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Classification of loops of generalized Bol-Moufang type

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Abstract. A loop identity $\alpha = \beta$ is of *Bol-Moufang type* if the same 3 variables appear on both sides of the equal sign in the same order, one of the variables appears twice on both sides and the remaining two variables appear once on both sides. One can generalize this definition by allowing different variable orders on either side of the identity, e.g. $((xx)y)z = x(y(xz))$. There are 1215 nontrivial identities of this type. Loop varieties axiomatized by a single identity of this type are said to be of *generalized Bol-Moufang type*. We show that there are 48 such varieties: the 14 varieties of Bol-Moufang type [13], the 6 varieties of commutative Bol-Moufang type, and 28 new varieties.

1. Introduction

The aim of this paper is to find and classify all loops that are a generalization of loops of Bol-Moufang type [3], [4], [8], [13], and [17].

A *quasigroup* is a set Q with a binary operation $*$ such that the equation $a * b = c$ has a unique solution in Q whenever two of a, b, and c are fixed elements of Q.

A *loop* is a quasigroup with a two-sided neutral element, which we will denote as 1. Standard references for loop theory are [1] and [15].

An identity $\alpha = \beta$ is of $Bol-Moufang$ type if it satisfies the following:

1. the only operation in α and β is $*$,
2. the same 3 variables appear in α and β,
3. one of the variables appears twice in α and β,
4. the remaining two variables appear once in α and β,

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5. the variables appear in the same order in α and β.

We generalize this by dropping the fifth condition, above. An identity $\alpha = \beta$ is of $generalized\ Bol - Moufang$ type if it satisfies the following:

1. the only operation in α and β is $*$,
2. the same 3 variables appear in α and β,
3. one of the variables appears twice in α and β,
4. the remaining two variables appear once in α and β.

This paper presents the classification of all varieties of loops of generalized Bol-Moufang type. Given the classification of loops of Bol-Moufang type [13], we examine those identities in which the variables do not appear in the same order in α and β. We show that these identities are one of 48 varieties: the 14 varieties of Bol-Moufang type [13], the 6 varieties of commutative Bol-Moufang type, and 28 new varieties. As 28 is a perfect number, we name the new varieties $Perfect$.

For our convenience, we let x be the double variable and y and z be the remaining two variables in this classification. Also, we omit $*$ when multiplying two elements together ($e.g.$ $x * y = xy$).

Throughout the course of this research, Prover9, an automated theorem prover, and Mace4, a finite model builder, were used [11]. For ease of reading, only several distinguishing proofs and several important counterexamples are found in this paper. Untranslated proofs from Prover9 are not included.

2. Notation and definitions

The following scheme is used to label each identity. This is an extension of the labeling scheme used by Phillips and Vojtěchovský [14].

Variable Order

A	xxyz	G	xxzy
B	xyxz	H	xzxy
C	yxxz	I	zxxy
D	xyzx	J	xzyx
E	yxzx	K	zxyx
F	yzxx	L	zyxx

Multiplication Order

1	$a(b(cd))$
2	$a((bc)c)$
3	$(ab)(cd)$
4	$(a(bc))d$
5	$((ab)c)d$

For example, the identity $(x(yx))z = z((xy)x)$ is called $B4K2$.

The following terminology is used to describe the generalized Bol-Moufang identities. The variable order of α is *normal* if y appears before z, i.e., α of variable order $A-F$. The remaining variable orders are created by flipping y and z and are thus called *flip* where G is the flip of A, and so on. An identity $\alpha=\beta$ is *normal – normal* if α and β are normal. We define *normal – flip*, *flip – flip*, and *flip – normal* similarly. The *dual* of an identity is the identity created when reading an identity from right to left. For example, the *dual* of $A1B2$, $x(x(yz))=x((yx)z)$, is $(z(xy))x=((zy)x)x$, or $E4F5$.

3. Identities of generalized Bol-Moufang type

In order to classify all varieties of loops of generalized Bol-Moufang type, we first count all possible identities of generalized Bol-Moufang type. We then find equivalencies among these identities, systematically examining and eliminating first equivalent commutative identities and then equivalent non-commutative identities. The remainder is the list of unique varieties, the loops of Perfect type.

Theorem 3.1. *There are* 1215 *non-trivial identities of generalized Bol-Moufang type.*

Proof. Note that each normal-flip identity has an equivalent flip-normal identity. For example, $A1H2$ is equivalent to $H2A1$. Likewise, each normal-normal identity has an equivalent flip-flip identity when the substitution $y=z$ is made. It is thus sufficient to count the normal-flip identities and normal-normal identities, i.e. identities where α has the variable order A, B, C, D, E, or F.

