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Geometry of semiabelian n-ary groups

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Abstract. Semiabelian n-ary groups are characterized by their parallelograms and symmetries of points with respect to the vertices of tetragons.

1. Introduction

If the standard (affine) geometry has a fixed point O, then any point P of this geometry is uniquely determined by the vector $\vec{p} = \overrightarrow{OP}$, and conversely, the vector $\overrightarrow{OP}$ uniquely determines the point P. Any interval $\overline{PQ}$ may be interpreted as the vector $\vec{q} - \vec{p}$ or as the vector $\vec{p} - \vec{q}$. In the second case,

$$\overline{AB} = \overline{CD} \iff \vec{a} - \vec{b} + \vec{d} = \vec{c},$$

or, in the other words

$$\overline{AB} = \overline{CD} \iff f(a, b, d) = c\,,$$

where any vector $\vec{v}$ is treated as an element v of a commutative group $(G, +)$. Then the operation f has the form $f(x, y, z) = x - y + z$. Groups (also non-commutative) with a ternary operation defined in this way were considered by J. Certaine (cf. [3]) as a special case of *ternary heaps* studied earlier by H. Prüfer (cf. [25]). Ternary heaps have interesting applications to projective geometry (cf. [1]), affine geometry (cf. [2]), theory of nets (webs), theory of knots and even to the differential geometry.

On the other hand, affine geometries may be treated as geometries defined by some ternary relations (cf. for example [31]). Such geometries may be defined also by some n-ary ($n > 3$) relations (cf. [32]). Basic properties of affine geometries defined by ternary groups were described by Vakarelov (cf. [34]). Rusakov extended these results to the case of affine geometries

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defined by n-ary groups (cf. [28] and [29]). Later, affine geometries induced by n-ary groups and various properties of n-ary groups connected with affine geometries were studied by many authors (see for example [7], [12], [14], [27]).

2. Preliminaries

We will use the following abbreviated notation: the sequence $x_i, \ldots, x_j$ will be denoted by x_i^j (for $j < i$ it will be the empty symbol). In the case $x_{i+1} = \ldots = x_{i+k} = x$ instead of x_{i+1}^{i+k} we will write $\overset{(k)}{x}$. In this convention the formula $f(x_1, \ldots, x_i, x, x, \ldots, x, x_{i+k+1}, \ldots, x_n)$ will be written in the form $f(x_1^i, \overset{(k)}{x}, x_{i+k+1}^n)$.

If $m = k(n-1)+1$, then the m-ary operation g of the form

$$g(x_1^{k(n-1)+1}) = \underbrace{f(f(\ldots, f(f}_{k}(x_1^n), x_{n+1}^{2n-1}), \ldots), x_{(k-1)(n-1)+2}^{k(n-1)+1})$$

is denoted by $f_{(k)}$. In certain situations, when the arity of g does not play a crucial role, or when it will differ depending on additional assumptions, we write $f_{(\cdot)}$, to mean $f_{(k)}$ for some $k = 1, 2, \ldots$

By an *n-ary group* (G, f) we mean a non-empty set G together with one n-ary operation $f : G^n \to G$ satisfying for all $i = 1, 2, \ldots, n$ the following two conditions:

1^0 *the associative law*:

$$f(f(x_1^n), x_{n+1}^{2n-1}) = f(x_1^{i-1}, f(x_i^{n+i-1}), x_{n+i}^{2n-1})$$

2^0 for all $a_1, a_2, \ldots, a_n, b \in G$ there exits a unique $x_i \in G$ such that

$$f(a_1^{i-1}, x_i, a_{i+1}^n) = b\,.$$

Such n-ary groups may also be considered as algebras with two or more operations (see for example [6]). In particular, an n-ary group may be treated as an algebra with one associative n-ary operation and one unary operation satisfying some identities.

