Quasigroups and Related Systems **19** (2011), 227 – 238

On ordered fuzzy Γ-groupoids

Niovi Kehayopulu

Abstract. This paper serves as an example to show the way we pass from fuzzy ordered groupoids (semigroups) to fuzzy ordered Γ-groupoids (semigroups). All the results on fuzzy ordered groupoids (semigroups) can be transferred to fuzzy ordered Γ-groupoids (semigroups) in the way indicated in the present paper.

1. Introduction and prerequisites

The notion of a Γ-ring, a generalization of the notion of associative rings, has been introduced and studied by N. Nobusawa in [10]. Γ-rings have been also studied by W. E. Barnes in [1]. J. Luh studied many properties of simple Γ-rings and primitive Γ-rings in [9]. The concept of a Γ-semigroup has been introduced by M. K. Sen in 1981 as follows: Given a nonempty set Γ, a nonempty set M is called a Γ-semigroup if the following assertions are satisfied: (1) $a\alpha b \in M$ and $\alpha a\beta \in \Gamma$ and (2) $(a\alpha b)\beta c = a(\alpha b\beta)c = a\alpha(b\beta c)$ for all $a, b, c \in M$ and all $\alpha, \beta \in \Gamma$ [12]. In 1986, M. K. Sen and N. K. Saha changed that definition as follows: Given two nonempty sets M and Γ, M is called a Γ-semigroup if (1) $a\alpha b \in M$ and (2) $(a\alpha b)\beta c = a\alpha(b\beta c)$ for all $a, b, c \in M$ and all $\alpha, \beta \in \Gamma$ [13]. One can find that definition of Γ-semigroups in [16], where the notion of radical in Γ-semigroups and the notion of $\Gamma\mathcal{S}$-act over a Γ-semigroup have been introduced, in [14] and [15], where the notions of regular and orthodox Γ-semigroups have been introduced and studied. With that second definition, a semigroup $(S, .)$ can be viewed as a particular case of a Γ-semigroup, considering $\Gamma = \{\gamma\}$ ($\gamma \notin S$) and defining $a\gamma b := a.b$. Moreover, let M be a Γ-semigroup, take a (fixed) $\gamma \in \Gamma$, and define $a\,.b := a\gamma b$, then $(M, .)$ is a semigroup. Later, in [11], Saha calls a nonempty set M a Γ-semigroup ($\Gamma \neq \emptyset$) if there is a mapping $M \times \Gamma \times M \to M \mid (a, \gamma, b) \to a\gamma b$ such that $(a\alpha b)\beta c = a\alpha(b\beta c)$

2010 Mathematics Subject Classification: 06F99.
Keywords: Fuzzy Γ-groupoid (semigroup), ordered Γ-groupoid (semigroup), fuzzy subset, fuzzy right (left) ideal, fuzzy ideal, fuzzy quasi-ideal, fuzzy bi-ideal, regular, intra-regular ordered fuzzy Γ-semigroup.

for all $a, b, c \in M$ and all $\alpha, \beta \in \Gamma$, and remarks that the most usual semigroup concepts, in particular regular and inverse Γ-semigroups have their analogous in Γ-semigroups. The uniqueness condition was missing from the definition of a Γ-semigroup given in [12], [13]. If we add the uniqueness condition in the definition of a Γ-semigroup given by Sen and Saha in [13] (that is, considering Γ as a set of binary relations on M) we do not need to define it via mapping. So the definition of a Γ-semigroup given by Sen and Saha in 1986 can be formulated as follows:

For two nonempty sets M and Γ, define $M\Gamma M$ as the set of all elements of the form $m_1\gamma m_2$, where $m_1, m_2 \in M$ and $\gamma \in \Gamma$. That is,

$$M\Gamma M := \{m_1\gamma m_2 \mid m_1, m_2 \in M,\ \gamma \in \Gamma\}.$$

Definition 1. (cf. [2]–[5]) Let M and Γ be two nonempty sets. The set M is called a Γ-*groupoid* if the following assertions are satisfied:

(1) $M\Gamma M \subseteq M$.

(2) If $m_1, m_2, m_3, m_4 \in M$, $\gamma_1, \gamma_2 \in \Gamma$ such that $m_1 = m_3$, $\gamma_1 = \gamma_2$ and $m_2 = m_4$, then $m_1\gamma_1 m_2 = m_3\gamma_2 m_4$.

