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Transversals in loops. 2. Structural theorems

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Abstract. An investigation of a new notion of a transversal in a loop to its subloop is continued in the present article. This notion generalized a well-known notion of a transversal in a group to its subgroup and can be correctly defined only in the case, when some specific condition (condition A) for a loop and its subloop is fulfilled. The connections between transversals in some loop to its subloop and transversals in multiplicative group of this loop to suitable subgroup are studied in this work.

1. Introduction

In the present work we continue the study of a variant of natural generalization of a notion of transversal in a group to its subgroup [1, 5, 6, 11] at the class of loops, begun in [10]. As the elements of a left (right) transversal in a group to its subgroup are the representatives of every left (right) coset to the subgroup, then a notion of a left (right) transversal in a loop to its subloop can be well defined only in the case when this loop admits a left (right) coset decomposition by its subloop (see Condition A, Definition 2.4, [10]).

In the part 2 the different structural theorems are proved. They demonstrate the correspondence between transversals in a loop to its subloop and transversals in a multiplicative group of this loop to its suitable subgroup. Also, we demonstrate the necessity of Condition A when we generalize a notion of transversal at the class of loops.

Further, we shall use the following notations: $\langle L, \cdot, e\rangle$ is an initial loop with the unit e; $\langle R, \cdot, e\rangle$ is its proper subloop; E is a set of indexes ($1 \in E$) of the left (right) cosets R_i in L to R, where $R_1 = R$.

All necessary definitions and preliminary statements may be found in [10].

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2. The Condition A and included subgroups

The following lemma is an explanation of the necessity of the Condition A in the investigation of transversals in loops.

Lemma 2.1. *Let G be a group, H be its proper subgroup. Let K be a subgroup of group G such that $H \subseteq K \subset G$. If $T = \{t_i\}_{i \in E}$ is a left transversal G to H, then:*

1. $T_1 = T|_K = \{t_j\}_{j \in E_1}$, *where* $E_1 = \{x \in E \,|\, t_x \in K\}$, *is a left transversal K to H,*
2. $< E_1, \overset{(T_1)}{\cdot}, 1 > \subset < E, \overset{(T)}{\cdot}, 1 >$,
3. *The left Condition A is fulfilled in the left loop* $< E, \overset{(T)}{\cdot}, 1 >$ *to its left subloop* $< E_1, \overset{(T)}{\cdot}, 1 >$: *for every* $a, b \in < E, \overset{(T)}{\cdot}, 1 >$ *and every* $u \in < E_1, \overset{(T)}{\cdot}, 1 >$ *there exist* $c \in < E, \overset{(T)}{\cdot}, 1 >$ *and* $u_1 \in < E_1, \overset{(T)}{\cdot}, 1 >$ *such that* $a \overset{(T)}{\cdot} (b \overset{(T)}{\cdot} u) = c \overset{(T)}{\cdot} u_1$.

Proof. 1. Let us denote $E_1 = \{x \in E \,|\, t_x \in K\}$. Then the transversal $T_1 = \{t_j\}_{j \in E_1}$ consists of those elements of the transversal T which belong to the subgroup K. Let us take an arbitrary element $g \in K$; since $T = \{t_i\}_{i \in E}$ is a transversal G to H, then $g = t_{i_0} \cdot h$, $t_{i_0} \in T$, $h \in H$. But $g \in K$, $h \in H \subseteq K$, so we obtain that $t_{i_0} \in K$. Then every element $g \in K$ can be represented in the form $g = t_x \cdot h$, where $h \in H$ and

$$x \in E_1 = \{z \in E \,|\, t_z \in K \cap T\}.$$

This representation is unique for every $g \in K$, because it is the same for the transversal T in G to H.

2. Let us consider the set E_1 introduced in 1. Let $t_a, t_b \in K$ (is equal $t_a, t_b \in T_1$), then $K \ni t_a \cdot t_b = (t_c h)$ $h \in H$. As $t_c \in K$, then $t_c \in T_1$ and we obtain: $c \in E_1$. Thus $a \overset{(T_1)}{\cdot} b = c$. But $t_a t_b \in K \subset G$, and $G \ni t_a \cdot t_b = (t_c h)$, $h \in H$, so $a \overset{(T)}{\cdot} b = c$. Therefore

$$(\overset{(T_1)}{\cdot}) \equiv (\overset{(T)}{\cdot})|_{E_1},$$

and finally $< E_1, \overset{(T_1)}{\cdot}, 1 > \subset < E, \overset{(T)}{\cdot}, 1 >$.