Theorem 2.1. *An algebra* $(G, f, ^{-})$ *with one associative n-ary* ($n > 2$) *operation f and one unary operation* $^{-} : x \mapsto \overline{x}$ *is an n-ary group if and only if the identities*

$$f(\overset{(i-2)}{x}, \overline{x}, \overset{(n-i)}{x}, y) = f(y, \overset{(n-j)}{x}, \overline{x}, \overset{(j-2)}{x}) = y \tag{1}$$

are satisfied for some $i, j \in \{2, \ldots, n\}$. □

Theorem 2.2. *An algebra* $(G, f, ^{[-2]})$ *with one associative n-ary* ($n \geqslant 2$) *operation f and one unary operation* $^{[-2]} : x \mapsto x^{[-2]}$ *is an n-ary group if and only if the identities*

$$f(x^{[-2]}, \overset{(n-2)}{x}, f(\overset{(n-1)}{x}, y)) = f(f(y, \overset{(n-1)}{x}), \overset{(n-2)}{x}, x^{[-2]}) = y \tag{2}$$

are satisfied. □

The first theorem is proved in [10], the second in [26]. Useful modifications of Theorem 2.1 one can find in [4, 6, 9].

An element $\overline{x}$ satisfying the identities (1) is called *skew* to x. It is uniquely determined as a solution of the equation $f(\overset{(n-1)}{x}, z) = x$. In general $\overline{x} \neq x$, but there are n-ary groups in which $\overline{x} = x$ for all or only for some x (cf. [5] and [8]). In some n-ary groups we have

$$\overline{f(x_1, x_2, \ldots, x_n)} = f(\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_n), \tag{3}$$

which means that in some n-ary groups the operation $x \to \overline{x}$ is an endomorphism (cf. [8, 11, 13, 30]). This situation take place in *semiabelian* n-ary groups, i.e., in n-ary groups satisfying the identity

$$f(x_1^n) = f(x_1, x_2^{n-1}, x_n), \tag{4}$$

for example (cf. [13]). The class of all semiabelian n-ary groups coincides with the class of *medial* n-ary groups, i.e., n-ary groups in which

$$f(f(x_{11}^{1n}), f(x_{21}^{2n}), \ldots, f(x_{n1}^{nn})) = f(f(x_{11}^{n1}), f(x_{12}^{n2}), \ldots, f(x_{1n}^{nn})) \tag{5}$$

holds for all $x_{ij} \in G$ (cf. [13]). This condition means that the value of the operation f applied to the matrix $[x_{ij}]_{n\times n}$ is the same if we apply it to rows (from left) or to columns (from top).

As a simple consequence of results proved in [4] we obtain the following lemma.

Lemma 2.3. *For $n \geqslant 3$ an n-ary group (G, f) is semiabelian if and only if there exists $a \in G$ such that*

$$f(x, \overline{a}, \overset{(n-3)}{a}, y) = f(y, \overline{a}, \overset{(n-3)}{a}, x) \tag{6}$$

holds for all $x, y \in G$. □

In the theory of n-ary groups an important role is played by the Post's coset theorem which says that any n-ary group (G, f) can be embedded (as a coset) into some ordinary group G^* called the *covering group* for (G, f) (cf. [24]). But theory of n-ary groups can not be reduced to the theory of such groups [9]. A nice construction of a covering group is presented in [22].

Theorem 2.4. (Post's coset theorem)
For any n-ary group (G, f) there exists a binary group $(G^*, \cdot)$ such that $G \subset G^*$ and

$$f(x_1^n) = x_1 \cdot x_2 \cdot x_3 \cdot \ldots \cdot x_n$$

for all $x_1, x_2, \ldots, x_n \in G$. In this group $\overline{x} = x^{2-n}$. □

3. Geometry of semiabelian n-ary groups

In the affine geometry defined on an n-ary group (G, f) (for details see [27] or [28]) elements of G are called *points*. Four points $a, b, c, d \in G$ define a *parallelogram* if and only if

