = p(x_2', b_1', b_2')$. We will examine whether this definition of Ω is consistent with (10). Assume that (10) is satisfied, i.e.,

$$(a_1, a_3)W(a_1', a_3'),\ (a_2, a_3)W(b_1, b_2) \text{ and } (a_2', a_3')W(b_1', b_2').$$

Let $x_3, x_3' \in P$ be such that $(x_2, x_3)W(b_1, b_2)$ and $(x_2', x_3')W(b_1', b_2')$. By the transitivity of W we have $(x_2, x_3)W(a_2, a_3)$ and $(x_2', x_3')W(a_2', a_3')$. Since (11) is satisfied, it follows from (RT) for W that $(x_1, x_3)W(a_1, a_3)$ and $(x_1', x_3')W(a_1', a_3')$. From the first assumption and transitivity we obtain $(x_1, x_3)W(x_1', x_3')$, which implies that (12) is satisfied. Next we examine whether Ω is a subquasigroup of $(P^2/W)^4$. Let $(y_1, y_2), (y_1', y_2') \in P^2$ be such that $(y_1, y_2)W(c_1, c_2)$, $y_2 \geqslant d_1$ and $(y_1', y_2')W(c_1', c_2')$, $y_2' \geqslant d_1'$, and such that

$$((y_1, y_2)^W, (d_1', d_2')^W)\Omega((y_1', y_2')^W, (d_1, d_2)^W)$$

is satisfied in the sense of (10), which implies that $((c_1, c_2)^W, (d_1', d_2')^W)\Omega$ $((c_1', c_2')^W, (d_1, d_2)^W)$. Accordingly we use the proof of Lemma 3.2 [9] to obtain

$$\begin{aligned}&((x_1, x_2)^W(y_1, y_2)^W, (b_1', b_2')^W(d_1', d_2')^W)\\ &\qquad\qquad\Omega\,((x_1', x_2')^W(y_1', y_2')^W, (b_1, b_2)^W(d_1, d_2)^W).\end{aligned}$$

Since $(x_1, x_2)^W(y_1, y_2)^W = (a_1, a_2)^W(c_1, c_2)^W$ and $(x_1', x_2')^W(y_1', y_2')^W = (a_1', a_2')^W(c_1', c_2')^W$, we have by (11) and (12)

$$\begin{aligned}&((a_1, a_2)^W(c_1, c_2)^W, (b_1', b_2')^W(d_1', d_2')^W)\\ &\qquad\qquad\Omega\,((a_1', a_2')^W(c_1', c_2')^W, (b_1, b_2)^W(d_1, d_2)^W).\end{aligned}$$

It is clearly seen that the definition of (10) does not depend on the choices of representatives of $(b_1, b_2)^W, (b_1', b_2')^W$. Hence we may assume that $b_i \geqslant d_i$, $b_i' \geqslant d_i'$ for $i = 1, 2$. Take $s \in P$ such that $sx_i \geqslant y_i$, $sx_i' \geqslant y_i'$ for $i = 1, 2, 3$. Since

$$((s, s)^W, (e, e)^W)\Omega((s, s)^W, (e, e)^W),$$

it follows that

$$((sx_1, sx_2)^W, (b_1', b_2')^W)\Omega((sx_1', sx_2')^W, (b_1, b_2)^W).$$

Note here that this is also valid in the context of (10). Hence the proof of Lemma 3.2 [9] is used again to obtain

$$((sx_1, sx_2)^W/(y_1, y_2)^W, (b'_1, b'_2)^W/(d'_1, d'_2)^W)$$
$$\Omega\, ((sx'_1, sx'_2)^W/(y'_1, y'_2)^W, (b_1, b_2)^W/(d_1, d_2)^W).$$

Since $(sx_1, sx_2)^W/(y_1, y_2)^W = (a_1, a_2)^W/(c_1, c_2)^W$, $(sx'_1, sx'_2)^W/(y'_1, y'_2)^W = (a'_1, a'_2)^W/(c'_1, c'_2)^W$ by (8), we have by (11) and (12)

$$((a_1, a_2)^W/(c_1, c_2)^W, (b'_1, b'_2)^W/(d'_1, d'_2)^W)$$
$$\Omega\, ((a'_1, a'_2)^W/(c'_1, c'_2)^W, (b_1, b_2)^W/(d_1, d_2)^W).$$

In view of (9), a similar method gives Ω being closed under left division. Using the operations on Ω and p_s, we can prove that Ω satisfies the properties of a centering congruence. Finally, by considering a mapping $P \to P^2/W$; $a \mapsto (a, e)^W$ and by using Lemma 3.5, it is seen that P is o-isomorphic to the positive cone of P^2/W. □

The following corollary corresponds to Corollary 3.7 in [9].