M is called a Γ-*semigroup* if, in addition, the following assertion holds:

(3) $(m_1\gamma_1 m_2)\gamma_2 m_3 = m_1\gamma_1(m_2\gamma_2 m_3)$

for all $m_1, m_2, m_3 \in M$ and all $\gamma_1, \gamma_2 \in \Gamma$. In other words, Γ is a set of binary operations on M satisfying (3).

According to that "associativity", each of the elements $(m_1\gamma_1 m_2)\gamma_2 m_3$, and $m_1\gamma_1(m_2\gamma_2 m_3)$ is denoted as $m_1\gamma_1 m_2\gamma_2 m_3$.

Using conditions (1) – (3) one can prove that for an element with more than 5 words, for example of the form $m_1\gamma_1 m_2\gamma_2 m_3\gamma_3 m_4$, one can put a parenthesis in any expression beginning with some m_i and ending in some m_j.

There are several examples of Γ-semigroups in the bibliography. However, the example below based on Definition 1 above, shows clearly what a Γ-semigroup is.

Example 2. (cf. [3]) Consider the set $M = \{a, b, c, d\}$, and let $\Gamma = \{\gamma, \mu\}$ be the set of two binary operations on M defined in the tables below:

γ	a	b	c	d
a	a	b	c	d
b	b	c	d	a
c	c	d	a	b
d	d	a	b	c

μ	a	b	c	d
a	b	c	d	a
b	c	d	a	b
c	d	a	b	c
d	a	b	c	d

Since $(x\rho y)\omega z = x\rho(y\omega z)$ for all $x, y, z \in M$ and all $\rho, \omega \in \Gamma$, M is a Γ-semigroup.

An *ordered* Γ-*groupoid* (shortly *po*-Γ-*groupoid*) is a Γ-groupoid M together with an order relation $\leqslant$ on M such that $a \leqslant b$ implies $a\gamma c \leqslant b\gamma c$ and $c\gamma a \leqslant c\gamma b$ for all $c \in M$ and all $\gamma \in \Gamma$ (cf. also Sen and Seth [17]).

We have already seen in [2]–[5] that all the results on ordered groupoids or ordered semigroups based on ideals or ideal elements can be transferred to ordered Γ-groupoids or ordered Γ-semigroups. In the same way all the results on groupoids or semigroups (without order) based on ideals can be transferred to Γ-groupoids or Γ-semigroups. In the present paper we show that all the results on fuzzy ordered groupoids or semigroups can be transferred to fuzzy ordered Γ-groupoids or semigroups, respectively. The present paper serves as an example to show the way we pass from fuzzy ordered groupoids or fuzzy ordered semigroups to fuzzy ordered Γ-groupoids or fuzzy ordered Γ-semigroups.

There are two equivalent definitions of fuzzy left ideals, fuzzy right ideals, fuzzy quasi-ideals and fuzzy bi-ideals in ordered semigroups. The first one is in term of the fuzzy subset f itself, the second is based on the multiplication of fuzzy sets. The second one shows how similar is the theory of ordered semigroups based on fuzzy ideals with the theory of ordered semigroups based on ideals or ideal elements and it is very useful for applications. Using that second definition the results on fuzzy ordered semigroups or on fuzzy semigroups (without order) can be drastically simplified (cf. also [2]). In the present paper we examine these equivalent definitions in case of ordered fuzzy Γ-groupoids and ordered fuzzy Γ-semigroups. Characterizations of regular and intra-regular ordered semigroups in terms of fuzzy sets have been given in [7]. In the present paper we also characterize the regular and intra-regular ordered Γ-semigroups in terms of fuzzy sets. In a similar way one can prove that the characterizations of π-regular and intra π-regular ordered semigroups considered in [7] have their analogue for ordered Γ-semigroups.

2. Main results

Following the terminology given by L.A. Zadeh [18], if $(M, ., \leqslant)$ is an ordered Γ-groupoid, we say that f is a fuzzy subset of M (or a fuzzy set in M) if f is a mapping of M into the real closed interval [0,1]. For a subset A of M, the fuzzy subset f_A is defined as follows:

$$f_A : M \to [0,1] \mid x \to f_A(x) := \begin{cases} 1 & if\ \ x \in A \\ 0 & if\ \ x \notin A. \end{cases}$$

For an element a of M, we clearly have

$$f_a : M \to [0,1] \mid x \to f_a(x) := \begin{cases} 1 & if\ \ x = a \\ 0 & if\ \ x \neq a. \end{cases}$$

For an element a of M, denote by A_a the relation on M defined by

$$A_a := \{(y,z) \mid a \leqslant y\gamma z \text{ for some } \gamma \in \Gamma\}.$$