$$f(f(a, b^{[-2]}, \overset{(n-2)}{b}), \overset{(n-2)}{b}, c) = d.$$

Two points a and c are *symmetric* if and only if there exists a uniquely determined point $x \in G$ such that

$$f(f(a, x^{[-2]}, \overset{(n-2)}{x}), \overset{(n-2)}{x}, c) = x.$$

Since the operation f is associative identities (2) used in Theorem 2.2 can be written in the form

$$f(f(x^{[-2]}, \overset{(n-1)}{x}), \overset{(n-2)}{x}, y) = f(y, \overset{(n-2)}{x}, f(\overset{(n-1)}{x}, x^{[-2]})) = y,$$

which together with Theorem 2.1 implies

$$f(x^{[-2]}, \overset{(n-1)}{x}) = f(\overset{(n-1)}{x}, x^{[-2]}) = \overline{x}\,, \tag{7}$$

where $\overline{x}$ denotes the element skew to x. Thus for $n \geqslant 3$ the above two definitions can be presented in the following more useful form (cf. [7]):

Definition 3.1. Four points a, b, c, d of an n-ary group (G, f), where $n \geqslant 3$, define a *parallelogram* if and only if $f(a, \bar{b}, \overset{(n-3)}{b}, c) = d$.

Definition 3.2. Two elements a and c of an n-ary group (G, f) are *symmetric* if there exists a uniquely determined point $x \in G$ such that

$$f(a, \overline{x}, \overset{(n-3)}{x}, c) = x. \tag{8}$$

Since for symmetric points a and c of G the element x is uniquely determined we can consider the map $S_x : G \to G$ with the property $S_x(a) = c$. This map will be called the *symmetry*.

Definition 3.3. The point x of an n-ary group (G, f) is *self-returning* with respect to the finite sequence of points $a, b, c, \ldots, v \in G$ if

$$S_v(\ldots S_c(S_b(S_a(x)))) = x.$$

From the definition of an n-ary group it follows that in (8) an element c is uniquely determined by elements a and x. Thus, using the same method as in [4] and [10], we can prove that for $n \geqslant 3$ the symmetry S_x has the form:

$$S_x(a) = f(x, \overline{a}, \overset{(n-3)}{a}, x).$$

The point

$$S_a(b) = (a, \bar{b}, \overset{(n-3)}{b}, a)$$

is called *symmetrical* to the point b with respect to the point a. The sequence of k arbitrary elements from G is called a *k-gon* (cf. [29]).

In view of Theorem 2.4 the symmetry of points of an n-ary group can be considered as an *external* symmetry in the corresponding covering group. Namely, *two points a and c of an n-ary group (G, f) are symmetric if there exists a uniquely determined point $x \in G$ such that $ax^{-1}c = x$ in the covering group G^* of (G, f).* Note that in general x^{-1} is not an element of G.

In this case, the symmetry S_a has the form

$$S_a(x) = ax^{-1}a. \tag{9}$$

Moreover, as a consequence of Lemma 2.3 we obtain

Corollary 3.4. *An n-ary group (G, f) is semiabelian if and only if its covering group we have*

$$ax^{-1}b = bx^{-1}a \tag{10}$$

for all $a, b \in G$ and some fixed $x \in G$. □

Lemma 3.5. *Let $a_1, a_2, a_3, \ldots, a_m$ be arbitrary points of a semiabelian n-ary group (G, f). Then the composition $S_{a_m}(...S_{a_4}(S_{a_3}(S_{a_2}(S_{a_1}(x))))...)$ is equal to*

$$f_{(m)}(x, \underbrace{\overline{a}_1, \overset{(n-3)}{a_1}, a_2}_{2}, \underbrace{\overline{a}_3, \overset{(n-3)}{a_3}, a_4}_{2}, \ldots, \underbrace{\overline{a}_{m-1}, \overset{(n-3)}{a_{m-1}}, a_m}_{2})$$

if m is even, or to

$$f_{(m)}(S_{a_1}(x), \underbrace{\overline{a}_2, \overset{(n-3)}{a_2}, a_3}_{2}, \underbrace{\overline{a}_4, \overset{(n-3)}{a_4}, a_5}_{2}, \ldots, \underbrace{\overline{a}_{m-1}, \overset{(n-3)}{a_{m-1}}, a_m}_{2})$$

if m is odd.