Note that α and β can be 1 of 5 possible multiplication orders; each identity of variable order $\alpha=\beta$ thus has 25 possible multiplication orders.

Consider first the normal-flip identities in which β is the flip of α; for example, $A1G3$. There are 25 such identities for A, B, C, D, E, and F. However, for each variable order, ten of the normal-flip identities are equivalent to one of the remaining 15 when the substitution $y=z$ is made. For example, $A1G2$ is equivalent to $A2G1$. Thus, there are 15 identities for the normal-flip identities in which β is the flip of α.

Consider the remaining normal-flip identities and the normal-normal identities. Let A be the variable order for α. Note that AA is Bol-Moufang and has thus been classified [13], and that AG has already been accounted for. There are 10 remaining possible variable pairings with A: AB, AC, AD, AE, AF, AH, AI, AJ, AK, and AL. Likewise, the following are possible variable pairings with B: BA, BC, BD, BE, BF, BG, BI, BJ,

BK, and BL. Note that BA is equivalent to AB and has thus already been counted. Therefore, B has 9 possible variable pairings. Similarly, C has 8 possible variable pairings, D has 7, E has 6, and F has 5.

Multiplying the normal-flip and normal-normal variable orders with the possible 25 multiplication orders and adding the specific case of the normal-flip identities in which β is the flip of α gives the total number of identities: $(10+9+8+7+6+5)*25+(6)*15 = 1215$ □

4. Commutative identities

First we examine the identities which imply commutativity. Any identity in which letting x, y, or $z = 1$ yields $zy = yz$, $xy = yx$, or $xz = zx$ will be commutative.

Theorem 4.1. *There are* 1092 *commutative identities of the generalized Bol-Moufang type.*

Proof. The 840 normal - flip identities are commutative because when $x = 1$, they become $yz = zy$.

In addition, 90 normal-normal identities were found to be commutative by letting x, y, or $z = 1$: $A1B1$, $A1B2$, $A1B4$, $A1D1$, $A1D2$, $A1D3$, $A1E1$, $A1E3$, $A2B1$, $A2B2$, $A2B4$, $A2D1$, $A2D2$, $A2D3$, $A2E1$, $A2E3$, $A4B1$, $A4B2$, $A4B4$, $A4D1$, $A4D2$, $B1D1$, $B1D2$, $B1D3$, $B1E1$, $B1E3$, $B2D1$, $B2D2$, $B2D3$, $B2E1$, $B2E3$, $B3C3$, $B3C5$, $B3D1$, $B3D2$, $B3D3$, $B3E1$, $B3E3$, $B3E4$, $B3E5$, $B3F4$, $B3F5$, $B5C3$, $B5C5$, $B5E3$, $B5E4$, $B5E5$, $B5F4$, $B5F5$, $C1D1$, $C1D2$, $C1D3$, $C1E1$, $C1E3$, $C3D1$, $C3D2$, $C3D3$, $C3D4$, $C3D5$, $C3E1$, $C3E3$, $C5D3$, $C5D4$, $C5D5$, $D3E3$, $D3E4$, $D3E5$, $D3F4$, $D3F5$, $D4E3$, $D4E4$, $D4E5$, $D4F2$, $D4F4$, $D4F5$, $D5E3$, $D5E4$, $D5E5$, $D5F2$, $D5F4$, $D5F5$, $E2F2$, $E2F4$, $E2F5$, $E4F2$, $E4F4$, $E4F5$, $E5F2$, $E5F4$, $E5F5$.

Example 4.2. For the equation $A4B2$, $(x(xy))z = x((yx)z)$ when setting the variable z as the identity the equation yields $x(xy) = x(yx)$, which is left cancelative. The resulting equation is $xy = yx$, which is commutative.

