A characterization of binary invertible algebras linear over a group

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Abstract. In this paper we define linear over a group and an abelian group binary invertible algebras and characterize the class of such algebras by second-order formulae, namely the $\forall\exists(\forall)$-identities.

1. Introduction

A quasigroup, $(Q; \cdot)$, of the form,

$$xy = \varphi x + a + \psi y,$$

where $(Q; +)$ is a group, $\varphi$, $\psi$ are automorphisms (antiautomorphisms) of $(Q; +)$, and $a$ is a fixed element of $Q$, is called linear (alinear) quasigroup over the group, $(Q; +)$, [2, 6].

All primitive linear (alinear) quasigroups form a variety [6].

A linear quasigroup over an abelian group is called a $T$-quasigroup [10].

An important subclass of the $T$-quasigroups is the class of medial quasigroups. A quasigroup $(Q; \cdot)$ is called medial, if the following identity holds:

$$xy \cdot uv = xu \cdot yv.$$ Any medial quasigroup is a $T$-quasigroup by Toyoda theorem, [3] - [8], with the condition, $\varphi \psi = \psi \varphi$.

Medial quasigroups have been studied by many authors, namely R.H. Bruck [8], T. Kepka, P. Nemec and J. Ježek [9]-[11], D.S. Murdoch [16], A.B. Romanowska and J.D.H. Smith [17], K. Toyoda [21] and others and this class plays a special role in the theory of quasigroups. $T$-quasigroups were introduced by T. Kepka and P. Nemec [10, 11]. Later G.B. Belyavskaya characterized the class of $T$-quasigroups by a system of two identities [5, 7].

A binary algebra $(Q; \Sigma)$ is called invertible, if $(Q; A)$ is a quasigroup for any operation, $A \in \Sigma$. The invertible algebras first were considered by
R. Schaufler in touch with coding theory [19, 20]. Later such algebras were investigated by J. Aczel [1], V.D. Belousov [2, 3], Yu.M. Movsisyan [12] - [15], A. Sade [18] and others.

By analogy with linear (alinear) quasigroups we introduce the notion of a linear (alinear) invertible algebra.

**Definition 1.1.** An invertible algebra \((Q; \Sigma)\) is called linear (alinear) over the group \((Q; +)\) if every operation \(A \in \Sigma\) has the form:

\[
A(x, y) = \phi_A x + t_A + \psi_A y,
\]

where \(\phi_A, \psi_A\) are automorphisms (antiautomorphisms) of \((Q; +)\) for all \(A \in \Sigma\), and \(t_A\) are fixed elements of \(Q\).

A linear invertible algebra over an abelian group is called an invertible T-algebra.

Let us recall, that the following absolutely closed second-order formulae:

\[
\forall X_1, \ldots, X_m \forall x_1, \ldots, x_n \quad (\omega_1 = \omega_2),
\]

\[
\forall X_1, \ldots, X_k \exists X_{k+1}, \ldots, X_m \forall x_1, \ldots, x_n \quad (\omega_1 = \omega_2),
\]

where \(\omega_1, \omega_2\) are words (terms) written in the functional variables \(X_1, \ldots, X_m\), and in the objective variables, \(x_1, \ldots, x_n\), are called \(\forall(\forall)-identity\) or \(hyper-identity\) and \(\forall\exists(\forall)-identity\). The satisfiability (truth) of these second order formulae in the algebra \((Q; \Sigma)\) is understood in the sense of functional quantifiers, \((\forall X_i)\) and \((\exists X_j)\), meaning: "for every value \(X_i = A \in \Sigma\) of the corresponding arity" and "there exists a value \(X_j = A \in \Sigma\) of the corresponding arity". It is assumed that such a replacement is possible, that is:

\[
\{|X_1|, \ldots, |X_m|\} \subseteq \{|A| \mid A \in \Sigma\},
\]

where \(|S|\) is the arity of \(S\). Generally, hyperidentities are written without a quantifier prefix: \(\omega_1 = \omega_2\). For details about such formulae see [12] - [15].

The binary algebra, \((Q; \Sigma)\), is called medial (abelian) if the following hyperidentity holds:

\[
X(Y(x, y), Y(u, v)) = Y(X(x, u), X(y, v)).
\]

Yu.M. Movsisyan proved that medial invertible algebras are a special class of invertible T-algebras, namely all automorphisms of the group \((Q; +)\),
which correspond the operations from $\Sigma$ are permutable:

$$\varphi_A \cdot \varphi_B = \varphi_B \cdot \varphi_A, \ \psi_A \cdot \psi_B = \psi_B \cdot \psi_A, \ \varphi_A \cdot \psi_B = \psi_B \varphi_A$$

for all $A, B \in \Sigma$.

In the present paper we characterize the class of invertible linear (alinear) algebras and the class of invertible $T$-algebras by second-order formulae, namely, $\forall \exists(\forall)$-identities. For proofs of these results we use the methods of the papers, [6, 5].

2. Linear and alinear invertible algebras

We denote by $L_{A,a}$ and $R_{A,a}$ the left and right translations of the binary algebra $(Q; \Sigma)$:

$L_{A,a} : x \mapsto A(a, x)$, $R_{A,a} : x \mapsto A(x, a)$. If the algebra $(Q; \Sigma)$ is an invertible algebra, then the translations, $L_{A,a}$ and $R_{A,a}$ are bijections for all $a \in Q$ and all $A \in \Sigma$.

The unique solution of the equality $B(a, x) = a (B(x, a) = a)$ is denoted by $e_a^B (f_a^B)$, i.e., $e_a^B (f_a^B)$ is the right (left) local identity of the element $a$ with respect to the operation $B$.