Proof. Indeed, for points $a, b, x \in G$ we have

$$\begin{aligned} S_bS_a(x) &= S_b(S_a(x)) \overset{(9)}{=} b(ax^{-1}a)^{-1}b = ba^{-1}xa^{-1}b \overset{(10)}{=} xa^{-1}ba^{-1}b \\ &\overset{(9)}{=} f_{(2)}(x, \overline{a}, \overset{(n-3)}{a}, b, \overline{a}, \overset{(n-3)}{a}, b). \end{aligned}$$

Similarly,

$$\begin{aligned} S_c(S_bS_a(x)) &\overset{(9)}{=} c(xa^{-1}ba^{-1}b)^{-1}c = cb^{-1}ab^{-1}ax^{-1}c \overset{(10)}{=} ax^{-1}ab^{-1}cb^{-1}c \\ &\overset{(9)}{=} f_{(3)}(a, \overline{x}, \overset{(n-3)}{x}, a, \overline{b}, \overset{(n-3)}{b}, c, \overline{b}, \overset{(n-3)}{b}, c) \\ &= f_{(3)}(S_a(x), \overline{b}, \overset{(n-3)}{b}, c, \overline{b}, \overset{(n-3)}{b}, c). \end{aligned}$$

Consequently,

$$\begin{aligned} S_d(S_c(S_bS_a(x))) &= S_d(S_c(xa^{-1}ba^{-1}b)) = (xa^{-1}ba^{-1}b)c^{-1}dc^{-1}d \\ &= f_{(4)}(x, \overline{a}, \overset{(n-3)}{a}, b, \overline{a}, \overset{(n-3)}{a}, b, \overline{c}, \overset{(n-3)}{c}, d, \overline{c}, \overset{(n-3)}{c}, d) \\ &= f_{(4)}(x, \underbrace{\overline{a}, \overset{(n-3)}{a}, b}_{2}, \underbrace{\overline{c}, \overset{(n-3)}{c}, d}_{2}) \end{aligned}$$

and so on. □

Proposition 3.6. *In any semiabelian n-ary group (G,f) for all elements $a_1,a_2,\ldots,a_m,x\in G$, where m is odd, we have*

$$T_{a_1^m}(T_{a_1^m}(x))=x\,, \tag{11}$$

where $T_{a_1^m}(x)=S_{a_m}(\ldots S_{a_3}(S_{a_2}(S_{a_1}(x)))\ldots)$.

Proof. According to Lemma 3.5 and (9) for odd m we have

$$T_{a_1^m}(x)=a_1x^{-1}a_1a_2^{-1}a_3a_2^{-1}a_3a_4^{-1}a_5a_4^{-1}a_5\ldots a_{m-1}^{-1}a_ma_{m-1}^{-1}a_m\,,$$

which together with (10) implies (11). □

Corollary 3.7. *Each point of a semiabelian n-ary group is self-returning with respect to double symmetry with respect to the vertex of an arbitrary its polygon with odd number of vertex.* □

These results give the possibility to make new short proofs of the theorems proved in $[15-21]$. Below we give some of them.

Theorem 3.8. *An n-ary group (G,f) is semiabelian if and only if*

$$S_b(S_c(S_d(S_a(x))))=x \tag{12}$$

for any parallelogram $\langle a,b,c,d\rangle$ of (G,f) and an arbitrary $x\in G$.