For two fuzzy subsets f and g of M, we define the multiplication of f and g as follows:

$$f \circ g : M \to [0,1] \mid a \to \begin{cases} \sup\limits_{(y,z)\in A_a} \{\min f(y), g(z)\} & \text{if } \ A_a \neq \emptyset \\ 0 \quad \text{if } \ A_a = \emptyset \end{cases}$$

and in the set of all fuzzy subsets of M we define the order relation as follows:

$$f \preceq g \text{ if and only if } f(x) \leqslant g(x) \text{ for all } x \in M.$$

For two fuzzy subsets f and g of M, let $f \wedge g$ be the fuzzy subset of M defined by:

$$f \wedge g : M \to [0,1] \mid x \to \min\{f(x), g(x)\}.$$

Denote by 1 the fuzzy subset of M defined by

$$1 : M \to [0,1] \mid x \to 1(x) := 1.$$

Denote by f^2 the composition $f \circ f$. If $F(M)$ is the set of fuzzy subsets of M, it is clear that the fuzzy subset 1 of M is the greatest element of the ordered set $(F(M), \preceq)$. In a similar way as in [6] (using the methodology of the present paper) one can prove that if M is an ordered Γ-semigroup and f, g, h fuzzy subsets of M, then $(f \circ g) \circ h = f \circ (g \circ h)$.

Definition 3. Let M be an ordered Γ-groupoid. A fuzzy subset f of M is called a *fuzzy right ideal* of M if

(1) $f(x\gamma y) \geqslant f(x)$ for all $x, y \in M$ and all $\gamma \in \Gamma$,

(2) if $x \leqslant y$, then $f(x) \geqslant f(y)$.

A fuzzy subset f of M is called a *fuzzy left ideal* of M if

(1) $f(x\gamma y) \geqslant f(y)$ for all $x, y \in M$ and all $\gamma \in \Gamma$,

(2) if $x \leqslant y$, then $f(x) \geqslant f(y)$.

Definition 4. Let M be an ordered Γ-groupoid. A fuzzy subset f of M is called a *fuzzy quasi-ideal* of M if

(1) $(f \circ 1) \wedge (1 \circ f) \preceq f$,

(2) if $x \leqslant y$, then $f(x) \geqslant f(y)$.

Definition 5. Let M be an ordered Γ-semigroup. A fuzzy subset f of M is called a *fuzzy bi-ideal* of M if

(1) $f(x\gamma y\mu z) \geqslant \min\{f(x), f(z)\}$ for all $x, y, z \in M$ and all $\gamma, \mu \in \Gamma$,

(2) if $x \leqslant y$, then $f(x) \geqslant f(y)$.

Theorem 6. *Let M be an ordered Γ-groupoid. A fuzzy subset f of M is a fuzzy right ideal of M if and only if*

(1) $f \circ 1 \preceq f$,

(2) *if $x \leqslant y$, then $f(x) \geqslant f(y)$.*

Proof. ($\Longrightarrow$) Let $a \in M$. Then $(f \circ 1)(a) \leqslant f(a)$. In fact: If $A_a = \emptyset$, then $(f \circ 1)(a) := 0 \leqslant f(a)$. Let $A_a \neq \emptyset$. Then

$$(f \circ 1)(a) := \sup_{(y,z)\in A_a} \{\min\{f(y), 1(z)\}\} = \sup_{(y,z)\in A_a} \{f(y)\}.$$

On the other hand, $f(y) \leqslant f(a)$ for every $(y, z) \in A_a$. Indeed: If $(y, z) \in A_a$, then $y, z \in M$ and $a \leqslant y\gamma z$ for some $\gamma \in \Gamma$. Since f is a fuzzy right ideal of M, we have $f(a) \geqslant f(y\gamma z) \geqslant f(y)$. Therefore we have

$$(f \circ 1)(a) = \sup_{(y,z)\in A_a} \{f(y)\} \leqslant f(a).$$

($\Longleftarrow$) Let $x, y \in M$ and $\gamma \in \Gamma$. By hypothesis, we have $(f \circ 1)(x\gamma y) \leqslant f(x\gamma y)$. On the other hand, since $(x, y) \in A_{x\gamma y}$, we have

$$(f \circ 1)(x\gamma y) := \sup_{(u,v)\in A_{x\gamma y}} \{\min\{f(u), 1(v)\}\} \geqslant \min\{f(x), 1(y)\} = f(x).$$

Hence we obtain $f(x\gamma y) \geqslant f(x)$, and f is a fuzzy right ideal of M. □

