Indicators of quasigroups

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Abstract. We present some useful conditions which are necessary for isotopy of two quasigroups of the same finite order.

Let \( Q = \{1, 2, 3, \ldots, n\} \) be a finite set, \( S_n \) - the set of all permutations of \( Q \). The multiplication (composition) of permutations \( \varphi \) and \( \psi \) of \( Q \) is defined as \( \varphi \psi(x) = \varphi(\psi(x)) \). All permutations will be written in the form of cycles and cycles will be separated by points, e.g.

\[
\varphi = \left(\begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 & 6 \\
3 & 1 & 2 & 5 & 4 & 6
\end{array}\right) = (132.45.6).
\]

By a cyclic type of a permutation \( \varphi \in S_n \) we mean the sequence \( l_1, l_2, \ldots, l_n \), where \( l_i \) denotes the number of cycles of the length \( i \). In this case we will write

\[
C(\varphi) = \{l_1, l_2, \ldots, l_n\}.
\]

Obviously, \( \sum_{i=1}^{n} i \cdot l_i = n \).

Definition 1. By the indicator of a permutation \( \varphi \) of type \( C(\varphi) = \{l_1, l_2, \ldots, l_n\} \) we mean the polynomial

\[
w(\varphi) = x_1^{l_1} x_2^{l_2} \cdots x_n^{l_n}.
\]

For example, for \( \varphi = (123.45.6) \) we have \( C(\varphi) = \{1, 1, 0, 0, 0\} \) and \( w(\varphi) = x_1 x_2 x_3 \); for \( \psi = (1.2536.47.80.9) \), \( C(\psi) = \{2, 2, 0, 1, 0, 0, 0, 0, 0, 0\} \) and \( w(\psi) = x_1^2 x_2^2 x_4 \).

As it is well-known, two permutations \( \varphi, \psi \in S_n \) are conjugate if there exists a permutation \( \rho \in S_n \) such that

\[
\rho \varphi \rho^{-1} = \psi.
\]

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Theorem 1. (Theorem 5.1.3 in [4]) Two permutations are conjugate if and only if they have the same cyclic type.

As a consequence we obtain

Corollary 1. Conjugated permutations have the same indicators.

As it is well-known, two quasigroups $Q(\circ)$ and $Q(\cdot)$ are isotopic if there are three permutations $\alpha, \beta, \gamma$ of $Q$ such that

$$\gamma(x \circ y) = \alpha(x) \cdot \beta(y).$$

(1)

In the case $\alpha = \beta = \gamma$ we say that quasigroups are autotopic.

A track (or a right middle translation) of a quasigroup $Q(\cdot)$ is a permutation $\varphi_i$ of $Q$ satisfying the identity

$$x \cdot \varphi_i(x) = i,$$

where $i \in Q$. Each quasigroup can be identified with the set $\{\varphi_1, \varphi_2, \ldots, \varphi_n\}$ of all its tracks (cf. [2]).

Tracks of $Q(\cdot)$ will be denoted by $\varphi_i$, track of $Q(\circ)$ by $\psi_i$. Similarly, left and right translations of $Q(\cdot)$ will be denoted by $L_a$ and $R_a$, left and right translations of $Q(\circ)$ by $L_a^\circ$ and $R_a^\circ$.

Proposition 1. (cf. [2]) Tracks of isotopic quasigroups satisfying (1) are connected by the formula

$$\varphi_{\gamma(i)} = \beta \psi_i \alpha^{-1}.$$

(2)

Similar results hold for left and right translations.

Theorem 2. Left and right translations of isotopic quasigroups satisfying (1) are connected by the conditions

$$L_{\alpha(a)} = \gamma L_a^\circ \beta^{-1}, \quad R_{\beta(b)} = \gamma R_b^\circ \alpha^{-1}.$$

(3)

Proof. Indeed, putting $x = a$ we obtain $\gamma L_a^\circ(y) = L_{\alpha(a)} \beta(y)$ for every $y \in Q$, which implies $\gamma L_a^\circ \beta^{-1} = L_{\alpha(a)}$. Similarly, putting in (1) $y = b$ we obtain $R_{\beta(b)} = \gamma R_b^\circ \alpha^{-1}$. □

Corollary 2. For autotopic quasigroups we have

$$\varphi_{\alpha(i)} = \alpha \psi_i \alpha^{-1}, \quad L_{\alpha(a)} = \alpha L_a^\circ \alpha^{-1}, \quad R_{\alpha(b)} = \alpha R_b^\circ \alpha^{-1}.$$
Consider the following three matrices:

\[ \Phi = [\varphi_{ij}], \quad L = [L_{ij}], \quad R = [R_{ij}], \]

where \( \varphi_{ij} = \varphi_i \varphi_j^{-1} \), \( L_{ij} = L_i L_j^{-1} \), \( R_{ij} = R_i R_j^{-1} \) for all \( i, j \in Q \). Obviously, \( \varphi_{ii}(x) = L_{ii}(x) = R_{ii}(x) = x \) and \( \varphi_{ij}(x) \neq x \), \( L_{ij}(x) \neq x \), \( R_{ij}(x) \neq x \) for all \( i, j, x \in Q \) and \( i \neq j \).