It is well known [3] that with each quasigroup $A$ the next five quasigroups are connected:

$$A^{-1}, -1A, -1(A^{-1}), (-1A)^{-1}, A^*,$$

where $A^*(x, y) = A(y, x)$. These quasigroups are called inverse quasigroups or parastrophies. Like this, with each invertible algebra $(Q; \Sigma)$ the next five invertible algebras are connected:

$$(Q; \Sigma^{-1}), (Q; -1 \Sigma), (Q; (\Sigma^{-1})^{-1}), (Q; (\Sigma^*)^*),$$

where

$$\Sigma^{-1} = \{A^{-1} | A \in \Sigma\},$$

$$-1 \Sigma = \{-1A | A \in \Sigma\},$$

$$-1(\Sigma^{-1}) = \{-1(A^{-1}) | A \in \Sigma\},$$

$$(\Sigma^{-1})^{-1} = \{(-1A)^{-1} | A \in \Sigma\},$$

$$\Sigma^* = \{A^* | A \in \Sigma\}.$$

Each of these invertible algebras are called parastrophies of $(Q; \Sigma)$.

**Lemma 2.1.** If an invertible algebra $(Q; \Sigma)$ satisfies the following equality:

$$A(B(x, y), B(u, v)) = A(B(x, u), B(\alpha y, v)),$$  \hspace{1cm} (2)
where $\alpha$ is a mapping from $Q$ into $Q$ and $A$, $B$ are some operations from $\Sigma$, then $\alpha$ depends on $u$, $A$, $B$ and on their inverse operations and has the form:

$$\alpha y = \alpha_u A B y = -1 B(A^{-1}(u, A(B^{-1}(u, y), u)), B^{-1}(u, u)).$$  \hspace{1cm} (3)

**Proof.** If in (2) $x = f_u^B$ and $v = e_u^B$, we obtain:

$$A(B(f_u^B, y), B(u, e_u^B)) = A(B(f_u^B, u), B(\alpha y, e_u^B));$$

$$A(B(f_u^B, y), u) = A(u, B(\alpha y, e_u^B));$$

$$A(L_B f_u^B y, u) = A(u, R_B e_u^B \alpha y);$$

$$R_{A,u} L_B f_u^B y = L_{A,u} R_B e_u^B \alpha y;$$

$$\alpha y = R_{B,e_u^B}^{-1} L_{A,u}^{-1} R_{A,u} L_B f_u^B y.$$

We have

$$\alpha y = R_{B,e_u^B}^{-1} L_{A,u}^{-1} R_{A,u} B(f_u^B, y) = R_{B,e_u^B}^{-1} L_{A,u}^{-1} A(B(f_u^B, y), u) =$$

$$R_{B,e_u^B}^{-1} A^{-1}(u, A(B(f_u^B, y), u)) =$$

$$-1 B(A^{-1}(u, A(B^{-1}(B(u, y), u)), B^{-1}(u, u))),$$

since $e_u^B = B^{-1}(u, u)$, $f_u^B = -1 B(u, u)$, $R_{B,y}^{-1} x = -1 B(x, y)$, $L_{B,y} x = B^{-1}(y, x).$ \hfill \Box

**Lemma 2.2.** If an invertible algebra $(Q; \Sigma)$ satisfies the following equality:

$$A(B(x, y), B(u, v)) = A(B(\beta v, y), B(u, x)), \hspace{1cm} (4)$$

where $\beta$ is a mapping from $Q$ into $Q$ and $A$, $B$ are some operations from $\Sigma$, then $\beta$ depends on $x$, $A$, $B$ and on their inverse operations and has the form:

$$\beta v = \beta_x A B v = -1 B(-1 A(x, B^{-1}(B(x, y), v)), x), B^{-1}(x, x)). \hspace{1cm} (5)$$

**Proof.** If in (4) $y = e_x^B$ and $u = f_x^B$, then we obtain as in Lemma 2.1. \hfill \Box

**Theorem 2.1.** The binary algebra $(Q; \Sigma)$ is an invertible linear algebra iff the following second order formula:

$$X(Y(x, y), Y(u, v)) = X(Y(x, u), Y(A_x Y y, v)), \hspace{1cm} (6)$$

where

$$\alpha_u A x Y y = -1 Y(X^{-1}(u, X(Y^{-1}(Y(u, u), y), u))), Y^{-1}(u, u)) \hspace{1cm} (7)$$

is valid in the algebra $(Q; \Sigma \cup \Sigma^{-1} \cup \Sigma^{-1})$ for all $X, Y \in \Sigma.$
Lemma 2.1 we obtain that
\[ X, Y = \text{prove that equality (6) is valid in the algebra } (Q; \Sigma \cup \Sigma^1 \cup \Sigma^{-1}) \text{ for all } X, Y \in \Sigma, \text{ when} \]
\[ \alpha_{XY}^{X,Y} y = -\alpha_0^{X,Y} u + \alpha_0^{XY} y + u, \]
where \( \alpha_0^{XY} y = \varphi_X^{-1} \psi_X^{-1} \tilde{L}_{cy} \tilde{R}_{cx} \varphi_X \psi_Y y, \tilde{L}_{cy} x = cy + x, \tilde{R}_{cx} x = x + cX. \)