In a similar way we prove the following theorem

Theorem 7. *Let M be an ordered Γ-groupoid. A fuzzy subset f of M is a fuzzy left ideal of M if and only if*

(1) $1 \circ f \preceq f$,

(2) *if* $x \leqslant y$, *then* $f(x) \geqslant f(y)$. $\square$

Theorem 8. *Let* M *be an ordered* Γ*-groupoid. A fuzzy subset* f *of* M *is a fuzzy quasi-ideal of* M *if and only if the following conditions are satisfied:*

(1) *if* $x \leqslant b\gamma s$ *and* $x \leqslant t\mu c$ *for some* $x, b, s, t, c \in M$ *and* $\gamma, \mu \in \Gamma$, *then* $f(x) \geqslant \min\{f(b), f(c)\}$,

(2) *if* $x \leqslant y$, *then* $f(x) \geqslant f(y)$.

Proof. $(\Longrightarrow)$ Let $x, b, s, t, c \in M$ and $\gamma, \mu \in \Gamma$ such that $x \leqslant b\gamma s$ and $x \leqslant t\mu c$. Since f is a fuzzy quasi-ideal of M, we have

$$f(x) \geqslant ((f \circ 1) \wedge (1 \circ f))(x) := \min\{(f \circ 1)(x), (1 \circ f)(x)\}.$$

Since $x \leqslant b\gamma s$, we have $(b, s) \in A_x$, then

$$(f \circ 1)(x) := \sup_{(u,v) \in A_x} \{\min\{f(u), 1(v)\}\} \geqslant \min\{f(b), 1(s)\} = f(b).$$

Similarly from $x \leqslant t\mu c$, we get $(1 \circ f)(x) \geqslant f(c)$. Hence we have

$$f(x) \geqslant \min\{(f \circ 1)(x), (1 \circ f)(x)\} \geqslant \min\{f(b), f(c)\}.$$

$(\Longleftarrow)$ Let $x \in M$. Then $((f \circ 1) \wedge (1 \circ f))(x) \leqslant f(x)$. In fact: We have

$$((f \circ 1) \wedge (1 \circ f))(x) := \min\{(f \circ 1)(x), (1 \circ f)(x)\}.$$

1. If $A_x = \emptyset$, then $(f \circ 1)(x) := 0$ and $(1 \circ f)(x) := 0$. Moreover $\min\{(f \circ 1)(x), (1 \circ f)(x)\} = 0$, and $((f \circ 1) \wedge (1 \circ f))(x) = 0 \leqslant f(x)$.

2. Let $A_x \neq \emptyset$. Then

$$(f \circ 1)(x) := \sup_{(y,s) \in A_x} \{\min\{f(y), 1(s)\}\} \qquad (\star)$$

$$(1 \circ f)(x) := \sup_{(t,z) \in A_x} \{\min\{1(t), f(z)\}\}.$$

2.1. If $f(x) \geqslant (f \circ 1)(x)$, then

$$\begin{aligned} f(x) &\geqslant (f \circ 1)(x) \geqslant \min\{(f \circ 1)(x), (1 \circ f)(x)\} \\ &= ((f \circ 1) \wedge (1 \circ f))(x). \end{aligned}$$

2.2. Let $f(x) < (f \circ 1)(x)$. By $(*)$, there exists $(y, s) \in A_x$ such that $\min\{f(y), 1(s)\} > f(x)$ (otherwise $(f \circ 1)(x) \leqslant f(x)$ which is impossible). Since $\min\{f(y), 1(s)\} = f(y)$, we have

$$f(y) > f(x). \tag{$\star\star$}$$

On the other hand, $f(x) \geqslant \min\{1(t), f(z)\}$ for every $(t, z) \in A_x$. Indeed: Let $(t, z) \in A_x$. Since $(y, s) \in A_x$, we have $y, s \in M$ and $x \leqslant y\gamma s$ for some $\gamma \in \Gamma$. Since $(t, z) \in A_x$, we have $t, z \in M$ and $x \leqslant t\mu z$ for some $\mu \in \Gamma$. Since $x, y, s, t, z \in M$ and $\gamma, \mu \in \Gamma$ such that $x \leqslant y\gamma s$ and $x \leqslant t\mu z$, by hypothesis, we have $f(x) \geqslant \min\{f(y), f(z)\}$. If $\min\{f(y), f(z)\} = f(y)$, then $f(x) \geqslant f(y)$ which is impossible by $(**)$. Thus we have $\min\{f(y), f(z)\} = f(z)$, and $f(x) \geqslant f(z) = \min\{1(t), f(z)\}$. Therefore we have

$$\begin{aligned} f(x) &\geqslant \sup_{(t,z)\in A_x} \{\min\{1(t), f(z)\} = (1 \circ f)(x) \\ &\geqslant \min\{(f \circ 1)(x), (1 \circ f)(x)\} = ((f \circ 1) \wedge (1 \circ f))(x), \end{aligned}$$

and the proof is complete. □

By Theorem 8, in a similar way as in [8], one can prove the following theorem.