Corollary 3.9. *If P is a central r-naturally f.o. groupoid with left identity, then it satisfies identities* (2), (3), *and* (4). □

A f.o. quasigroup Q with left identity is said to *be generated by* P if it is a quasigroup generated by P on which a full order is introduced such that it is an extension of the full order of P. If, in addition, identities (2) to (4) are satisfied, then Q is the f.o. central quasigroup with left identity generated by P. Henceforth $B(P)$ denotes an algebra $(P, +, e)$ where $+$ is a binary operation defined by (1). Since $a \geqslant e$ for all $a \in P$ (Lemma 3.5), by right solvability we have $a/e \in P$, and thus $B(P)$ is actually a subgroupoid of Q, or $B(Q)$.

Proposition 3.10. *Let P be an r-naturally f.o. groupoid with left identity and let Q be the f.o. central quasigroup with left identity generated by P. Then every element $x \in Q$ is written in the form $x = (a/b)e$ where $a, b \in P$.*

Proof. Let $(A, +, e)$ be the subgroup of $B(Q)$ generated by $B(P)$. Since $B(Q)$ is an abelian group, every element $x \in A$ can be written in the form $x = a - b$, where $a, b \in B(P)$. Using the fact that R_e is an automorphism of $B(Q)$, we can write $xe = (a-b)e = ae - be$. Hence $xe \in A$. Analogously, $x/e \in A$. Since $xy = xe + y$, $x/y = (x-y)/e$, and $y\backslash x = x - ye$, it follows that xy, x/y, $y\backslash x \in A$ for all $x, y \in A$. Thus A is a subquasigroup of Q

that contains P. This means that $A = Q$ because Q is generated by P. Therefore for every $x \in Q$ we have $x = (a/b)e$, where $a, b \in P$. $\square$

In view of Proposition 3.10, it is easy to see that an extended order on Q from the full order on P is uniquely determined.

Proposition 3.11. *An r-naturally f.o. groupoid P with left identity is central if and only if it is the positive cone of the f.o. central quasigroup Q with left identity generated by P.*

Proof. Assume that $P = Q^+$. Then it is obvious that P is an r-naturally f.o. groupoid with left identity for which (2) to (4) are satisfied. According to [9], define the subtraction mapping $F : P^2 \to Q$ by

$$F(a, a') = a - a'.$$

With the aid of (4), F is a homomorphism with respect to groupoid multiplication and the two partial divisions. We show only the homomorphic property with groupoid multiplication. Since $ab = R_e(a) + b$, it follows that

$$\begin{aligned} F((a, a')(b, b')) &= (R_e(a) + b) - (R_e(a') + b') \\ &= R_e(a - a') + (b - b') \qquad (R_e \text{ is an automorphism}) \\ &= (a - a')(b - b') \qquad\qquad (\text{by } (1)) \\ &= F(a, a')F(b, b'). \end{aligned}$$

Hence $\ker F$ is a congruence, and it must have the diagonal $\widehat{P}$ as a congruence class. This means that P is central.

Assume that P is central. By Corollary 3.9, P has the same algebraic properties (i.e., (2) to (4)) as Q, and hence P is embedded o-isomorphically in Q. From Lemma 3.5 we conclude that P is o-isomorphic to Q^+. $\square$

Note that the quotient P^2/W of Theorem 3.7 is o-isomorphic to this Q. Indeed, in view of Proposition 3.10, it can be verified that the mapping $P^2/W \to Q;\ (a, b)^W \mapsto (a/b)e (= a - b)$ is an o-isomorphism.

4. Embedding in the non-negative real numbers

Henceforth assume that a central r-naturally f.o. groupoid P with left identity has no smallest strictly positive element. We will use the fact that P is o-isomorphic to the positive cone of the f.o. central quasigroup generated by P (Proposition 3.11) to prove the lemmas and theorem in this section.

A relaxed version [7] of the Archimedean property is required for the embedding of P in the non-negative real numbers. Let $a \in P$ be an arbitrary element. We will define the n-th left multiplication of a as $a^n = a \cdot a^{n-1}$ for $n = 2, 3, \ldots$ and $a^1 = a$. An r-naturally f.o. groupoid P is called *left Archimedean* if for every strictly positive elements $a,\ b \in P$ there is a positive integer n such that $a^n > b$.