Indeed,
\[ X(Y(x, y), Y(u, v)) = \varphi_X(\varphi_Y x + cy + \psi_Y y) + cX + \psi_X(\varphi_Y u + cy + \psi_Y v) = \]
\[ = \varphi_X \psi_Y x + \varphi_X cy + \varphi_X \psi_Y y + cX + \psi_X \varphi_Y u + \psi_X \psi_Y v, \]
on the other hand, using the expressions for \( \alpha_0^{XY} y \), we obtain
\[ X(Y(x, u), Y(\alpha_{XY}^{X,Y} y, v)) = \varphi_X(\varphi_Y x + cy + \psi_Y u) + cX + \]
\[ + \psi_X(\varphi_Y \alpha_{XY}^{X,Y} y + cy + \psi_Y v) = \varphi_X \varphi_Y x + \varphi_X cy + \varphi_X \psi_Y u + cX + \]
\[ + \psi_X \varphi_Y (-\alpha_0^{X,Y} u + \alpha_0^{X,Y} y + u) + \psi_X cy + \psi_X \psi_Y v = \varphi_X \varphi_Y x + \varphi_X cy + \]
\[ + \psi_X \psi_Y u + cX - \psi_X \varphi_Y \varphi_Y^{-1} \psi_X^{-1} \tilde{L}_{cy} \tilde{R}_{cx} \varphi_X \psi_Y u + \]
\[ + \psi_X \varphi_Y \varphi_Y^{-1} \psi_X^{-1} \tilde{L}_{cy} \tilde{R}_{cx} \varphi_X \psi_Y y + \psi_X \varphi_Y u + \psi_X cy + \psi_X \psi_Y v = \]
\[ = \varphi_X \varphi_Y x + \varphi_X cy + \varphi_X \psi_Y u + cX - \tilde{L}_{cy} \tilde{R}_{cx} \varphi_X \psi_Y u + \tilde{L}_{cy} \tilde{R}_{cx} \varphi_X \psi_Y y + \]
\[ + \psi_X \varphi_Y u + \psi_X cy + \psi_X \psi_Y v = \varphi_X \psi_Y x + \varphi_X cy + \varphi_X \psi_Y u + cX - \psi_X \varphi_Y u + \psi_X cy + \]
\[ - (\psi_X \varphi_Y u + cX) - cY + \varphi_X \psi_Y y + cX + \psi_X \varphi_Y u + \psi_X cy + \]
\[ + \psi_X \psi_Y v = \varphi_X \psi_Y x + \varphi_X cy + \varphi_X \psi_Y y + cX - \psi_X \varphi_Y u + \psi_X cy + \]
\[ - \psi_X \varphi_Y u + \psi_X cy + \psi_X \varphi_Y u + \psi_X cy + \psi_X \psi_Y v = \]
\[ = \varphi_X \varphi_Y x + \varphi_X cy + \varphi_X \psi_Y y + cX + \psi_X \varphi_Y u + \psi_X cy + \psi_X \psi_Y v. \]

Thus, the right and left sides of equality (6) are equal. According to Lemma 2.1 we obtain that \( \alpha_0^{XY} y \) has the form of (7).

Conversely, let formula (6) be valid in the algebra \((Q; \Sigma \cup \Sigma^1 \cup \Sigma^{-1})\) for all \( X, Y \in \Sigma \). We prove that the algebra \((Q; \Sigma)\) is an invertible linear algebra. Let us fix (in (6)) the element \( u = a \) and the operations \( X = A, Y = B \), where \( A, B \in \Sigma \), then we obtain:
\[ A(B(x, y), B(a, v)) = A(B(x, a), B(\alpha_{a}^{A,B} y, v)), \]
\[ A(B(x, y), L_{B,a} v) = A(R_{B,a} x, B(\alpha_{a}^{A,B} y, v)), \]
or

\[ A_1(x, y) = A_3(x, A_4(y, v)), \]

where

\[ A_1(x, y) = A(x, L_{B,a}y), \quad A_2(x, y) = B(x, y), \quad A_3(x, y) = A(R_{B,a}x, y), \]

\[ A_4(x, y) = B(\alpha_{A,B}x, y). \]

From the last equality, according to Belousov’s theorem about four quasigroups which are connected through the associative law [18], all the operations \( A_i \) \((i = 1, 2, 3, 4)\) are isotopic to the same group. Hence, the operations \( A \) and \( B \), are isotopic to the same group, and since the operations \( A \) and \( B \) are arbitrary we obtain that all the operations from \( \Sigma \) are isotopic to the same group \((Q; *)\).

For every \( X \in \Sigma \), let us define the operations:

\[ x + y = X(R_{X,a}^{-1}x, L_{X,b}^{-1}y), \tag{8} \]

where \( a, b \) are some elements from \( Q \). These operations are loops with the identity element \( 0_X = X(b, a) \)[3], and they are isotopic to the group \((Q; *)\). Hence, by Albert’s theorem [3], they are groups for every \( X \in \Sigma \).

Let us rewrite equality (6) (where \( X = A \), \( Y = B \)), (in terms of the operations \( + \) and \( + \)) in the following way:

\[ R_{A,a}(R_{B,a}x + L_{B,b}y) + L_{A,b}(R_{B,a}u + L_{B,b}v) = \]

\[ R_{A,a}(R_{B,a}x + L_{B,b}u) + L_{A,b}(R_{B,a}\alpha_{A,B}y + L_{B,b}v), \]

\[ R_{A,a}(x + y) + L_{A,b}(u + v) = \]

\[ R_{A,a}(x + L_{B,b}R_{B,a}^{-1}u) + L_{A,b}(R_{B,a}\alpha_{A,B}y + L_{B,b}R_{B,a}^{-1}y + u). \]