Theorem 9. *Let M be an ordered Γ-groupoid. A fuzzy subset f of M is a fuzzy quasi-ideal of M if and only if the following conditions are satisfied:*

(1) *if $x \leqslant b\gamma s$ and $x \leqslant t\mu c$ for some $x, b, s, t, c \in M$ and $\gamma, \mu \in \Gamma$, then $f(x) \geqslant \max\{\min\{f(b), f(c)\}, \min\{f(t), f(s)\}\}$,*

(2) *if $x \leqslant y$, then $f(x) \geqslant f(y)$.* □

Theorem 10. *Let M be an ordered Γ-semigroup. A fuzzy subset f of M is a fuzzy bi-ideal of M if and only if the following assertions are satisfied:*

(1) *$f \circ 1 \circ f \preceq f$,*

(2) *if $x \leqslant y$, then $f(x) \geqslant f(y)$.*

Proof. ($\Longrightarrow$) Let $a \in S$. Then $(f \circ 1 \circ f)(a) \leqslant f(a)$. In fact: If $A_a = \emptyset$ then $(f \circ 1 \circ f)(a) = ((f \circ 1) \circ f)(a) := 0 \leqslant f(a)$. Let $A_a \neq \emptyset$. Then

$$(f \circ 1 \circ f)(a) := \sup_{(y,z)\in A_a} \{\min\{(f \circ 1)(y), f(z)\}\}.$$

It is enough to prove that

$$\min\{(f\circ 1)(y), f(z)\} \leqslant f(a) \text{ for every } (y,z)\in A_a. \tag{$\star$}$$

Let now $(y,z)\in A_a$. If $A_y=\emptyset$, then $(f\circ 1)(y):=0$, and

$$\min\{(f\circ 1)(y), f(z)\} = 0 \leqslant f(a).$$

Let $A_y\neq\emptyset$. Then

$$(f\circ 1)(y) := \sup_{(s,t)\in A_y}\{\min\{f(s), 1(t)\}.\} \tag{$\star\star$}$$

We consider the following two cases:

1. Let $f(a)\geqslant (f\circ 1)(y)$. Then

$$f(a)\geqslant (f\circ 1)(y)\geqslant \min\{(f\circ 1)(y), f(z)\},$$

and condition $(*)$ is satisfied.

2. Let $f(a)<(f\circ 1)(y)$. Then, by $(**)$, there exists $(x,w)\in A_y$ such that $f(a)<\min\{f(x),1(w)\}$ (otherwise, $f(a)\geqslant (f\circ 1)(y)$ which is impossible). Since $\min\{f(x),1(w)\}=f(x)$, we have

$$f(a)<f(x).$$

Since $(y,z)\in A_a$, we have $y,z\in M$ and $a\leqslant y\mu z$ for some $\mu\in\Gamma$. Since $(x,w)\in A_y$, we have $x,w\in M$ and $y\leqslant x\gamma w$ for some $\gamma\in\Gamma$. Since $a\leqslant y\mu z\leqslant x\gamma w\mu z$ and f is a fuzzy bi-ideal of S, by the definition of fuzzy bi-ideal, we have

$$f(a)\geqslant f(x\gamma w\mu z)\geqslant \min\{f(x), f(z)\}.$$

If $f(x)\leqslant f(z)$, then $\min\{f(x),f(z)\}=f(x)$, and $f(a)\geqslant f(x)$ which is impossible. Hence we have $f(x)>f(z)$. Then $\min\{f(x),f(z)\}=f(z)$, and $f(a)\geqslant f(z)$. Since $(x,w)\in A_y$, by $(**)$, we have

$$\min\{f(x),1(w)\}\leqslant \sup_{(s,t)\in A_y}\{\min\{f(s),1(t)\}\} = (f\circ 1)(y).$$

Then we have $(f\circ 1)(y)\geqslant \min\{f(x),1(w)\}=f(x)>f(z)$. Consequently $\min\{(f\circ 1)(y), f(z)\}=f(z)\leqslant f(a)$, and condition $(*)$ is satisfied.