**Theorem 3.** For isotopic quasigroups \( Q(\circ) \) and \( Q(\cdot) \) with the isotopy of the form (1) we have

\[ \varphi_{\gamma(i)\gamma(j)} = \beta \psi_{ij} \beta^{-1}, \quad L_{\alpha(i)\alpha(j)} = \gamma L_{ij}^\alpha \gamma^{-1}, \quad R_{\beta(i)\beta(j)} = \gamma R_{ij}^\beta \gamma^{-1}. \]

**Proof.** Indeed, using (2) we obtain

\[ \varphi_{\gamma(i)\gamma(j)} = \varphi_{\gamma(i)} \varphi_{\gamma(j)}^{-1} = (\beta \psi_i \alpha^{-1}) \beta \psi_j \alpha^{-1} \beta^{-1} = \beta \psi_i \psi_j^{-1} \beta^{-1} = \beta \psi_{ij} \beta^{-1}. \]

In a similar way, using (3), we obtain the other two equations. \( \square \)

**Definition 2.** By the *indicator of the matrix* \( \Phi \) we mean the polynomial

\[ w(\Phi) = \sum_{i=1}^{n} w(\Phi_i), \]

where \( \Phi_i = \{ \varphi_{i1}, \varphi_{i2}, \ldots, \varphi_{in} \} \) and \( w(\Phi_i) = \sum_{j=1, j \neq i}^{n} w(\varphi_{ij}) \).

Indicators of the matrices \( L \) and \( M \) are defined analogously.

**Example 1.** Consider two quasigroups defined by the following tables:

\[ \begin{array}{ccc|ccc}
1 & 2 & 3 & 4 & 5 & 6 \\
1 & 4 & 1 & 6 & 2 & 5 & 3 \\
2 & 5 & 3 & 2 & 6 & 4 & 1 \\
3 & 2 & 6 & 5 & 3 & 1 & 4 \\
4 & 3 & 5 & 1 & 4 & 6 & 2 \\
5 & 6 & 2 & 4 & 1 & 3 & 5 \\
6 & 1 & 4 & 3 & 5 & 2 & 6 \\
\end{array} \quad \begin{array}{ccc|ccc}
\circ & 1 & 2 & 3 & 4 & 5 & 6 \\
1 & 1 & 2 & 3 & 4 & 5 & 6 \\
2 & 2 & 1 & 5 & 6 & 4 & 3 \\
3 & 3 & 5 & 4 & 2 & 6 & 1 \\
4 & 4 & 6 & 2 & 3 & 1 & 5 \\
5 & 5 & 4 & 6 & 1 & 3 & 2 \\
6 & 6 & 3 & 1 & 5 & 2 & 4 \\
\end{array} \]

For the quasigroup \( Q(\circ) \) we have:

\[ \varphi_1 = (126.354.) \quad \varphi_2 = (146523.) \quad \varphi_3 = (16342.5.) \quad \varphi_4 = (1.25364.) \quad \varphi_5 = (15642.3.) \quad \varphi_6 = (13245.6.). \]

Thus,

\[ \begin{array}{ccc}
\varphi_{11} = (1.2.3.4.5.6.) & \varphi_{12} = (15.24.36.) & \varphi_{13} = (13.26.45.) \\
\varphi_{14} = (12.34.56.) & \varphi_{15} = (164.235.) & \varphi_{16} = (146.253.). \\
\end{array} \]
Consequently,
\[ w(\varphi_{11}) = x_1^6, \quad w(\varphi_{12}) = w(\varphi_{13}) = w(\varphi_{14}) = x_2^3, \quad w(\varphi_{15}) = w(\varphi_{16}) = x_3^2. \]

Hence \( w(\Phi_1) = 3x_3^2 + 2x_3^2. \)

By analogous computations we can see that for this quasigroup
\[ w(\Phi) = w(L) = w(R) = 6(3x_2^3 + 2x_3^2). \]

For the second quasigroup we obtain:
\[ w(\Phi) = (2x_2x_4 + x_3^3 + 2x_6) + (x_2^3 + 2x_3^3 + 2x_6) + 2(x_2x_4 + x_3^2 + 3x_6) + 2(2x_3^2 + 3x_6), \]
\[ w(L) = w(R) = 2(x_2x_4 + 4x_6) + 4(2x_2x_4 + x_3^3 + 2x_6). \]

As a consequence of our Theorem 3 and Corollary 1 we obtain

**Theorem 4.** Isotopic quasigroups have the same indicators of the matrices \( \Phi, L \) and \( R. \)

This theorem shows that quasigroups from the above example are not isotopic.

**Corollary 3.** For quasigroups of order \( n \) isotopic to a group we have \( w(\Phi) = w(\Phi_1). \)

**Proof.** In [2] it is proved that for a quasigroup isotopic to a group all its \( \Phi_i \) are groups isomorphic to \( \Phi_1. \) Hence \( w(\Phi_i) = w(\Phi_1) \) for every \( i \in Q. \)

There are examples proving that the converse statement is not true.

**References**


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