If we take \( u = 0_B \) and \( v = L_{A,b}^{-1}0_A \) in the last equality, then we have:

\[ R_{A,a}(x + y) + L_{A,b}(0_B + L_{A,b}^{-1}0_A) = \]

\[ R_{A,a}(x + L_{B,b}R_{B,a}^{-1}0_B) + L_{A,b}(R_{B,a}\alpha_{A,B}y + L_{B,b}^{-1}y + L_{B,b}^{-1}0_A), \]

\[ R_{A,a}(x + y) = \alpha_{A,B}x + \beta_{A,B}y, \tag{9} \]

where

\[ \alpha_{A,B}x = R_{A,a}(x + L_{B,b}R_{B,a}^{-1}0_B), \]

\[ \beta_{A,B}y = L_{A,b}(R_{B,a}\alpha_{A,B}^{-1}y + L_{B,b}^{-1}y + L_{A,b}^{-1}0_A). \]
Since the operations $A$ and $B$ are arbitrary, we can take $A = B$ in (9), then we obtain:
\[ R_{A,a}(x + y)_A = \alpha_{A,A}x + \beta_{A,A}y. \] (10)

From (9) and (10), we have:
\[ x + y = R_{A,a}(\alpha_{A,A}^{-1}x + \beta_{A,A}^{-1}y), \]
\[ x + y = R_{A,a}(\alpha_{A,B}^{-1}x + \beta_{A,B}^{-1}y), \]
\[ \alpha_{A,A}^{-1}x + \beta_{A,A}^{-1}y = \alpha_{A,B}^{-1}x + \beta_{A,B}^{-1}y, \]

thus, we obtain:
\[ x + y = \gamma_{A,B}x + \delta_{A,B}y, \] (11)

where $\gamma_{A,B} = \alpha_{A,B}^{-1}\alpha_{A,A}$ and $\delta_{A,B} = \beta_{A,B}^{-1}\beta_{A,A}$ are the permutations of the set $Q$. Hence, from (9), according to (11), we get:
\[ R_{A,a}(x + y)_B = \gamma_{A,B}x + \delta_{A,B}y, \]

i.e., $R_{A,a}$ is a quasiautomorphism of the group $(Q; +)$ and since the operation $A$ is arbitrary, we have that $R_{A,a}$ is the quasiautomorphism of the group $(Q; +)$ for all operations $A$ from $\Sigma$. We fix the operation $+$ and further will be denote it by $+$.

According to (8), for the operations $A \in \Sigma$ we have:
\[ A(x, y) = R_{A,a}x + L_{A,b}y. \]

According to (11), from the last equality, we get:
\[ A(x, y) = \theta_1^{A,B}x + \theta_2^{A,B}y, \] (12)

where $\theta_1^{A,B} = \gamma_{A,B}R_{A,a}$ and $\theta_2^{A,B} = \delta_{A,B}L_{A,b}$ are the permutations of $Q$.

We prove that $\theta_1^{A,B}$ and $\theta_2^{A,B}$ are quasiautomorphisms of the group $(Q; +)$. To do it we take $v = a$, $u = f^B_a$, $X = A$, $Y = B$ in equality (6) and rewrite this equality in terms of the operation $+$:
\[ A(B(x, y), a) = A(B(x, f^B_a), B(\alpha_{f^B_a,y}^A, a)), \]
\[ \theta_1^{A,B}(R_{B,a}x + L_{B,b}y) + \theta_2^{A,B}a = \theta_1^{A,B}R_{B,f^B_a}x + \theta_2^{A,B}(R_{B,a}\alpha_{f^B_a,y}^A + L_{B,b}a), \]
\[ \theta_1^{A,B}(R_{B,a}x + L_{B,b}y) = \theta_1^{A,B}R_{B,f^B_a}x + \theta_2^{A,B}(R_{B,a}\alpha_{f^B_a,y}^A + L_{B,b}a) - \theta_2^{A,B}a, \]
\[
\theta_1^{A,B}(x + y) = \theta_1^{A,B} R_B f_B^y R_B^{-1} x + \theta_2^{A,B} (R_B s_B^\alpha A^B L_B b y + L_B b a) - \theta_2^{A,B} a,
\]
\[
\theta_1^{A,B}(x + y) = \sigma_{A,B} x + \mu_{A,B} y,
\]
where
\[
\sigma_{A,B} x = \theta_1^{A,B} R_B f_B^y R_B^{-1} x \text{ and } \mu_{A,B} y = \theta_2^{A,B} (R_B s_B^\alpha A^B L_B b y + L_B b a) - \theta_2^{A,B} a
\]
are the permutations of \( Q \) and therefore \( \theta_1^{A,B} \) is a quasiautomorphism of the group \((Q; +)\).

Now, we take \( x = f_B^y, u = b, X = A, Y = B \) in (6) and rewrite this equality in terms of the operation \(+\):
\[
A(B(f_B^y, B(b, v)) = A(b, B(\alpha_{A,B}^B y, v)),
\]
\[
\theta_1^{A,B} L_B f_B^y + \theta_2^{A,B} L_B b v = \theta_1^{A,B} b + \theta_2^{A,B} (R_B s_B^\alpha A^B y + L_B b v),
\]
\[
\theta_2^{A,B} (R_B s_B^\alpha A^B y + L_B b v) = -\theta_1^{A,B} b + \theta_1^{A,B} L_B f_B^y + \theta_2^{A,B} L_B b v,
\]
\[
\theta_2^{A,B} (y + v) = \sigma'_{A,B} v + \mu'_{A,B} v,
\]
where \( \sigma'_{A,B} v = -\theta_1^{A,B} b + \theta_1^{A,B} L_B f_B^y (\alpha_{A,B}^B)^{-1} R_B^{-1} y \) and \( \mu'_{A,B} v = \theta_2^{A,B} v \) are the permutations of the set \( Q \) and therefore \( \theta_2^{A,B} \) is a quasiautomorphism of the group \((Q; +)\).