($\Longleftarrow$) Let $x, y, z \in M$ and $\gamma, \mu \in \Gamma$. Then $f(x\gamma y\mu z) \geqslant \min\{f(x), f(z)\}$. Indeed: Since $f \circ 1 \circ f \preceq f$ and $x\gamma y\mu z \in M$, we have $(f \circ 1 \circ f)(x\gamma y\mu z) \leqslant f(x\gamma y\mu z)$. Since $x\gamma y\mu z \leqslant (x\gamma y)\mu z$, $\mu \in \Gamma$, we have $(x\gamma y, z) \in A_{x\gamma y\mu z}$. Then

$$\begin{aligned}(f \circ 1 \circ f)(x\gamma y\mu z) &:= \sup_{(u,v)\in A_{x\gamma y\mu z}} \{\min\{(f \circ 1)(u), f(v)\}\} \\ &\geqslant \min\{(f \circ 1)(x\gamma y),\ f(z)\}.\end{aligned}$$

Since $(x, y) \in A_{x\gamma y}$, we have

$$(f \circ 1)(x\gamma y) := \sup_{(z,w)\in A_{x\gamma y}} \{\min\{(f(z), 1(w)\}\} \geqslant \min\{f(x),\ 1(y)\} = f(x).$$

Thus $f(x\gamma y\mu z) \geqslant \min\{(f \circ 1)(x\gamma y), f(z)\} \geqslant \min\{f(x), f(z)\}$. □

Definition 11. An ordered Γ-semigroup M is called *regular* if for every $a \in M$ there exist $x \in M$ and $\gamma, \mu \in \Gamma$ such that $a \leqslant a\gamma x\mu a$.

In a similar way as in [8] we prove the following lemma and the two corollaries below:

Lemma 12. *Let M be an ordered Γ-groupoid, f, g fuzzy subsets of M and $a \in M$. The following are equivalent:*

(1) $(f \circ g)(a) \neq 0$.

(2) *There exists $(x, y) \in A_a$ such that $f(x) \neq 0$ and $g(y) \neq 0$.* □

Corollary 13. *Let M be an ordered Γ-groupoid, f a fuzzy subset of M and $a \in M$. The following are equivalent:*

(1) $(f \circ 1)(a) \neq 0$.

(2) *There exists $(x, y) \in A_a$ such that $f(x) \neq 0$.* □

Corollary 14. *Let M be an ordered Γ-groupoid, g a fuzzy subset of M and $a \in M$. The following are equivalent:*

(1) $(1 \circ g)(a) \neq 0$.

(2) *There exists $(x, y) \in A_a$ such that $g(y) \neq 0$.* □

Theorem 15. *An ordered Γ-semigroup M is regular if and only if for every fuzzy subset f of M we have $f \preceq f \circ 1 \circ f$.*

Proof. ($\Longrightarrow$) Let f be a fuzzy subset of M and $a \in M$. Since M is regular, there exist $x \in M$ and $\gamma, \mu \in \Gamma$ such that $a \leqslant a\gamma x\mu a$. Since $a \leqslant (a\gamma x)\mu a$, where $\mu \in \Gamma$, we have $(a\gamma x, a) \in A_a$. Then we have

$$(f \circ 1 \circ f)(a) := \sup_{(y,z)\in A_a} \{\min\{(f \circ 1)(y), f(z)\}\} \geqslant \min\{(f \circ 1)(a\gamma x), f(a)\}.$$

Since $(a, x) \in A_{a\gamma x}$, we have

$$(f \circ 1)(a\gamma x) := \sup_{(u,v)\in A_{a\gamma x}} \{\min\{f(u), 1(v)\}\} \geqslant \min\{f(a), 1(x)\} = f(a).$$

Hence we have $(f \circ 1 \circ f)(a) \geqslant \min\{(f \circ 1)(a\gamma x), f(a)\} = f(a)$.

$(\Longleftarrow)$ Let $a \in M$. Since f_a is a fuzzy subset of M, by hypothesis, we have $1 = f_a(a) \leqslant (f_a \circ 1 \circ f_a)(a)$. Since $f_a \circ 1 \circ f_a$ is a fuzzy subset of M, we have $(f_a \circ 1 \circ f_a)(a) \leqslant 1$. Then $((f_a \circ 1) \circ f_a)(a) = (f_a \circ 1 \circ f_a)(a) = 1$. By Lemma 12, there exists $(x, y) \in A_a$ such that $(f_a \circ 1)(x) \neq 0$ and $f_a(y) \neq 0$. If $y \neq a$, then $f_a(y) = 0$ which is impossible. Thus we have $y = a$, $(x, a) \in A_a$, and $a \leqslant x\mu a$ for some $\mu \in \Gamma$. If $A_x = \emptyset$, then $(f_a \circ 1)(x) := 0$ which is impossible. Thus we have $A_x \neq \emptyset$, and

$$(f_a \circ 1)(x) := \sup_{(b,c)\in A_x} \{\min\{f_a(b),\, 1(c)\}\} = \sup_{(b,c)\in A_x} \{f_a(b)\}.$$