162 remaining commutative identities were found using Prover9 [11]: $A1B3$, $A1B5$, $A1D4$, $A1D5$, $A1E2$, $A1E4$, $A1E5$, $A2B3$, $A2B5$, $A2D4$, $A2D5$, $A2E2$, $A2E4$, $A2E5$, $A3B1$, $A3B2$, $A3B3$, $A3D1$, $A3D3$, $A3D5$, $A3E3$, $A3E4$, $A3E5$, $A4B3$, $A4B5$, $A4D3$, $A4D4$, $A4D5$, $A4E1$, $A4E2$, $A4E3$, $A4E4$, $A4E5$, $A5B1$, $A5B2$, $A5B3$, $A5D1$, $A5D3$, $A5D5$, $A5E3$, $A5E4$, $A5E5$, $B1C1$, $B1C2$, $B1C3$, $B1C4$, $B1C5$, $B1D4$, $B1D5$, $B1E2$, $B1E4$, $B1E5$, $B1F1$, $B1F2$, $B1F3$,

$B1F4$, $B1F5$, $B2C1$, $B2C2$, $B2C3$, $B2C4$, $B2C5$, $B2D4$, $B2D5$, $B2E2$, $B2E4$, $B2E5$, $B2F1$, $B2F2$, $B2F3$, $B2F4$, $B2F5$, $B3C1$, $B3C2$, $B3C4$, $B3D4$, $B3D5$, $B3E2$, $B3F1$, $B3F2$, $B3F3$, $B4C1$, $B4C3$, $B4C5$, $B4D1$, $B4D3$, $B4D5$, $B4E3$, $B4E4$, $B4E5$, $B4F2$, $B4F4$, $B4F5$, $B5C1$, $B5D1$, $B5D3$, $B5D5$, $B5F2$, $C1D4$, $C1D5$, $C1E2$, $C1E4$, $C1E5$, $C2D1$, $C2D3$, $C2D5$, $C2E3$, $C2E4$, $C2E5$, $C3E2$, $C3E4$, $C3E5$, $C4D1$, $C4D3$, $C4D5$, $C4E3$, $C4E4$, $C4E5$, $C5D1$, $C5D2$, $C5E1$, $C5E2$, $C5E3$, $C5E4$, $C5E5$, $D1E1$, $D1E2$, $D1E3$, $D1E4$, $D1E5$, $D1F1$, $D1F2$, $D1F3$, $D1F4$, $D1F5$, $D2E3$, $D2E4$, $D2E5$, $D2F2$, $D2F4$, $D2F5$, $D3E1$, $D3E2$, $D3F1$, $D3F2$, $D3F3$, $D5E1$, $D5E2$, $D5F1$, $D5F3$, $E1F2$, $E1F4$, $E1F5$, $E3F1$, $E3F2$, $E3F3$, $E3F4$, $E3F5$, $E4F1$, $E4F3$, $E5F1$, $E5F3$.

Thus, since $840+90+162 = 1092$, there are 1092 commutative identities of Bol-Moufang type. □

Theorem 4.3. *Any commutative identity of the generalized Bol-Moufang type can either be commuted to be of the Bol-Moufang type (i.e. the variables appear in the same order in α and β) or is of the commutative Moufang variety.*

Proof. Many commutative identities of the generalized Bol-Moufang type can be commuted to be of the Bol-Moufang type and have thus been classified [13]. Using the following table, we generated a list of the only commutative identities that cannot be commuted to the Bol-Moufang type. Letting α have the variable order of the left-most column and the multiplication order of the top-most row, all possible commutations of α are listed, such that α commutes to some multiplication order of the listed variable orders.

Table 1: Possible Commutations

	1	2	3	4	5
A	D,F,G,J,L	B,D,E,H,J,K,L	F,G,L	B,C,I,K,L	C,I,L
B	D,E,F,G,H,J,K	A,D,E,H,J,K,L	C,D,E,H,I,J,K	A,C,I,K,L	A,C,I,K,L
C	E,F,G,H,I,	F,G,I	B,D,E,H,I,J,K	A,I,L	A,D,E,H,I,K,L
D	B,E,F,G,H,J,K	A,F,G,J,L	B,C,E,H,I,J,K	A,F,G,J,L	A,B,E,H,J,K,L
E	C,F,G,H,I	C,F,G,H,I	B,C,D,H,I,J,K	B,D,F,G,H,J,K	A,B,D,H,J,K,L
F	C,G,I	C,E,G,H,I	A,G,L	B,D,E,G,H,J,K	A,D,G,J,L
G	A,D,F,J,L	B,D,E,F,H,J,K	A,F,L	C,E,F,H,I	C,F,I
H	A,B,D,E,J,K,L	B,D,E,F,G,J,K	B,C,D,E,I,J,K	C,E,F,G,I	C,E,F,G,I
I	A,B,C,K,L	A,C,L	B,C,D,E,H,J,K	C,F,G	C,E,F,G,H
J	A,B,D,E,H,K,L	A,D,F,G,L	B,C,D,E,H,I,K	A,D,F,G,L	B,D,E,F,G,H,K
K	A,B,C,I,L	A,B,C,I,L	B,C,D,E,H,I,J	A,B,D,E,J,H,L	B,D,E,F,G,H,J
L	A,C,I	A,B,C,I,K	A,F,G	A,B,D,E,J,H,K	A,D,F,G,J

Example 4.4. $A3 = (xx)(yz)$, can be commuted into the variable order F, G, or L, namely $(yz)(xx)$, $(xx)(zy)$, or $(zy)(xx)$.