According to [3, lemma 2.5] we have:
\[
\theta_1^{A,B} x = \varphi_A x + s_A,
\]
\[
\theta_2^{A,B} x = t_A + \psi_A y,
\]
where \( \varphi_A, \psi_A \) are automorphisms of the group \((Q; +)\) and \( t_A, s_A \) are some elements of the set \( Q \). Hence, from (12), it follows that
\[
A(x, y) = \varphi_A x + c_A + \psi_A y, \tag{13}
\]
where \( c_A = s_A + t_A \).

Since the operation \( A \) is arbitrary, we obtain that all the operations from \( \Sigma \) can be presented in the form of (13) through the operation \(+\).

**Theorem 2.2.** The binary algebra \((Q; \Sigma)\) is an invertible alinear algebra iff the following second order formula:
\[
X(Y(x, y), Y(u, v)) = X(Y(\beta^X_Y v, y), Y(u, x)), \tag{14}
\]
where
\[
\beta^X_Y v = -1 Y(-1 X(x, Y(-1 Y(x, x), v)), x), Y^{-1}(x, x) \tag{15}
\]
is valid in the algebra \((Q; \Sigma \cup \Sigma^{-1} \cup -1 \Sigma)\) for all \( X, Y \in \Sigma \).
Proof. Let \((Q; \Sigma)\) be an invertible alinear algebra, then for every \(X \in \Sigma\)

\[ X(x, y) = \varphi_X x + c_X + \psi_X y, \]

where \(\varphi_X, \psi_X\) are anti-automorphisms of the group \((Q; +)\) and \(c_X \in Q\). We prove that equality (14) is valid in the algebra \((Q; \Sigma \cup \Sigma^{-1} \cup -1 \Sigma)\) for all \(X, Y \in \Sigma\), if:

\[ \beta_{X Y}^v = x + \beta_0^{X Y} v - \beta_0^{X Y} x, \]

where \(\beta_0^{X Y} v = \varphi_Y^{-1} \tilde{R}_{c y}^{-1} \tilde{L}_{c x} \psi_X \psi_Y v, \tilde{R}_{c y} x = x + c_Y, \tilde{L}_{c x} x = c_X + x.\)

Indeed,

\[
X(Y(x, y), Y(u, v)) = \varphi_X(\varphi_Y x + c_Y + \psi_Y y) + c_X + \psi_X(\varphi_Y u + c_Y + \psi_Y v) = \\
= \varphi_X \psi_Y y + \varphi_X c_Y + \varphi_X \varphi_Y x + c_X + \psi_X \psi_Y v + \psi_X c_Y + \psi_X \varphi_Y u,
\]

on the other hand, using the expressions for \(\beta_0^{X Y}\), and taking into account that \(\varphi_X \varphi_Y\) is an automorphism of the group \((Q; +)\) we obtain:

\[
X(Y(\beta_{X Y}^v, y), Y(u, x)) = \varphi_X(\varphi_Y \beta_{X Y}^v + c_Y + \psi_Y y) + c_X + \\
+ \psi_X(\varphi_Y u + c_Y + \psi_Y x) = \varphi_X \psi_Y y + \varphi_X c_Y + \varphi_X \varphi_Y \beta_{X Y}^v + c_X + \\
+ \psi_X \psi_Y x + \psi_X c_Y + \psi_X \varphi_Y u = \varphi_X \psi_Y y + \varphi_X c_Y + \varphi_X \varphi_Y x + \\
+ \varphi_X \psi_Y \varphi_Y^{-1} \varphi_X^{-1} \tilde{R}_{c y}^{-1} \tilde{L}_{c x} \varphi_X \psi_Y x = \varphi_X \psi_Y y + \varphi_X c_Y + \varphi_X \varphi_Y x + c_X + \\
+ \psi_X \psi_Y v - c_Y - (c_X + \psi_X \psi_Y - c_Y) + c_X + \psi_X \psi_Y x + \psi_X c_Y + \psi_X \varphi_Y u = \\
= \varphi_X \psi_Y y + \varphi_X c_Y + \varphi_X \varphi_Y x + c_X + \psi_X \psi_Y v - c_Y - \psi_X \psi_Y x + \\
- c_X + c_X + \varphi_X \psi_Y x + \psi_X c_Y + \psi_X \varphi_Y u = \\
= \varphi_X \psi_Y y + \varphi_X c_Y + \varphi_X \varphi_Y x + c_X + \psi_X \psi_Y v + \psi_X c_Y + \psi_X \varphi_Y u.
\]

Thus, the right and left sides of equality (14) are equal. According to Lemma 2.2, we get that \(\beta_{X Y}^v\) has the form of (15).

Conversely, let the formula (14) be valid in the algebra \((Q; \Sigma \cup -1 \Sigma)\) for all \(X, Y \in \Sigma\). We prove that the algebra \((Q; \Sigma)\) is an invertible alinear algebra. Fixing the element \(x = p\) and the operations \(X = A, Y = B,\)
where \( A, B \in \Sigma \) in (14), we obtain:

\[
A(B(p, y), B(u, v)) = A(B(\beta^{A,B}_p v, y), B(u, p)),
\]
\[
A(L_{B,p} y, B(u, v)) = A(B(\beta^{A,B}_p v, y), R_{B,p} u),
\]
\[
A^*(B(u, v), L_{B,p} y) = A^*(R_{B,p} u, B(\beta^{A,B}_p v, y))
\]
or

\[
A_1(A_2(u, v), y) = A_3(u, A_4(v, y)),
\]
where \( A_1(x, y) = A^*(x, L_{B,p} y), A_2(x, y) = B(x, y), A_3(x, y) = A^*(R_{B,p} x, y), A_4(x, y) = B(\beta^{A,B}_p x, y). \)

From the last equality, according to Belousov’s theorem about four quasigroups which are connected with the associative law [18], all the operations \( A_i (i = 1, 2, 3, 4) \) are isotopic to the same group. Since the operation \( B \) is arbitrary, we obtain that all the operations from \( \Sigma \) are isotopic to the same group \( (Q; *) \).