If $b \neq a$ for every $(b, c) \in A_x$, then $f_a(b) = 0$ for every $(b, c) \in A_x$, then $(f_a \circ 1)(x) = 0$ which is impossible. Hence there exists $(b, c) \in A_x$ such that $b = a$. Then $(a, c) \in A_x$, so $x \leqslant a\gamma c$ for some $\gamma \in \Gamma$. Then we obtain $a \leqslant x\mu a \leqslant a\gamma c\mu a$, where $c \in M$ and $\gamma, \mu \in \Gamma$, and M is regular. $\square$

Definition 16. An ordered Γ-semigroup M is called *intra-regular* if for every $a \in M$ there exist $x, y \in M$ and $\gamma, \mu, \rho \in \Gamma$ such that $a \leqslant x\gamma a\mu a\rho y$.

Proposition 17. *Let M be an ordered Γ-groupoid and $a, b \in M$. Then we have $b \leqslant a\gamma a$ for some $\gamma \in \Gamma$ if and only if $f_a^2(b) \neq 0$.*

Proof. $(\Longrightarrow)$ Let $b \leqslant a\gamma a$ for some $\gamma \in \Gamma$. Since $(a, a) \in A_b$, we have

$$(f_a \circ f_a)(b) := \sup_{(x,y)\in A_b} \{\min\{f_a(x), f_a(y)\}\} \geqslant \min\{f_a(a), f_a(a)\} = 1.$$

$(\Longleftarrow)$ Since $f_a^2(b) \neq 0$, by Lemma 12, there exists $(x, y) \in A_b$ such that $f_a(a) \neq 0$ and $f_a(y) \neq 0$. Since $f_a(x) \neq 0$, we have $x = a$. Since $f_a(y) \neq 0$, we have $y = a$. Since $b \leqslant x\gamma y$ for some $\gamma \in \Gamma$, we have $b \leqslant a\gamma a$, where $a \in \Gamma$. $\square$

Theorem 18. *An ordered* Γ*-semigroup* M *is intra-regular if and only if for every fuzzy subset* f *of* M *we have* $f \preceq 1 \circ f^2 \circ 1$.

Proof. ($\Longrightarrow$) Let f be a fuzzy subset of M and $a \in M$. Since M is intra-regular we have $a \leqslant x\gamma a\mu a\rho y$ for some $x, y \in M$ and $\gamma, \mu, \rho \in \Gamma$. Since $(x\gamma a\mu a, y) \in A_a$, we have

$$\begin{aligned}(1 \circ f^2 \circ 1)(a) &:= \sup_{(u,v)\in A_a} \{\min\{(1 \circ f^2)(u), 1(v)\}\} \\ &\geqslant \min\{(1 \circ f^2)(x\gamma a\mu a), 1(y)\} = (1 \circ f^2)(x\gamma a\mu a).\end{aligned}$$

Since $(x, a\mu a) \in A_{x\gamma a\mu a}$, we have

$$\begin{aligned}(1 \circ f^2)(x\gamma a\mu a) &:= \sup_{(s,t)\in A_{a\gamma\alpha\mu\alpha}} \{\min\{1(s), f^2(t)\}\} \\ &\geqslant \min\{1(x), f^2(a\mu a)\} = f^2(a\mu a).\end{aligned}$$

Since $(a, a) \in A_{a\mu a}$, we have

$$\begin{aligned}f^2(a\mu a) = (f \circ f)(a\mu a) &:= \sup_{(w,z)\in A_{\alpha\mu\alpha}} \{\min\{f(w), f(z)\}\} \\ &\geqslant \{f(a), f(a)\} = f(a).\end{aligned}$$

Hence we have

$$f(a) \leqslant f^2(a\mu a) \leqslant (1 \circ f^2)(x\gamma a\mu a) \leqslant (1 \circ f^2 \circ 1)(a).$$

Since $f(a) \leqslant (1 \circ f^2 \circ 1)(a)$ for every $a \in M$, we have $f \preceq 1 \circ f^2 \circ 1$.