Using this table, we found all generalized Bol-Moufang identities which could not be commuted into the Bol-Moufang type and eliminated any which did not axiomatize a commutative variety. The following identities are commutative but cannot be commuted into the Bol-Moufang type: $A2G5$, $A2I4$, $A3H3$, $A3I3$, $A3J3$, $A3K3$, $A5H2$, $A5J5$, $A5K5$, $B1I2$, $B1L1$, $B2G5$, $B2I4$, $B3G3$, $C2H1$, $C2J1$, $C2K4$, $C2L4$, $C3G3$, $C3L3$, $C4G2$, $C4G2$, $C4J5$, $C4K5$, $D1L1$, $D3G3$, $D3L3$, $D5G5$, $D5I4$, $E3G3$, $E3L3$, $E4I2$, $E4L1$, $E5G5$, $E5I4$, $F1H1$, $F1J1$, $F1K4$, $F1L4$, $F3H3$, $F3I3$, $F3J3$, $F3K3$, and $F4I2$. □

Using Prover9, we found these identities to be equivalent to the commutative Moufang identity, $(xx)(yz) = (xy)(xz)$, and have thus been classified [11]. An example of one of these proofs follows.

Theorem 4.5. *$A2I4$ is of the commutative Moufang variety.*

Proof. Letting $x = 1$ in $A2I4$, $x((xy)z) = (z(xx))y$, gives communtativity, $yz = zy$. Similarly, by setting $y = 1$, we have $x(xz) = z(xx)$. Using these,

$$\begin{aligned} x(z(xy)) &= x((xy)z) && \text{(by commutativity)} \\ &= (z(xx))y && \text{(assumption)} \\ &= (x(xz))y && \\ &= ((xz)x)y && \text{(by commutativity)} \end{aligned}$$

Thus, $A2I4$ is commutative Moufang variety. □

Similarly, it can be shown that all commutative identities that do not commute to be of the Bol-Moufang type are of the commutative Moufang variety. Thus, all commutative identities of the generalized Bol-Moufang type have already been classified [13].

5. Non-commutative identities

With 1092 identities that have already been classified, there are 123 remaining non-commutative identities of the generalized Bol-Moufang type. Using the automated theorem prover, Prover9, and the finite model builder, Mace4, we eliminated any identity that was equivalent to another, finding the following 28 Perfect varieties. The first identity listed is used in future structural analysis and was chosen such that its dual is also in the list of 28 varieties. The equivalencies are as follows, with several notable counterexamples and proofs:

Theorem 5.1. *$A4F2$ is not equivalent to any identity.*

Example 5.2. This is a loop that is of the $A4F2$ variety but is not of the $A2C3$ variety.

*	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	2	4	5	0	6	7	3
2	2	3	0	1	6	7	4	5
3	3	0	6	7	2	4	5	1
4	4	5	1	2	7	3	0	6
5	5	6	7	4	3	0	1	2
6	6	7	3	0	5	1	2	4
7	7	4	5	6	1	2	3	0

Theorem 5.3. *The following identities are not equivalent to any other identities; $A2C3$, $A2F4$, $A4C2$, $C3F4$, $C4F2$.*

Theorem 5.4. *$A4F5$ and $A1C5$ are equivalent.*

Proof. $A1C5 \to A4F5$

Letting $z = 1$ in $A1A4$, $x(x(yz)) = x(x(yz))$ gives $x(xy) = x(xy)$. By contradiction, we assume $A1 \neq A4$. Then, $x(xy) \neq x(xy)$ which is a contradiction so $A1 = A4$. It remains to show $F5 = A4$. Letting $z = 1$ in $((xy)y)z = y)(y(xz))$ gives $(xy)y = y(yx)$.

$$\begin{aligned} &x(x(yz)) = ((yx)x)z \\ &((xy)y)z) = y(y(xz)) && \text{(let } x = y) \\ &(xy)y = y(yx) && \text{(by assumption)} \end{aligned}$$