For every \( X \in \Sigma \) let us define the operations:

\[
x + y = X(R^{-1}_{X,a} x, L^{-1}_{X,b} y), \tag{16}
\]
where \( a, b \) are some elements from \( Q \). These operations are loops with the identity element \( 0_X = X(b, a) \) [3], and they are isotopic to the group \( (Q; *) \).

Hence by Albert’s theorem [3] they are groups for every \( X \in \Sigma \).

Let us rewrite the equality (14) (where \( X = A, Y = B \)) in terms of the operations \(+ \) and \(- \):

\[
R_{A,a}(R_{B,a} x + L_{B,b} y) + L_{A,b}(R_{B,a} u + L_{B,b} v) = R_{A,a}(R_{B,a} \beta^{A,B}_x v + L_{B,b} y) + L_{A,b}(R_{B,a} u + L_{B,b} x).
\]

If we take \( y = a \) and \( x = R^{-1}_{B,a} b = d \) in the last equality, we have:

\[
R_{A,a}(R_{B,a} R^{-1}_{B,a} b + L_{B,b} a) + L_{A,b}(R_{B,a} u + L_{B,b} v) = R_{A,a}(R_{B,a} \beta^{A,B}_d v + L_{B,b} a) + L_{A,b}(R_{B,a} u + L_{B,b} d),
\]
\[
R_{A,a}(b + 0_B) + L_{A,b}(R_{B,a} u + L_{B,b} v) = R_{A,a}(R_{B,a} \beta^{A,B}_d v + 0_B) + L_{A,b} B(u, d),
\]
\[
R_{A,a} b + L_{A,b}(R_{B,a} u + L_{B,b} v) = R_{A,a} R_{B,a} \beta^{A,B}_d v + L_{A,b} R_{B,d} u,
\]
A characterization of binary algebras

\begin{align*}
L_{A,b}(R_{B,a}u + L_{B,b}v) &= R_{A,b}R_{B,a}A^{A,B}v + L_{A,b}R_{B,d}u, \\
\text{or} \quad L_{A,b}(u + v) &= \alpha_{A,B}v + \beta_{A,B}u \quad (17)
\end{align*}

where

\begin{align*}
\alpha_{A,B} &= R_{A,b}R_{B,a}A^{A,B}v_{B,b}^{-1} \\
\beta_{A,B} &= L_{A,b}R_{B,d}R_{B,a}^{-1}
\end{align*}

are permutations of the set \( Q \).

Since the operations \( A \) and \( B \) are arbitrary, we can take \( A = B \) in (17), and get:

\begin{align*}
L_{A,b}(u + v) &= \alpha_{A,A}v + \beta_{A,A}u. \quad (18)
\end{align*}

From (17) and (18) we have:

\begin{align*}
v + u &= L_{A,b}(\beta_{A,B}^{-1}u + \alpha_{A,B}^{-1}v), \\
v + u &= L_{A,b}(\beta_{A,A}^{-1}u + \alpha_{A,A}^{-1}v), \\
\beta_{A,B}^{-1}u + \alpha_{A,B}^{-1}v &= \beta_{A,A}^{-1}u + \alpha_{A,A}^{-1}v,
\end{align*}

and thus, we obtain:

\begin{align*}
u = \gamma_{A,B}u + \delta_{A,B}v, \quad (19)
\end{align*}

where \( \gamma_{A,B} = \beta_{A,B}^{-1} \beta_{A,A} \) and \( \delta_{A,B} = \alpha_{A,B}^{-1} \alpha_{A,A} \) are the permutations of the set \( Q \).

According to (16), for the operations \( A \in \Sigma \), we have:

\begin{align*}
A(x, y) &= R_{A,a}x + L_{A,b}y.
\end{align*}

According to (19), from the last equality, we get:

\begin{align*}
A(x, y) &= \theta_1^{A,B}x + \theta_2^{A,B}y, \quad (20)
\end{align*}

where \( \theta_1^{A,B} = \gamma_{A,B}R_{A,a} \) and the \( \theta_2^{A,B} = \delta_{A,B}L_{A,b} \) are the permutations of the set \( Q \). Thus, we can represent every operations from \( \Sigma \) by the operation \( + \). We fix the operation \( + \) and further denote it by \( + \).

We shall prove that \( \theta_1^{A,B} \) and \( \theta_2^{A,B} \) are antiquasiomorphisms of the group \( (Q; +) \). To do it we take \( x = a, u = f_a^B, X = A, Y = B \), in equality (14) and rewrite this equality in terms of the operation, \( + \):
\[ A(B(a, y), B(f^B_v, v)) = A(B(\beta^{AB}_a v, y), a), \]
\[ \theta^{AB}_1 (\beta^{AB}_a v + L_{B, b} v) + \theta^{AB}_2 \beta^{AB}_a = \theta^{AB}_1 (\beta^{AB}_a v + L_{B, b} y) + \theta^{AB}_2 a, \]
\[ \theta^{AB}_1 (\beta^{AB}_a v + L_{B, b} y) = \theta^{AB}_1 (\beta^{AB}_a v + L_{B, b} y) + \theta^{AB}_2 L_{B, f^B_v} v - \theta^{AB}_2 a, \]
\[ \theta^{AB}_1 (v + y) = \theta^{AB}_1 (\beta^{AB}_a v + y) + \theta^{AB}_2 L_{B, f^B_v} (\beta^{AB}_a v^{-1} R_{B, a} v - \theta^{AB}_2 a, \]
\[ \theta^{AB}_1 (v + y) = \sigma^{AB} y + \mu^{AB} v, \]