($\Longleftarrow$) Let $a \in M$. Since f_a is a fuzzy subset of M, by hypothesis, we have $f_a \preceq 1 \circ f_a^2 \circ 1$. Then $1 = f_a(a) \leqslant (1 \circ f_a^2 \circ 1)(a)$. Since $1 \circ f_a^2 \circ 1$ is a fuzzy subset of M, we have $(1 \circ f_a^2 \circ 1)(a) \leqslant 1$, so $(1 \circ f_a^2 \circ 1)(a) \neq 0$. Then, by Corollary 13, there exists $(w, y) \in A_a$ such that $(1 \circ f_a^2)(w) \neq 0$. By Corollary 14, there exists $(x, t) \in A_w$ such that $f_a^2(t) \neq 0$. By Proposition 17, we have $t \leqslant a\mu a$, for some $\mu \in \Gamma$. Since $(x, t) \in A_w$, $w \leqslant x\gamma t$ for some $\gamma \in \Gamma$. Since $(w, y) \in A_a$, $a \leqslant w\rho y$ for some $\rho \in \Gamma$. Then we obtain $a \leqslant w\rho y \leqslant x\gamma t\rho y \leqslant x\gamma a\mu a\rho y$, where $\gamma, \mu, \rho \in \Gamma$, and M is intra-regular. □

References

[1] **W. E. Barnes**, *On the* Γ*-rings of Nobusawa*, Pacific J. Math. **18** (1966), 411 − 422.

[2] **N. Kehayopulu**, *On prime, weakly prime ideals in po-*Γ*-semigroups*, Lobachevskii J. Math. **30** (2009), no. 4, 257 – 262.

[3] **N. Kehayopulu**, *On le-*Γ*-semigroups*, Int. Math. Forum **4** (2009), 1915 – 1922.

[4] **N. Kehayopulu**, *On ordered* Γ*-semigroups*, Sci. Math. Jpn. **71** (2010), 179 – 185.

[5] **N. Kehayopulu**, *On regular duo po-*Γ*-semigroups*, Math. Slovaca, to appear.

[6] **N. Kehayopulu, M. Tsingelis**, *The embedding of an ordered groupoid into a poe-groupoid in terms of fuzzy sets*, Inform. Sci. **152** (2003), 231 – 236.

[7] **N. Kehayopulu, M. Tsingelis**, *Characterization of some types of ordered semigroups in terms of fuzzy sets*, Lobachevskii J. Math. **29** (2008), no. 1, 14 – 20.

[8] **N. Kehayopulu, M. Tsingelis**, *Fuzzy right, left, quasi-ideals, bi-ideals in ordered semigroups*, Lobachevskii J. Math. **30** (2009), no. 1, 17 – 22.

[9] **J. Luh**, *On the theory of simple* Γ*-rings*, Michigan Math. J. **16** (1969), 65 – 75.

[10] **N. Nobusawa**, *On a generalization of the ring theory*, Osaka J. Math. **1** (1964), 81 – 89.

[11] **N. K. Saha**, *The maximum idempotent-separating congruence on an inverse* Γ*-semigroup*, Kyungpook Math. J. **34** (1994), 59 – 66.

[12] **M. K. Sen**, *On* Γ*-semigroup*, Algebra and its applications (New Delhi, 1981), 301 – 308, Lecture Notes in Pure and Appl. Math. **91**, Dekker, New York, 1984.

[13] **M. K. Sen, N. K. Saha**, *On* Γ*-semigroups I*, Bull. Calcutta Math. Soc. **78** (1986), no. 3, 180 – 186.

[14] **M. K. Sen, N. K. Saha**, *The maximum idempotent-separating congruence on an orthodox* Γ*-semigroup*, J. Pure Math. **7** (1990), 39 – 47.

[15] **M. K. Sen, N. K. Saha**, *Orthodox* Γ*-semigroups*, Internat. J. Math. Math. Sci. **13** (1990), 527 – 534.

[16] **M. K. Sen, A. Seth**, *Radical of* Γ*-semigroup*, Bull. Calcutta Math. Soc. **80** (1988), no. 3, 189 – 196.

[17] **M. K. Sen, A. Seth**, *On po-*Γ*-semigroups*, Bull. Calcutta Math. Soc. **85** (1993), no. 5, 445 – 450.

[18] **L. A. Zadeh**, *Fuzzy sets*, Information and Control **8** (1965), 338 – 353.

Received May 22, 2011

University of Athens, Department of Mathematics, 15784 Panepistimiopolis, Athens, Greece
E-mail: nkehayop@math.uoa.gr