Proof. From the above results it follows that points a,b,c,d form a parallelogram of an n-ary group (G,f) if and only if $ab^{-1}c=d$ holds in a covering group of (G,f). This together with (9) reduces (12) to the form

$$bc^{-1}ab^{-1}ca^{-1}x=x.$$

Thus $bc^{-1}a=(b^{-1}ca^{-1})^{-1}=ac^{-1}b$. So, $f(b,\overline{c},\overset{(n-3)}{c},a)=f(a,\overline{c},\overset{(n-3)}{c},b)$, which by Lemma 2.3 means that (G,f) is semiabelian.

The converse statement is obvious. □

Theorem 3.9. *An n-ary group (G,f) is semiabelian if and only if*

$$S_d(S_c(S_b(S_a(x))))=x$$

for any parallelogram $\langle a,b,c,d\rangle$ of (G,f) and an arbitrary $x\in G$.

Proof. The proof is analogous to the proof of Theorem 3.8. □

Corollary 3.10. *An n-ary group is semiabelian if and only if each its point is self-returning with respect to the vertex of each its parallelogram.* □

Theorem 3.11. *An n-ary group (G, f) is semiabelian if and only if for any three points $a, b, c \in G$ the tetragon $\langle a, S_b(a), S_c(a), S_d(a)\rangle$, where $d = f(a, \overline{b}, \overset{(n-3)}{b}, c)$, is a parallelogram.*

Proof. Indeed, in the covering group G^* of (G, f) we have $S_d(a) = ab^{-1}cb^{-1}c$ and $a(S_b(a))^{-1}S_c(a) = ab^{-1}ab^{-1}ca^{-1}c$. Thus $a(S_b(a))^{-1}S_c(a) = S_d(a)$ if and only if $ab^{-1}c = cb^{-1}a$. Corollary 3.4 completes the proof. □

Analogously we can prove the following two theorems.

Theorem 3.12. *An n-ary group is semiabelian if and only if for any three its points a, b, c (at least) one of the following tetragons $\langle a, b, S_c(a), S_c(b)\rangle$, $\langle S_b(a), S_c(a), S_c(b), b\rangle$, $\langle S_{S_c(b)}(a), S_c(b), b, S_c(a)\rangle$ is a parallelogram.* □

Theorem 3.13. *An n-ary group (G, f) is semiabelian if and only if for each $a, b, c \in G$ all points $x \in G$ are self-returning with respect to the vertex of the hexagon $\langle S_b(a), S_c(a), S_c(b), S_a(b), S_a(c), S_b(c)\rangle$.* □

4. Vectors of semiabelian n-ary groups

According to [28] an ordered pair $\langle a, b\rangle$ of points $a, b \in G$ is called a *directed segment* of an n-ary group (G, f).In the set of all directed segments of an n-ary group we introduce the binary relation $=$ by putting

$$\langle a, b\rangle = \langle c, d\rangle \Longleftrightarrow f(a, \overline{b}, \overset{(n-3)}{b}, c) = d\,,$$

i.e., $\langle a, b\rangle = \langle c, d\rangle$ if and only if $\langle a, b, c, d\rangle$ is a parallelogram of G. Such defined relation is an equivalence and divides the set of all directed segments into disjoint classes $\langle a, b\rangle_=$. The class $\langle a, b\rangle_=$ is called a *vector* and is denoted by $\overrightarrow{ab}$. Hence

$$\overrightarrow{ab} = \overrightarrow{cd} \Longleftrightarrow f(a, \overline{b}, \overset{(n-3)}{b}, c) = d \Longleftrightarrow ab^{-1}c = d \tag{13}$$

in the covering group of (G, f).

On the set $V(G)$ of all vectors defined on an n-ary group (G, f) one can define the addition $+$ of vectors (cf. [28]). It is not difficult to verify that $(V(G), +)$ is a group. It is Abelian if and only if an n-ary group (G, f) is semiabelian (for details see [28] or [29]).