where
\[ \sigma^{AB} y = \theta^{AB}_1 (R_{B, a} v + y) \] and \( \mu^{AB} v = \theta^{AB}_2 L_{B, f^B_v} (\beta^{AB}_a v^{-1} R_{B, a} v - \theta^{AB}_2 a \)

are the permutations of the set \( Q \) and therefore, \( \theta^{AB}_1 \) is an antquasiautomorphism of the group \( (Q; +) \).

If we take \( x = a, y = e^B_a, X = A, Y = B \) in the equality (14), we can similarly prove that \( \theta^{AB}_2 \) is an antquasiautomorphism of the group \( (Q; +) \).

Thus, we have [2]
\[ \theta^{AB}_1 x = \varphi_A x + s_A, \]
\[ \theta^{AB}_2 x = t_A + \psi_A y, \]

where \( \varphi_A, \psi_A \) are antiautomorphisms of the group \( (Q; +) \) and \( t_A, s_A \) are some elements of the set \( Q \). Hence, from (20) we get that:
\[ A(x, y) = \varphi_A x + c_A + \psi_A y, \quad (21) \]

where \( c_A = s_A + t_A \).

Since the operation \( A \) is arbitrary, we obtain that all the operations from \( \Sigma \) can be presented in the form of (21).

\[ \square \]

3. Invertible \( T \)-algebras

It is known [10, 11] that \( T \)-quasigroups are invariant under parastrophies. We have the same result for parastrophies of invertible \( T \)-algebras.

**Proposition 3.1.** Let \((Q; \Sigma)\) be an invertible \( T \)-algebra. Then all parastrophies of the algebra, \((Q; \Sigma)\), are invertible \( T \)-algebras.

Also, as in the case of quasigroups [6], we have the following result:

**Proposition 3.2.** If an invertible algebra is linear and alinear then it is \( T \)-algebra.
Lemma 3.1. If the algebra \((Q; \Sigma \cup \Sigma^{-1} \cup \Sigma^{-1})\), where \((Q; \Sigma)\) is an invertible T-algebra, satisfies equality (6) for all \(X, Y \in \Sigma\), then this equality is also valid in the algebra \((Q; \Sigma \cup \Sigma^{-1} \cup \Sigma^{-1} \cup \Sigma^{-1} \cup \Sigma^{-1} \cup \Sigma^{-1} \cup \Sigma^{-1} \cup \Sigma^{-1})\) for all \(X, Y \in \Sigma \cup \Sigma^{-1} \cup \Sigma^{-1} \cup \Sigma^{-1} \cup \Sigma^{-1} \cup \Sigma^{-1} \cup \Sigma^{-1} \cup \Sigma^{-1}\).

Proof. We must check equalities for all \(A, B \in \Sigma \cup \Sigma^{-1} \cup \Sigma^{-1} \cup \Sigma^{-1} \cup \Sigma^{-1} \cup \Sigma^{-1} \cup \Sigma^{-1} \cup \Sigma^{-1}\). For example, let us check the following equality:

\[
A((-1)B(x, y), -1B(u, v)) = A((-1)B(x, u), -1B(A^{-1}B(y, v), v))
\]

In this case, we have:

\[
\alpha_u^{-1}B y = B(A^{-1}(u, A^{-1}B(B(u, u), y)), (-1B)^{-1}(u, u)).
\]

It follows from (1):

\[
A^{-1}(x, y) = \psi_A^{-1}(-c_A - \varphi_A x + y),
\]

\[
-1B(x, y) = \varphi_B^{-1}(x - \psi_B y - c_B),
\]

\[
(1)^{-1}(x, y) = \psi_B^{-1}(-c_B - \varphi_B y + x).
\]

Let us calculate \(\alpha_u^{-1}B y\):

\[
\alpha_u^{-1}B y = \varphi_B \psi_A^{-1}((\varphi_A \varphi_B^{-1} \psi_B u - \varphi_A \varphi_B^{-1} \psi_B u + \psi_A u) + u - \varphi_B u - c_B + c_B
\]

\[
= \varphi_B \psi_A^{-1} \varphi_A \varphi_B^{-1} \psi_B u - \varphi_B \psi_A^{-1} \varphi_A \varphi_B^{-1} \psi_B u + \varphi_B u + u - \varphi_B u
\]

\[
= \varphi_B \psi_A^{-1} \varphi_A \varphi_B^{-1} (\psi_B u - \psi_B y) + u.
\]

Therefore

\[
A((-1)B(x, u), -1B(\alpha_u^{-1}B y, v))
\]

\[
= A(\varphi_B^{-1}(x - \psi_B u - c_B), \varphi_B^{-1}(\alpha_u^{-1}B y - \psi_B v - c_B))
\]

\[
= \varphi_A \varphi_B^{-1}(x - \psi_B u - c_B) + \psi_A \varphi_B^{-1}(\alpha_u^{-1}B y - \psi_B v - c_B) + c_A
\]

\[
= \varphi_A \varphi_B^{-1} x - \varphi_A \varphi_B^{-1} \psi_B u - \varphi_A \varphi_B^{-1} c_B + \psi_A \varphi_B^{-1} \psi_B \psi_A \varphi_B^{-1} (\psi_B u - \psi_B y)
\]

\[
+ \psi_A \varphi_B^{-1} u - \psi_A \varphi_B^{-1} \psi_B v - \psi_A \varphi_B^{-1} c_B + c_A
\]

\[
= \varphi_A \varphi_B^{-1} x - \varphi_A \varphi_B^{-1} \psi_B u + \psi_A \varphi_B^{-1} u - \psi_A \varphi_B^{-1} \psi_B v - \psi_A \varphi_B^{-1} c_B + c_A
\]