Similarly, letting $z = 1$ in $((xz)y)y = (y(yx))z$ gives $(xy)y = y(yx)$. By contradiction, assume $F5 \neq A4$.

$$\begin{aligned} &((yz)x)x \neq (x(xy))z \\ &((xz)y)y \neq (y(yx))z && \text{(let } x = y) \\ &(xy)y \neq y(yx) && \text{(by assumption)} \end{aligned}$$

But this is a contradiction so $F5 = A4$. So $A4 = F5 = A1$ and $A1 = C5$ by assumption. Therefore $A1C5 \to A4F5$

$A4F5 \to A1C5$

Letting $z = 1$ in $A4A1$, $(x(xy))z = x(x(yz))$ gives $x(xy) = x(xy)$. By contradiction, we assume $A4 \neq A1$. Then, $x(xy) \neq x(xy)$ which is a contradiction so $A4 = A1$. Since $A4 = A1$, it remains to show that $C5 = A1$. Letting $y = 1$ in $A4F5$, $(x(xy))z = ((yz)x)x$, gives $(xx)z = (zx)x$.

$$\begin{array}{ll} (x(xy))z = ((yz)x)x & \\ (x(xz))y = ((zy)x)x & (\text{let } y = z) \\ x(xz) = (zx)x & (\text{let } y = 1) \\ (xx)z = x(xz) & (\text{by assumption}) \end{array}$$

Similarly, in $C5 = A1$,$((yx)x)z = x(x(yz))$, letting $y = 1$ gives $(xx)z = x(xz)$. By contradiction, we assume $C5 \neq A1$. Then $(xx)z \neq x(xz)$, which is a contradiction. Therefore, $C5 = A1$. So $C5 = A1 = A4$ and $A4 = F5$ by assumption. Therefore $A4F5 \rightarrow A1C5$.

Therefore, because $A4F5 \rightarrow A1C5$ and $A1C5 \rightarrow A4F5$, $A4F5$ and $A1C5$ are equivalent. □

Theorem 5.5. *The following sets of loop varieties are equivalent;*

1. $C4F5$ *and* $C5F3$ *are equivalent.*
2. $A1F2$ *and* $C1F5$ *are equivalent.*
3. $A1C2$ *and* $A3C1$ *are equivalent.*
4. $A3F1$, $A3C2$, *and* $C2F3$ *are equivalent.*
5. $A5F1$, $A5C2$, *and* $C4F1$ *are equivalent.*
6. $A5F3$, $A3C4$, *and* $C4F3$ *are equivalent.*
7. $A5F5$, $A3C5$, *and* $C5F5$ *are equivalent.*
8. $B4C4$, $D2F3$, *and* $E1F1$ *are equivalent.*
9. $B5C4$, $D4F3$, *and* $E2F1$ *are equivalent.*
10. $C1F1$, $A4C4$, *and* $A1F3$ *are equivalent.*
11. $A1F5$, $C1F2$, *and* $A4C5$ *are equivalent.*
12. $A1F1$, $C1F3$, *and* $A1C1$ *are equivalent.*
13. $C2E1$, $A5B4$, *and* $A3D2$ *are equivalent.*
14. $C2E2$, $A5B5$, *and* $A3D4$ *are equivalent.*
15. $A3F3$, $A5C4$, *and* $C2F1$ *are equivalent.*
16. $A5C5$, $A3F5$, *and* $C2F2$ *are equivalent.*
17. $A3F2$, $A5F2$, $C2F5$, $C5F1$, *and* $C5F2$ *are equivalent.*
18. $A4F1$, $A5C1$, $A1C4$, $A4C1$ *and* $A4F3$ *are equivalent.*
19. $A4F4$, $A2C1$, $A2C2$, $A3C3$, $A1C3$, $A2F1$, $A2F2$, $A5F4$, $C3F3$, $C3F5$, $C4F4$, *and* $C5F4$ *are equivalent.*
20. $A4C3$, $A2C4$, $A2C5$, $A5C3$, $A1F4$, $A2F3$, $A2F5$, $A3F4$, $C1F4$, $C2F4$, $C3F1$, *and* $C3F2$ *are equivalent.*

21. *$A5D4$, $A5E1$, $A3E2$, $A5D2$, $A5E2$, $A3E1$, $A3B4$, $A3B5$, $B4C2$, $B5C2$, $B4D2$, $B4D4$, $B5D2$, $B5D4$, $B4E1$, $B4E2$, $B5E1$, $B5E2$, $B4F1$, $B4F3$, $B5F1$, $B5F3$, $C2D2$, $C2D4$, $C4D2$, $C4D4$, $C4E1$, $C4E2$, $D2E1$, $D2E2$, $D4E1$, $D4E2$, $D2F1$, $D4F1$, $E1F3$, and $E2F3$ are equivalent.*

Example 5.6. This is a loop that is of the $A4F5$ variety but is not of the $A4F4$ variety.