3. Let $a, b \in E$ and $x \in E_1$ (is equal $t_a, t_b \in G$ and $t_x \in K$), then

$$\begin{aligned} t_a \cdot t_b \cdot t_x &= t_a t_{b \overset{(T)}{\cdot} x} h' = t_{a \overset{(T)}{\cdot} (b \overset{(T)}{\cdot} x)} h'', \\ t_a \cdot t_b \cdot t_x &= t_{a \overset{(T)}{\cdot} b} h_1 t_x, \quad h_1 \in H, \quad h', h'' \in H. \end{aligned} \tag{1}$$

But $K \ni h_1 t_x = (t_u h'_1)$, $h'_1 \in H$, $u = \hat{t}_u(1) = \hat{t}_u \hat{h}'_1(1) = \hat{h}_1 \hat{t}_x(1) = \hat{h}_1(x)$, $h_1 t_x = (t_{\hat{h}_1(x)} h'_1) \in K$, $t_{\hat{h}_1(x)} \in K$, $\hat{h}_1(x) \in E_1$. So, (1) can be rewritten in the form

$$t_{a \overset{(T)}{\cdot} (b \overset{(T)}{\cdot} x)} h'' = t_{a \overset{(T)}{\cdot} b} t_{\hat{h}_1(x)} h'_1 = t_{(a \overset{(T)}{\cdot} b) \overset{(T)}{\cdot} \hat{h}_1(x)} h''_1.$$

Hence

$$a \overset{(T)}{\cdot} (b \overset{(T)}{\cdot} x) = (a \overset{(T)}{\cdot} b) \overset{(T)}{\cdot} \hat{h}_1(x).$$

Then for the left loop $< E, \overset{(T)}{\cdot}, 1 >$ and its left subloop $< E_1, \overset{(T_1)}{\cdot}, 1 >$ the left Condition A is fulfilled. □

Lemma 2.2. *Let $H \subseteq K \subset G$ be groups and let $T^* = \{t_x\}_{x \in E_0}$ be a left transversal G to K. Then T^* (as a set) can be always supplemented up to some left transversal $T = \{t_x\}_{x \in E}$ of G to H.*

Proof. If $T^* = \{t_x\}_{x \in E_0}$ is the left transversal G to K, then

$$(t_x K) \cap (t_y K) = \varnothing \quad \forall x, y \in E_0, \quad x \neq y.$$

Since $H \subseteq K$, we have $(t_x H) \cap (t_y H) = \varnothing$ for all $x, y \in E_0$, $x \neq y$.

If $K \equiv H$ then everything is proven. Let $H \subset K$ and we shall consider a union

$$S_0 = \underset{x \in E_0}{\cup} (t_x H).$$

Since

$$S_0 = \underset{x \in E_0}{\cup} (t_x H) \subset \underset{x \in E_0}{\cup} (t_x K) = G,$$

then S_0 is a subset in G consisting of a collection of left cosets in G to H. Supplementing S_0 up to G by left cosets in G to H, which consists in $(G - S_0)$, and choosing in every coset an unique representative, we obtain a required left transversal $T = \{t_x\}_{x \in E}$. Moreover, $T^* \subset T$ and $E_0 \subseteq E$. □

Lemma 2.3. *Let the assumptions of Lemma 2.2 be satisfied. Let $T^* = \{t_x\}_{x \in E_0}$ be a left transversal G to K, and $T = \{t_x\}_{x \in E}$ be a such left transversal G to H, for which $T^* \subseteq T$ and $E_0 \subseteq E$. Then $T_1 = T \cap K = \{t_x\}_{x \in E_1}$ is a left transversal K to H and the following statements are true:*

1. *All elements of the subset E_0 form a left transversal the left loop $< E, \overset{(T)}{\cdot}, 1 >$ to its left subloop $< E_1, \overset{(T)}{\cdot}, 1 >$.*
2. *The operations $< E_0, \overset{(T^*)}{\cdot}, 1 >$ and $< E_0, \overset{(E_0)}{\cdot}, 1 >$ are isomorphic (the first operation is a transversal operation that corresponds to the*

left transversal T^* in G to K, the second corresponds to a left transversal E_0 in the left loop $< E, \overset{(T)}{\cdot}, 1 >$ to its left subloop $< E_1, \overset{(T)}{\cdot}, 1 >$).