Lemma 4.1. *In an n-ary group (G,f) for any four points a,b,c,d of G we have*

$$\overrightarrow{a\,b}+\overrightarrow{c\,d}=\overrightarrow{a\,g}=\overrightarrow{h\,d}\,,$$

where $g=f(b,\overline{c},\overset{(n-3)}{c},d)$ and $h=f(c,\overline{b},\overset{(n-3)}{b},a)$.

Proof. Since in the covering group $g=f(b,\overline{c},\overset{(n-3)}{c},d)=bc^{-1}d$, thus

$$f(c,\overline{b},\overset{(n-3)}{b},g)=cb^{-1}g=cb^{-1}(bc^{-1}d)=d\,.$$

So, $\langle c,b,g,d\rangle$ is a parallelogram. Hence $\overrightarrow{cd}=\overrightarrow{bg}$. Consequently, $\overrightarrow{ab}+\overrightarrow{cd}=\overrightarrow{ab}+\overrightarrow{bg}=\overrightarrow{ag}$.

The proof of the second identity is analogous. □

Corollary 4.2. *In an n-ary group (G,f) we have*

$$\overrightarrow{ab}+\overrightarrow{cd}=\overrightarrow{a(bc^{-1}d)} \tag{14}$$

for all a,b,c,d of G. □

Theorem 4.3. *An n-ary group (G,f) is semiabelian if and only if*

$$\overrightarrow{ab}+\overrightarrow{cd}=\overrightarrow{ad}+\overrightarrow{cb} \tag{15}$$

for all $a,b,c,d\in G$.

Proof. Indeed, by Lemma 4.1

$$\overrightarrow{ab}+\overrightarrow{cd}=\overrightarrow{ag_1},\qquad \overrightarrow{ad}+\overrightarrow{cb}=\overrightarrow{ag_2}\,,$$

where $g_1=f(b,\overline{c},\overset{(n-3)}{c},d)$, $g_2=f(d,\overline{c},\overset{(n-3)}{c},b)$. Thus $\overrightarrow{ag_1}=\overrightarrow{ag_2}$ if and only if $\langle a,a,g_1,g_2\rangle$ is a parallelogram, i.e., if and only if $f(a,\overline{a},\overset{(n-3)}{a},g_1)=g_2$. The last means that $g_1=g_2$. This, by Lemma 2.3, means that an n-ary group (G,f) is semiabelian. □

Theorem 4.4. *An n-ary group (G,f) is semiabelian if and only if*

$$\overrightarrow{S_b(a)S_d(c)}=2\,\overrightarrow{bd}+\overrightarrow{ca} \tag{16}$$

for all $a,b,c,d\in G$.

Proof. Indeed, if an n-ary group (G, f) is semiabelian, then

$$\overrightarrow{S_b(a)S_d(c)} \overset{(9)}{=} \overrightarrow{(ba^{-1}b)(dc^{-1}d)} \overset{(14)}{=} \overrightarrow{(ba^{-1}b)d} + \overrightarrow{cd} \overset{(14)}{=} (\overrightarrow{ba} + \overrightarrow{bd}) + \overrightarrow{cd}$$
$$= \overrightarrow{bd} + (\overrightarrow{ba} + \overrightarrow{cd}) \overset{(15)}{=} \overrightarrow{bd} + (\overrightarrow{bd} + \overrightarrow{ca}) = 2\overrightarrow{bd} + \overrightarrow{ca}\,,$$

which proves (16).

Conversely, if (16) holds for all points $a, b, c, d \in G$, then

$$2\overrightarrow{bd} + \overrightarrow{ca} = \overrightarrow{S_b(a)S_d(c)} = \overrightarrow{ba} + \overrightarrow{bd} + \overrightarrow{cd}\,,$$

which, in view of (14), implies

$$\overrightarrow{b(db^{-1}d)} + \overrightarrow{ca} = \overrightarrow{b(ab^{-1}d)} + \overrightarrow{cd}\,.$$

Thus

$$\overrightarrow{b(db^{-1}dc^{-1}a)} = \overrightarrow{b(ab^{-1}dc^{-1}d)}\,.$$

So, that the tetragon $\langle b, b, (db^{-1}dc^{-1}a), (ab^{-1}dc^{-1}d)\rangle$ is a parallelogram. Hence $bb^{-1}(db^{-1}dc^{-1}a) = ab^{-1}dc^{-1}d$. From this, for $c = d$, we obtain

$$db^{-1}a = ab^{-1}d\,.$$

This by Lemma 2.3 means that an n-ary group (G, f) is semiabelian. □

Using the above method we can give a short proof of the following two theorems proved in [21].