*	0	1	2	3	4	5	6	7	8	9
0	0	1	2	3	4	5	6	7	8	9
1	1	0	3	2	5	4	9	8	7	6
2	2	3	0	1	6	8	4	9	5	7
3	3	2	1	0	7	9	8	4	6	5
4	4	5	6	7	0	1	2	3	9	8
5	5	4	8	9	1	0	7	6	2	3
6	6	9	4	8	2	7	0	5	3	1
7	7	8	9	4	3	6	5	0	1	2
8	8	7	5	6	9	2	3	1	0	4
9	9	6	7	5	8	3	1	2	4	0

This loop is not of the $A4F4$ variety since $(1 \cdot (1 \cdot 2)) \cdot 4 \neq (2 \cdot (4 \cdot 1)) \cdot 1$.

This demonstrates that there are 28 varieties of the Perfect type which axiomatize the 123 non-commutative identities. It should be noted that 28 is a perfect number. It should also be noted that 6 (also a perfect number) of these identities are not equivalent to any identity of generalized Bol-Moufang type. Three of these 6, $A2C3$, $A2F4$ and $C3F4$, have already been classified as Cheban I, Cheban II and the dual of Cheban I respectively [2].

6. Varieties of loops of Bol-Moufang, commutative and perfect type

The following are the 14 varieties of loops of Bol-Moufang type, the 6 commutative varieties, and the 28 varieties of the Perfect type.

Varieties of Bol-Moufang type

variety	abbrev.	defining identity	its name	ref.
Groups	GR	$x(yz) = (xy)z$	$A1A2$	[14]
Extra	EL	$x(y(zx)) = ((xy)z)x$	$D1D5$	[14]
Moufang	ML	$(xy)(zx) = (x(yz))x$	$D3D4$	[14]
Left Bol	LB	$x(y(xz)) = (x(yx))z$	$B1B4$	[14]
Right Bol	RB	$y((xz)x) = ((yx)z)x$	$E2E5$	[14]
C-loops	CL	$y(x(xz)) = ((yx)x)z$	$C1C5$	[14]
LC-loops	LC	$(xx)(yz) = (x(xy))z$	$A3A4$	[14]
RC-loops	RC	$y((zx)x) = (yz)(xx)$	$F2F3$	[14]
Left Alternative	LA	$x(xy) = (xx)y$	$A4A5$	[14]
Right Alternative	RA	$y(xx) = (yx)x$	$C4C5$	[14]
Flexible Loops	FL	$x(yx) = (xy)x$	$B4B5$	[14]
Middle Nuclear Square	MN	$y((xx)z) = (y(xx))z$	$C2C4$	[14]
Right Nuclear Square	RN	$y(z(xx)) = (yz)(xx)$	$F1F3$	[14]
Left Nuclear Square	LN	$((xx)y)z = (xx)(yz)$	$A5A3$	[14]

Varieties of commutative Bol-Moufang type

variety	abbrev.	defining identity	its name
Comm. Moufang	CM	$(xy)(xz) = (xx)(zy)$	$B3G3$
Abelian Group	AG	$x(yz) = (yx)z$	$A1B2$
Comm. C-loop	CC	$(y(xy))z = x(y(yz))$	$B4C1$
Comm. Alternative	CA	$((xx)y)z = z(x(yx))$	$A5K1$
Comm. Nuclear square	CN	$((xx)y)z = (xx)(zy)$	$A5G3$
Comm. loops	CP	$((yx)x)z = z(x(yx))$	$C5K1$

New varieties (presented below) are primarily named according to the number of Perfect identities that axiomatize them; i.e. *Lonely* for a single identity, *Mate* for two, and *Triad* for three. The six cancellative identities are named 2*can* as they are cancellative and leave two variables. The *Frute* variety is an acronym of the structural properties of $A4F4$, discussed later. $A4C3$ is named because it implies all of the Bol-Moufang varieties, which some may consider crazy. There are historical references to the name as well. Moldova is known for both Valentin Danilovitsch Belousov, who introduced quasigroup and loop theory to much of Eastern Europe, and a musical artist who goes by "Crazy Loop" [16]. $A5D4$ is named Krypton because the variety is axiomatized by 36 identities of the Perfect type and the atomic number of Krypton is 36.