Proof. According to Lemma 2.1 $T_1 = T \cap K$ is a left transversal K to H.

1. Let g be an arbitrary element of G. Then

$$g = t_x k, \quad t_x \in T^* \subseteq T, \quad k \in K, \quad x \in E_0,$$

and, on the other hand,

$$g = t_y h_1, \quad t_y \in T, \quad h_1 \in H, \quad y \in E.$$

Also $k = t_z h_2,\ t_z \in T_1 \subset T,\ z \in E_1,\ h_2 \in H$. Using the above we obtain

$$t_y h_1 = g = t_x k = t_x t_z h_2 = t_{x \overset{(T)}{\cdot} z} h_2', \quad h_2' \in H,$$

and so

$$y = x \overset{(T)}{\cdot} z. \tag{2}$$

Since $g \in G$ is arbitrary, (2) means that for every $y \in E$ there exist $x \in E_0$ and $z \in E_1$ such that $y = x \overset{(T)}{\cdot} z$. So, it is sufficient to show the uniqueness of the representation (2).

Let us assume, that this representation is not unique, then there exists $y \in E$ such that

$$y = x_1 \overset{(T)}{\cdot} z_1 = x_2 \overset{(T)}{\cdot} z_2, \quad x_1, x_2 \in E_0, \quad z_1, z_2 \in E_1.$$

Then

$$\begin{aligned} t_y &= t_{x_1 \overset{(T)}{\cdot} z_1} = t_{x_1} t_{z_1} h' = t_{x_1}(t_{z_1} h') \in t_{x_1} K, \\ t_y &= t_{x_2 \overset{(T)}{\cdot} z_2} = t_{x_2} t_{z_2} h'' = t_{x_2}(t_{z_2} h'') \in t_{x_2} K, \end{aligned} \tag{3}$$

(where $h', h'' \in H$). Since $T^* = \{t_x\}_{x \in E_0}$ is the left transversal G to K, then $x_1 = x_2$. Thus (3) may be rewritten in the form

$$t_{x_1} t_{z_1} h' = t_y = t_{x_2} t_{z_2} h'', \ t_{z_1} h' = t_{z_2} h''.$$

Since $T_1 = \{t_z\}_{z \in E_1}$ is the left transversal K to H, we have $z_1 = z_2$. Hence the representation (2) is unique, and elements of the set E_0 form a left transversal $< E, \overset{(T)}{\cdot}, 1 >$ to $< E_1, \overset{(T)}{\cdot}, 1 >$.

2. Let $< E_0, \overset{(T^*)}{\cdot}, 1 >$ be a transversal operation corresponding to a left transversal $T^* = \{t_x\}_{x\in E_0}$ the group G to its subgroup K. Then

$$a \overset{(T^*)}{\cdot} b = c \ \Rightarrow \ t_a t_b = t_c k, \ t_a, t_b, t_c \in T^* \subset T, a, \, b, \, c \in E_0, \ k \in K,$$

and $k = t_z h, \ t_z \in T_1 \subset T, \ z \in E_1, \ h \in H$.

From the above we have $t_a t_b = t_c k = t_c t_z h$, i,e., $t_{a \overset{(T)}{\cdot} b} h' = t_{c \overset{(T)}{\cdot} z} h'' h$, $h', h'' \in H$. Thus $a \overset{(T)}{\cdot} b = c \overset{(T)}{\cdot} z$. Since $a, b, c \in E_0$, $z \in E_1$, from 1 we obtain $a \overset{(E_0)}{\cdot} b = c$, (see also (8) from [10]). Consequently, $a \overset{(T^*)}{\cdot} b = c = a \overset{(E_0)}{\cdot} b$, which completes the proof. □

Corollary 2.4. *Let $H \subseteq K \subset G$ be groups. Then there exists a one-to-one correspondence between each left transversal $T^* = \{t_x\}_{x\in E_0}$ of G to K and some left transversal E_0 the left loop $< E, \overset{(E_0)}{\cdot}, 1 >$ to its left subloop $< E_1, \overset{(E_0)}{\cdot}, 1 >$ (where T is a left transversal G to H, $T^* \subset T$, and $T_1 = \{t_z\}_{z\in E_1}$ is a left transversal K to H, $T_1 = T \cap K$) such that corresponding transversal operations $\overset{(T^*)}{\cdot}$ and $\overset{(E_0)}{\cdot}$ are isomorphic.* □

This correspondence can be converted, as it will be shown further in the next paragraph.