Theorem 4.5. *An n-ary group (G, f) is semiabelian if and only if for each its parallelogram $\langle a, b, c, d\rangle$ and each point $x \in G$ we have*

$$\overrightarrow{x\,a} + \overrightarrow{S_a(x)\,b} + \overrightarrow{S_bS_a(x)\,c} + \overrightarrow{S_cS_bS_a(x)\,d} = \overrightarrow{0}\,. \tag{17}$$

Proof. For any four pints $a, b, c, d \in G$ we have

$$\begin{aligned}
&\overrightarrow{x\,a} + \overrightarrow{S_a(x)\,b} + \overrightarrow{S_bS_a(x)\,c} + \overrightarrow{S_cS_bS_a(x)\,d} \\
&\overset{(9)}{=} \overrightarrow{x\,a} + \overrightarrow{(ax^{-1}a)\,b} + \overrightarrow{(ba^{-1}xa^{-1}b)\,c} + \overrightarrow{(cb^{-1}ax^{-1}ab^{-1}c)\,d} \\
&\overset{(14)}{=} \overrightarrow{x\,(xa^{-1}b)} + \overrightarrow{(ba^{-1}xa^{-1}b)\,c} + \overrightarrow{(cb^{-1}ax^{-1}ab^{-1}c)\,d} \\
&\overset{(14)}{=} \overrightarrow{x\,(ab^{-1}c)} + \overrightarrow{(cb^{-1}ax^{-1}ab^{-1}c)\,d} \overset{(14)}{=} \overrightarrow{x\,(xa^{-1}bc^{-1}d)}\,.
\end{aligned}$$

So, according to (13),

$$\overrightarrow{x\,(xa^{-1}bc^{-1}d)} = \overrightarrow{0} = \overrightarrow{x\,x} \Longleftrightarrow xd^{-1}cb^{-1}a = x \Longleftrightarrow d = cb^{-1}a\,.$$

Hence any four points a, b, c, d satisfying (17) form a parallelogram if and only if $d = cb^{-1}a = ab^{-1}c$. Corollary 3.4 completes the proof. □

Theorem 4.6. *An n-ary group (G, f) is semiabelian if and only if*

$$\overrightarrow{x\,a}+\overrightarrow{S_a(x)\,b}+\overrightarrow{S_bS_a(x)\,c}+\overrightarrow{S_cS_bS_a(x)\,d}+\overrightarrow{S_dS_cS_bS_a(x)\,e}+\overrightarrow{S_eS_dS_cS_bS_a(x)\,a} = \overrightarrow{0} \tag{18}$$

for all points $a, b, c, d, x \in G$, where $e = f(d, \overline{c}, \overset{(n-3)}{c}, b)$.

Proof. Similarly as in the previous proof

$$\begin{aligned}\overrightarrow{xa} + \overrightarrow{S_a(x)\,b} + \overrightarrow{S_bS_a(x)\,c} + \overrightarrow{S_cS_bS_a(x)\,d} + \overrightarrow{S_dS_cS_bS_a(x)\,e} + \overrightarrow{S_eS_dS_cS_bS_a(x)\,a}\\ = \overrightarrow{x\,(xa^{-1}bc^{-1}d)} + \overrightarrow{S_dS_cS_bS_a(x)\,e} + \overrightarrow{S_eS_dS_cS_bS_a(x)\,a}\\ = \overrightarrow{x\,(ab^{-1}cd^{-1}e)} + \overrightarrow{S_eS_dS_cS_bS_a(x)\,a} = \overrightarrow{x\,(xa^{-1}bc^{-1}de^{-1}a)}\,.\end{aligned}$$