Varieties of Perfect type

variety	abbrev.	defining identity	its name
Cheban 1	C1	$x((xy)z) = (yx)(xz)$	$A2C3$
Cheban 2	C2	$x((xy)z) = (y(zx))x$	$A2F4$
Lonely I	L1	$(x(xy))z = y((zx)x)$	$A4F2$
Cheban I Dual	CD	$(yx)(xz) = (y(zx))x$	$C3F4$
Lonely II	L2	$(x(xy))z = y((xx)z)$	$A4C2$
Lonely III	L3	$(y(xx))z = y((zx)x)$	$C4F2$
Mate I	M1	$(x(xy))z = ((yz)x)x$	$A4F5$
Mate II	M2	$(y(xx))z = ((yz)x)x$	$C4F5$
Mate III	M3	$x(x(yz)) = y((zx)x)$	$A1F2$
Mate IV	M4	$x(x(yz)) = y((xx)z)$	$A1C2$
Triad I	T1	$(xx)(yz) = y(z(xx))$	$A3F1$
Triad II	T2	$((xx)y)z = y(z(xx))$	$A5F1$
Triad III	T3	$((xx)y)z = (yz)(xx)$	$A5F3$
Triad IV	T4	$((xx)y)z = ((yz)x)x$	$A5F5$
Triad V	T5	$x(x(yz)) = y(z(xx))$	$A1F1$
Triad VI	T6	$(xx)(yz) = (yz)(xx)$	$A3F3$
Triad VII	T7	$((xx)y)z = ((yx)x)z$	$A5C5$
Triad VIII	T8	$(xx)(yz) = y((zx)x)$	$A3F2$
Triad IX	T9	$(x(xy))z = y(z(xx))$	$A4F1$
2can I	2C1	$x(yx) = y(xx)$	$B4C4$
2can II	2C2	$(xy)x = y(xx)$	$B5C4$
2can III	2C3	$x(xz) = z(xx)$	$C1F1$
2can IV	2C4	$x(xz) = (zx)x$	$C1F2$
2can V	2C5	$(xx)z = x(zx)$	$C2E1$
2can VI	2C6	$(xx)z = (xz)x$	$C2E2$
Frute	FR	$(x(xy))z = (y(zx))x$	$A4F4$
Crazy Loop	CR	$(x(xy))z = (yx)(xz)$	$A4C3$
Krypton	KL	$((xx)y)z = (x(yz))x$	$A5D4$

7. Structure of the 28 non-commutative perfect identities

Six Perfect varieties are axiomatized by an identity which is left or right cancellative:

Cancellative identities axiomatizing Perfect varieties

$B4C4$	$(x(yx))z = (y(xx))z$	$x(yx) = y(xx)$
$B5C4$	$((xy)x)z = (y(xx))z$	$(xy)x = y(xx)$
$C1F1$	$y(x(xz)) = y(z(xx))$	$x(xz) = z(xx)$
$C1F2$	$y(x(xz)) = y((zx)x)$	$x(xz) = (zx)x$
$C2E1$	$y((xx)z) = y(x(zx))$	$(xx)z = x(zx)$
$C2E2$	$y((xx)z) = y((xz)x)$	$(xx)z = (xz)x$

For the purpose of this paper, the structures of the six cancellative varieties have not been examined, as they are structurally less interesting.

Cheban I, Cheban II, and the dual of Cheban I, $C3F4$, have not been examined, as they have been already classified [2].

The following chart demonstrates which Perfect varieties imply which Bol-Moufang varieties. **None** of the Bol-Moufang varieties implied the Perfect varieties.