Analogous results may be proved for the right transversals and two-sided transversals in loops to its proper subloops.

3. Semidirect products of loops

Let us remind a definition of semidirect product of a left loop $L =< E, \cdot, 1 >$ with two-sided unit 1 and a suitable permutation group H acting on the set E ($H \subseteq St_1(S_E)$) (see [8], [13]).

Definition 3.1. Let the following two conditions be fulfilled for some left loop $L =< E, \cdot, 1 >$ and the permutation group H:

1. $\forall a, \, b \in E : \ \ l_{a,\,b} = (L^{-1}_{a\cdot b} L_a L_b) \in H$,
2. $\forall u \in E$ and $\forall h \in H : \ \varphi(u, h) = (L^{-1}_{h(u)} h \, L_u h^{-1}) \in H$, where L_a is a left translation in $< E, \cdot, 1 >$.

Then the set $E \times H$ with the operation

$$(u, h_1) * (v, h_2) = (u \cdot h_1(v), \, l_{u,h_1(v)} \varphi(v, h_1) h_1 h_2)$$

is a group denoted by $L \leftthreetimes H =< E \times H, *, (1, id) >$ and called a *semidirect product* of L and H. The group H is called a *transassociant* of L.

It is easy to show (see [8, 13]) that for the *left multiplicative group* $LM(L)$ and the *left inner permutation* group $LI(L)$ of L we have

$$LI(L) = St_1(LM(L)) \subset LM(L) \quad \text{and} \quad LM(L) = L \times LI(L).$$

Lemma 3.2. *Let $L =< E, \cdot, 1 >$ be a loop and $R =< E_1, \cdot, 1 >$ be its proper subloop, and the left Condition A be fulfilled. If $T = \{t_x\}_{x \in E_0}$ is a left transversal L to R and $H \subseteq St_1(S_L)$ is a permutation group such that $LI(L) \subseteq H$ and $\varphi(u, h) \in H$ for all $u \in L$ and all $h \in H$, then*

1. *a semidirect product $G = L \leftthreetimes H$ can be defined,*
2. *$K = \{(r, h) \,|\, r \in R,\ h \in H\}$ is a subgroup of the group G and $H \subset K$,*
3. *$T^* = \{(t_x, id) \,|\, t_x \in T_0,\ x \in E_0\}$ is a left transversal the group G to its subgroup K,*
4. *the transversal operations $< E_0, \overset{(T)}{\cdot}, 1 >$ and $< E_0, \overset{(T^*)}{\cdot}, 1 >$ (corresponding to the left transversal T the loop L to its subloop R, and to the left transversal the group G to its subgroup K, respectively) coincide.*

Proof. 1. If the conditions of the Lemma are satisfied, then we can define the semidirect product $G = L \leftthreetimes H = \{(a, h) \,|\, a \in L,\ h \in H\}$, where $H = \{(1, h) \,|\ h \in H\} \subseteq \{(a, h) \,|\, a \in L,\ h \in H\} = G$.

2. Since $R \subseteq L$, then according to the assumptions of our lemma, we have $l_{a,b} \in LI(L) \subseteq H$ for all $a, b \in R$. This implies

$$\varphi(u, h) \in \{\varphi(u, h) \,|\, u \in R,\ h \in H\} \subseteq \{\varphi(u, h) \,|\, u \in L,\ h \in H\} \subseteq H$$

for all $u \in R$ and $h \in H$. Thus, we can define a semidirect product

$$K = R \leftthreetimes H = \{(r, h) \,|\, r \in R,\ h \in H\}.$$

Clearly, $H = \{(1, h) \,|\ h \in H\} \subset K \subseteq \{(a, h) \,|\, a \in L,\ h \in H\} = G$.