On the other hand

\[
A((-1)B(x, u), -1B(u, v)) = \varphi_A \varphi_B^{-1}(x - \psi_B y - c_B) + \psi_A \varphi_B^{-1}(u - \psi_B v - c_B) + c_A
\]

\[
= \varphi_A \varphi_B^{-1} x - \varphi_A \varphi_B^{-1} \psi_B y - \varphi_A \varphi_B^{-1} c_B + \psi_A \varphi_B^{-1} u - \psi_A \varphi_B^{-1} \psi_B v - \psi_A \varphi_B^{-1} c_B + c_A.
\]

Thus, the right and left sides are equal. Similarly, we can check the other cases. \(\square\)
Lemma 3.2. Let \((Q; \Sigma)\) be an invertible \(T\)-algebra. If the algebra, \((Q; \Sigma \cup \Sigma^{-1} \cup \Sigma^{-1})\), satisfies equality (14) for all \(X, Y \in \Sigma\), then this equality is valid in the algebra \((Q; \Sigma \cup \Sigma^{-1} \cup (-1)\Sigma^{-1} \cup (-1)\Sigma^{-1} \cup \Sigma^*)\) for all \(X, Y \in \Sigma \cup \Sigma^{-1} \cup (-1)\Sigma^{-1} \cup (-1)\Sigma^{-1} \cup \Sigma^*)\).

Proof. Similarly as Lemma 3.1.

Theorem 3.1. \((Q; \Sigma)\) is an invertible \(T\)-algebra iff (6) and (14) are valid in the algebra \((Q; \Sigma \cup \Sigma^{-1} \cup (-1)\Sigma^{-1} \cup (-1)\Sigma^{-1} \cup \Sigma^*)\) for all \(X, Y \in \Sigma \cup \Sigma^{-1} \cup (-1)\Sigma^{-1} \cup (-1)\Sigma^{-1} \cup \Sigma^*)\).

Proof. As in the proof of Theorems 2.1 and 2.2, the invertible \(T\)-algebra satisfies formulae (6) and (14). The rest follows from Lemmas 3.1 and 3.2. The converse statement is a consequence of Proposition 3.2.

Corollary 3.1. Let \((Q; \Sigma)\) be an invertible \(T\)-algebra. If \((Q; \Sigma)\) satisfies the following second-order formula:

\[
\forall X_1, X_2 \, \forall x_1, x_2, x_3 \exists x_4 \\
(X_1(X_2(x_1, x_2), X_2(x_4, x_3))) = X_1(X_2(x_1, x_4), X_2(x_2, x_3)),
\]

then in \((Q; \Sigma)\) the following hyperidentity is valid:

\[
X_1(X_2(x_1, x_2), X_2(x_4, x_3)) = X_1(X_2(x_1, x_4), X_2(x_2, x_3)).
\]

Proof. Let \((Q; \Sigma)\) be an invertible \(T\)-algebra. Then it satisfies (6). If we rewrite (6), in terms of the operation \(+\), then after cancellations we obtain

\[
\psi_X \psi_Y u + \varphi_X \psi_Y y = \varphi_X \psi_Y u + \psi_X \varphi_Y \alpha^{X,Y}_u y,
\]

which for \(u = 0\) gives \(\varphi_X \psi_Y = \psi_X \varphi_Y \alpha^{X,Y}_0\). This together with (23) implies

\[
u + \alpha^{X,Y}_0 y = \alpha^{X,Y}_0 u + \alpha^{X,Y}_u y,
\]

where \(\alpha^{X,Y}_0\) is the permutation which corresponds to the identity element of the group, \((Q; +)\).

If (22) is valid in \((Q; \Sigma)\), then for every \(X, Y \in \Sigma\) and every \(x, y, v \in Q\) there exists an element \(h \in Q\) such that the following equality is valid:

\[
X(Y(x, y, Y(h, v))) = X(Y(x, h), Y(y, v)).
\]
Therefore, $\alpha_{h}^{X,Y}$ is the identity permutation of the set $Q$.

From the proof of Theorem 2.1, it follows that the loops $x \pm y = X(R_{X,a}^{-1}x, L_{X,b}^{-1}y)$ are groups for all $a, b \in Q$ and all operations $X \in \Sigma$ and also, we can take any of the groups, $\pm (X \in \Sigma)$ as a group $\pm$.

Let us choose the elements $a, b$ such that $h = Y(b, a)$ is an identity element of the group $(Q; +)$, then $\alpha_{h}^{X,Y}$ is the identity permutation of the set $Q$. Therefore, from (24), we have $\alpha_{u}^{X,Y} y = y$ since $\alpha_{0}^{X,Y} = \alpha_{h}^{X,Y}$ is the identity permutation. Hence $\alpha_{u}^{X,Y}$ is the identity permutation for all $u \in Q$ and all $X, Y \in \Sigma$.

**Corollary 3.2.** The quasigroup, $(Q; \cdot)$, is a $T$-quasigroup iff formulae (6) and (14) are valid in the quasigroup, $(Q; \cdot, /, \backslash)$, for all $X, Y \in \{\cdot, /, \backslash\}$.

**References**


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