By (3), we may define the n-th addition of a in the left sided manner: $na = a + (n-1)a$ for $n = 2, 3, \ldots$ and $1 \cdot a = a$. From (1) it is seen that $na = L_{a/e}^{n-1}(a)$ for all $n \geqslant 1$ where $L_{a/e}^{0} = L_e$.

Lemma 4.1. *Let P be a central r-naturally f.o. groupoid with left identity. If P is left Archimedean, then $B(P)$ is an Archimedean f.o. monoid.*

Proof. As was stated in Section 2, it is clear that $B(P)$ is a (commutative) monoid. Note that by (M) and (D) of P

$$x \geqslant y \Leftrightarrow (x/e)z \geqslant (y/e)z \text{ and } x \geqslant y \Leftrightarrow (z/e)x \geqslant (z/e)y.$$

Hence $B(P)$ is a f.o. monoid. We show that $B(P)$ is Archimedean. Let $a,\ b \in P$ be strictly positive. Without loss of generality we can assume that $b > a$. Since $a/e > e$ if $a > e$, the left Archimedean property guarantees the existence of $n > 1$ such that $L_{a/e}^{n-1}(a/e) > b$. Since the map $L_{a/e}^{n-1}$ is order preserving, it follows from the r-positivity property that $L_{a/e}^{n-1}((a/e)a) > b$, or $(n+1)a > b$, as required. □

Theorem 4.2. *Let P be a left Archimedean, central r-naturally f.o. groupoid with left identity. Then P is o-isomorphic to a subgroupoid of the groupoid of all non-negative real numbers of Example* 3.1.

Proof. Since $B(P)$ is an Archimedean f.o. monoid by Lemma 4.1, it is seen from Hölder's [6] theorem that there exists an o-isomorphism f of $B(P)$ to a submonoid of the additive f. o. monoid of all non-negative real numbers. Since $ab = ae + b$, $f(ab) = f(ae) + f(b)$. To complete the proof, it suffices to show that $f(ae) = \alpha f(a)$ for some $\alpha \geqslant 1$. For this the following lemma is provided.

Lemma 4.3. *Let $P_e = \{ae \,|\, a \in P\}$. Then P_e is equal to P, and hence $B(P_e) = (P_e, +, e)$ is an Archimedean f.o. monoid.*

Proof. Since it is obvious that $P_e \subset P$, we show only that $P \subset P_e$. Let $x \geqslant e$ be an arbitrary element of P. Then since $x = ae$ where $a = x/e \in P$ by right solvability, we have $x \in P_e$. It is clear from Lemma 4.1 that $B(P_e)$ is an Archimedean f.o. monoid. □

Combining this lemma with Hölder's theorem, we obtain the result that $f(B(P_e))$ is a submonoid of the additive f.o. monoid of all non-negative real numbers. Since $f((a+b)e) = f(ae) + f(be)$ by (4), there is a strictly positive real number α such that $f(ae) = \alpha f(a)$ (e.g. see the proof of Proposition 2.2.1 in [8]). Moreover, since $a \leqslant ae$ for all $a \in P$ by r-positivity, $f(a) \leqslant f(ae) = \alpha f(a)$. Thus $\alpha \geqslant 1$. □

The hypothesis of the following corollaries is that P is a left Archimedean, central r-naturally f.o. groupoid with left identity.

Corollary 4.4. *If $P = \mathbb{R}^+$, then $ab = \alpha a + b$ $(\alpha \geqslant 1)$* for all $a, b \in \mathbb{R}^+$.

Proof. It suffices to show that the o-isomorphism f in the proof of Theorem 4.2 is continuous. Indeed, if so, then since f is additive and continuous on $\mathbb{R}^+$, it is well known [1] that $f(a) = sa$ for some $s \in \mathbb{R}$. Setting $s = 1$, we obtain $f(ab) = \alpha a + b$. To prove continuity, assume that $a > b$. By right solvability $a = xb$ for some $x \in P$. Since P has no smallest strictly positive element, we have $a > x'b > b$ for $x' < x$, and hence $f(a) > f(x'b) > f(b)$. This means that f has no gap in its range. Hence we conclude from Debreu's [2] open gap lemma that f is continuous. □

Corollary 4.5. *If e is a two-sided identity, then P is o-isomorphic to a submonoid of the additive f.o. monoid of all non-negative real numbers.*

Proof. Since $a/e = a$, identities (2) and (3) reduce to $(ab)c = a(bc)$ and $ab = ba$, respectively. Also it is obvious that P satisfies (M) and the Archimedean property. □

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