3. Let $T = \{t_x\}_{x \in E_0}$ be a left transversal L to R and let

$$T^* = \{(t_x, id) \,|\, t_x \in T,\ x \in E_0\} \subset G.$$

For an arbitrary element $x \in E_0$ we consider the set

$$K_x = (t_x, id) * K = \{(t_x, id) * (r, h) \,|\, r \in R, h \in H\} \subset G. \tag{4}$$

K_x is a left coset in G to K. Indeed, any $g \in G$ can be written in the form $g = (u_0, h_0)$, where $u_0 \in L$, $h_0 \in H$. Since $T = \{t_x\}_{x \in E_0}$ is a left transversal L to R, we have $u_0 = t_{x_0} \cdot r_0$ for some $t_{x_0} \in T_0$ and $r_0 \in R$. Thus for $h_1 = (l^{-1}_{t_{x_0}, r_0} h_0) \in H$ we have

$$(t_{x_0}, id) * (r_0, h_1) = (t_{x_0} \cdot r_0, l_{t_{x_0}, r_0} h_1) = (u_0, h_0) = g,$$

which gives $g \in (t_{x_0}, id) * K$. Since $t_x \cdot R = \{t_x \cdot r | r \in R\}$ is a left coset in L to R, in view of (4), for $x_1 \neq x_2$ we obtain $t_{x_1} \cdot R) \cap (t_{x_2} \cdot R) = \varnothing$. So for $x_1 \neq x_2$ we have

$$\begin{aligned} K_{x_1} \cap K_{x_2} &= ((t_{x_1}, id) * K) \cap ((t_{x_2}, id) * K) \\ &= \{(t_{x_1}, id) * (r, h) \,|\, r \in R, h \in H\} \cap \{(t_{x_2}, id) * (r, h) \,|\, r \in R, h \in H\} \\ &= \{(t_{x_1} \cdot r, l_{t_{x_1}, r} h) \,|\, r \in R, h \in H\} \cap \{(t_{x_2} \cdot r, l_{t_{x_2}, r} h) \,|\, r \in R, h \in H\} = \varnothing. \end{aligned}$$

Hence $K_x = (t_x, id) * K$, $x \in E_0$ is the left cosets in G to K. So, $T^* = \{(t_x, id) \,|\, t_x \in T, \ x \in E_0\}$ is a left transversal G by K.

4. Let us consider the transversal operation $< E_0, \overset{(T^*)}{\cdot}, 1 >$ which corresponds to the left transversal $T^* = \{(t_x, id)\}|_{x \in E_0}$. Then $x \overset{(T^*)}{\cdot} y = z$ iff $(t_x, id) * (t_y, id) = (t_z, id) * (r, h)$, $(r, h \in K)$. Thus $(t_x \overset{(L)}{\cdot} t_y, l_{t_x, t_y}) = (t_z \overset{(L)}{\cdot} r, l_{t_z, r} h)$. Hence $t_x \overset{(L)}{\cdot} t_y = t_z \overset{(L)}{\cdot} r$, $r \in R$, $t_x, t_y, t_z \in T$. Consequently, $x \overset{(T)}{\cdot} y = z$, i.e., $x \overset{(T^*)}{\cdot} y = x \overset{(T)}{\cdot} y$ for all $x, y \in E_0$. □

Corollary 3.3. *If the conditions of Lemma* 3.2 *are satisfied, then for every* $h \in H$ *we have* $\hat{h}(R) \subseteq R$.

Proof. The previous lemma shows that for any two elements (r_1, h_1) and (r_2, h_2) from K holds

$$(r_1, h_1) * (r_2, h_2) = (r_1 \overset{(R)}{\cdot} \hat{h}_1(r_2), l_{r_1, \hat{h}_1(r_2)} \varphi(r_2, h_1) h_1 h_2).$$

Because K is a subgroup of the group G, for all $r_1, r_2 \in R$ and $h \in H$ we have $(r_1 \overset{(R)}{\cdot} \hat{h}(r_2)) \in R$. Hence $\hat{h}(R) \subseteq R$ for all $h \in H$. □

In the case $H = LI(R)$ the inclusion $\hat{h}(R) \subseteq R$ is equivalent to the fact that $l_{a,b}(R) \subseteq R$ for all $a, b \in L$. The last condition is equivalent to the left Condition A for the loop L and its subloop R (Lemma 2.8 in [10]).

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