Bol-Moufang varieties implied by Perfect varieties

Groups	($\Leftarrow$) A4C3
Extra	($\Leftarrow$) A4C3
Moufang	($\Leftarrow$) A4C3, A4F4
Left Bol	($\Leftarrow$) A4C3, A4F4
Right Bol	($\Leftarrow$) A4C3, A4F4
C-loops	($\Leftarrow$) A4C3
LC-loops	($\Leftarrow$) A4C3, A4F1, A4F5
RC-loops	($\Leftarrow$) A4C3, A1F2, A3F2
L. Alt.	($\Leftarrow$) A4C3, A3C1, A4F1, A4F4, A4F5
R. Alt.	($\Leftarrow$) A4C3, A1F2, A3F2, A4F4, C4F5
Flexible	($\Leftarrow$) A4C3, A4F4, A5D4
L. Nuclear	($\Leftarrow$) A4C3, A3F2, A4F1, A4F5, A5D4, A5F3, A5F5, C4F5
M. Nuclear	($\Leftarrow$) A4C3, A3F2, A4C2, A4F1, A4F2, A4F5, A5D4, A5F1, C4F2
R. Nuclear	($\Leftarrow$) A4C3, A1F2, A3C1, A3F1, A3F2, A4F1, A5D4, C1F3
3-Power	($\Leftarrow$) A4C3, A1F2, A3C1, A3F1, A3F2, A3F3, A4F1, A4F4, A4F5, A5D4, A5F1, A5F3, B5C4, C1F2, C2E1, C4F5

Theorem 7.1. *$A4C3$ implies groups.*

Proof. Letting $y = y/x$ and $z = 1$ in $A4C3$ gives $x(x(y/x)) = ((y/x))x)x$. Since $y = (y/x)x$, $x(x(y/x)) = yx$. Furthermore, letting $x = y$ and $z = 1$ in $A4C3$ gives $(xy)y = y(yx)$. We prove by contradiction, assuming group, $(xy)z = x(yz)$, is not true.

$$(xy)z \neq x(yz)$$
$$(yx)z \neq y(xz) \qquad \text{(let } y = x)$$
$$(((y/x)x)x)z \neq ((y/x)x)(xz) \qquad \text{(let } y = (y/x)x)$$

$$\begin{array}{ll}
((tx)x)z \neq (tx)(xz) & (\text{let } t=(y/x)) \\
(x(xt))z \neq (tx)(xz) & (\text{by assumption}) \\
(x(x(y/x))z \neq ((y/x)x)(xz) & (\text{let } t=(y/x)) \\
(x(x(y/x))z \neq y(xz) & (\text{by assumption}) \\
x(x(y/x)) \neq yx & (\text{let } z=1)
\end{array}$$

This is a contradiction. Thus, $A4C3$ implies $(xy)z = x(yz)$, all groups. □

Example 7.2. This is a loop that is of the $C4F2$ variety but is not an extra loop because $1 \cdot (2 \cdot (3 \cdot 1)) \neq ((1 \cdot 2) \cdot 3) \cdot 1$.

*	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	0	3	4	5	2
2	2	4	0	5	3	1
3	3	5	1	0	2	4
4	4	2	5	1	0	3
5	5	3	4	2	1	0

In addition to the implications, some Perfect varieties satisfied other structural properties. Recall the following definitions:

A loop is *right conjugacy closed* (RCC-loop) if it satisfies $z(yx) = ((zy)/z)(zx)$. A loop is *left conjugacy closed* (LCC-loop) if it satisfies $(xy)z = (xz)(z\backslash(yz))$. A loop is *conjugacy closed* if it is both RCC and LCC. A loop is *Osborn* if it satisfies $x((yz)x) = (x\backslash y)(zx)$. The *center* of a loop L, $Z(L)$, is the set such that $y \in Z(L)$ implies $xy = yx$ for all $x \in L$. A loop L is *nilpotent of class 2* if $L/Z(L)$ is abelian.

Using Prover9, we found that these are the only Perfect varieties to satisfy such conditions:

Theorem 7.3. *$A4C3$ is conjugacy closed and Osborn, $A4F4$ is Osborn, $A4C3$ is nilpotent of class 2.*

$A4F4$ contains the most interesting structural properties of any variety of Perfect type. We call $A4F4$ the *Frute* variety since $A4F4$ is *F*lexible, *R*ight bol and left bol, *U*nity of R. Alt and L. Alt, *T*hree-power associative and *E*ntails both Osborn and Moufang properties. Loops of the Frute variety will be examined further by the authors of this paper and our advisor Dr. J.D. Phillips in a future paper.

Historical Remarks. The classification of varieties of loops of the Bol-Moufang type was initiated by Fenyves and continued by Phillips and Vojtěchovský [6], [7], [13], [14]. In classifying loops of generalized Bol-Moufang type, we partially respond to Drápal and Jedlička's call to classify all varieties of loops that include all quasigroup binary operations, *, /, and \ of generalized Bol-Moufang type [5].

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