Hence
$$\overrightarrow{x\,(xa^{-1}bc^{-1}de^{-1}a)} = \overrightarrow{x\,x} \Longleftrightarrow xa^{-1}ed^{-1}cb^{-1}a = x\,.$$

The last means that $a^{-1}e = (d^{-1}cb^{-1}a)^{-1} = a^{-1}bc^{-1}d$, i.e., $e = bc^{-1}d$. But by the assumption $e = dc^{-1}b$. So, (18) holds if and only if $bc^{-1}d = dc^{-1}b$ for all $a, b, c \in G$. □

5. Flocks

Flocks are ternary quasigroups with a *para-associative operation*, i.e., algebras of the form $(G, [\,])$, where $[[x, y, z], u, v] = [x, [u, z, y], v] = [x, y, [z, u, v]]$ for all $x, y, z, u, v \in G$, and for all $a, b \in G$ there are uniquely determined $x, y, z \in G$ such that $[x, a, b] = [a, y, b] = [a, b, z] = c$.

Such flocks are a special case of heaps and semiheaps considered by Vagner [33]. Similar structures are investigated also by Prüfer [25]. Baer (cf. [1]) has investigated a connection linking Brandt groupoids and mixed groups with *idempotent flocks*, i.e., flocks satisfying the identity $[x, x, x] = x$.

As it was observed in [7] flocks and ternary groups have very similar properties. Moreover, the affine geometry induced by n-ary groups ($n > 3$)

can be described by flocks. Namely, if (G, f) is an n-ary group with $n > 3$ then G with the operation

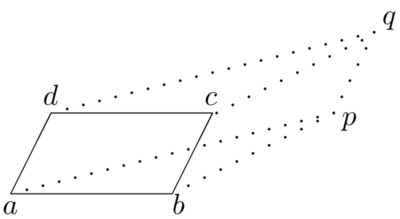
$$[x, y, z] = f(x, \overline{y}, \overset{(n-3)}{y}, z)$$

is a flock. Thus, in the covering group of (G, f) we have $[x, y, z] = xy^{-1}z$. This means that flocks induced by semiabelian n-ary groups are idempotent ternary group.

Theorem 5.1. *Let $\langle a, b, c, d\rangle$ be a parallelogram on an n-ary group (G, f). Then for all $p, q, x, y \in G$*

(1) *$\langle b, p, q, c\rangle$ is a parallelogram if and only if $\langle a, p, q, d\rangle$ is a parallelogram.*
(2) *$\langle d, c, x, y\rangle$ is a parallelogram if and only if $\langle a, b, x, y\rangle$ is a parallelogram.*

The geometrical sense of this theorem is illustrated by the picture:



This theorem is a consequence of Proposition 5.5 proved in [7]. Below we give the equivalent proof based on the above connections.

Proof. Let $\langle a, b, c, d\rangle$ and $\langle b, p, q, c\rangle$ be parallelograms. Then $d = [a, b, c] = ab^{-1}c$ and $c = [b, p, q] = bp^{-1}q$. Thus $[a, p, q] = (ab^{-1}b)p^{-1}q = ab^{-1}(bp^{-1}q) = d$. Hence $\langle a, p, q, d\rangle$ is a parallelogram.

Conversely, if $\langle a, b, c, d\rangle$ and $\langle a, p, q, d\rangle$ are parallelograms, then $ab^{-1}c = d$ and $ap^{-1}q = d$. Thus, $[b, p, q] = (ba^{-1}a)p^{-1}q = ba^{-1}(ap^{-1}q) = ba^{-1}d = c$, which completes the proof of (1). The proof of (2) is analogous. □

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