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Transversals in loops. 1. Elementary properties

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Devoted to the memory of Valentin D. Belousov (1925-1988)

Abstract. A new notion of a transversal in a loop to its subloop is introduced and studied. This notion generalized a well-known notion of a transversal in a group to its subgroup and can be correctly defined only in the case, when some specific condition (condition A) for a loop and its subloop is fulfilled. Elementary properties of the transversals in a loop to its subloop are investigated and proved. With the help of the notion of transversal in a loop to its subloop a new notion of permutational representation of a loop by left (right) cosets to its subloop is introduced and studied.

1. Introduction

In group theory, in group representation theory and in quasigroup theory the following notion is well-known – the notion of a left (right) transversal in a group to its subgroup [1, 5, 6, 10].

Definition 1.1. Let G be a group and H be a subgroup in G. A complete system $T = \{t_i\}_{i \in E}$ of representatives of the left (right) cosets of H in G $(e = t_1 \in H)$ is called a *left (right) transversal in G to H*.

In the present work a variant of natural generalization of the notion of transversal at the class of loops is proposed and studied. As the elements of a left (right) transversal in a group to its subgroup are the representatives of every left (right) coset to the subgroup, then a notion of a left (right) transversal in a loop to its subloop can be correctly defined only in a case when this loop admits a left (right) coset decomposition by its subloop (see [11] and the Condition A below).

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In the part 2 of this article we start studying a class of loops which admits a left (right) coset decomposition by its subloop (admits the left (right) condition A). Elementary properties of those loops are proved. One of these properties (for finite loops) is an analogue of Lagrange theorem for groups.

In the part 3 of this article at the investigated class of loops we introduce the notion of left (right) transversals to its subloops. Some elementary properties of the transversals are investigated and proved.

In the part 4 of this article at this class of loops we introduce and study a notion of a permutational representation of loop by the left (right) cosets to its subloop. Elementary properties of this new notion are proved. Also we will prove an equivalence of this notion and a notion of permutation loop from [3].

Further we shall use the following notations:

$\langle L, \cdot, e\rangle$ is an initial loop with the unit e;

$\langle R, \cdot, e\rangle$ is its proper subloop;

E is a set of indexes ($1 \in E$) of the left (right) cosets R_i in L to R (assume $R_1 = R$).

2. Preliminaries

Definition 2.1. The system $\langle E, \cdot\rangle$ is called [2] a *right* (*left*) *quasigroup* if for arbitrary $a, b \in E$ the equation $x \cdot a = b$ ($a \cdot y = b$) has a unique solution in E. If $\langle E, \cdot\rangle$ is both a right and left quasigroup, then it is called a *quasigroup*. If in a right (left) quasigroup $\langle E, \cdot\rangle$ there exists an element $e \in E$ such that

$$x \cdot e = e \cdot x = x$$

for every $x \in E$, then $\langle E, \cdot\rangle$ is called a *right* (*left*) *loop* (the element e is called a *unit* or an *identity element*). If $\langle E, \cdot\rangle$ is both a right and left loop, then it is called a *loop*.

Definition 2.2. Let $\langle L, \cdot\rangle$ be a loop and $\langle R, \cdot\rangle$ be its proper subloop. Then a *left coset* of R is a set of the form

$$xR = \{xr \mid r \in R\},$$

and a *right coset* has the form

$$Rx = \{rx \mid r \in R\}.$$

The cosets in a loop to its subloop do not necessarily form a partition of the loop. This leads us to the following definition.

Definition 2.3. A loop L has a *left* (*right*) *coset decomposition by its proper subloop* R, if the left (right) cosets form a partition of the loop L, is equal for some set of indexes E

1. $\bigcup_{i\in E}(a_iR) = L$;
2. for every $i, j \in E$, $i \neq j$ $\quad (a_iR) \cap (a_jR) = \varnothing$.

In order to define correctly a notion of a left (right) transversal in a loop to its proper subloop, it is necessary that the following condition be fulfilled.

Definition 2.4 (see [9]). (*Left Condition A*) Let R be a subloop of a loop L. For all $a, b \in L$ there exists $c \in L$ such that

$$a(bR) = cR. \tag{1}$$

The *right condition A* is defined analogously.

In [11] the following theorem was proved.

Lemma 2.5. *The following conditions are equivalent:*

1. *A loop L has a left cosets decomposition by its proper subloop R.*
2. *The following condition takes place (it can be named the weak left condition A): for every $a \in L$*

$$(aR)R = aR. \tag{2}$$

Proof. See in [11], Theorem I.2.12. □

Below we shall prove all statements only for a case of the left cosets (if the *left condition A* take place); in a case of the right cosets all proofs are similar.

Lemma 2.6. *Let the left condition A in a loop L to its subloop R be satisfied. Then*

$$(a \cdot R) \cdot R = a \cdot R \tag{3}$$

for all $a \in L$.

Proof. By the left condition A for all $a, b \in L$ there exists an element $c = c(a, b) \in L$ such that $a \cdot (b \cdot R) = c \cdot R$. In the loop L always it is possible to find an element $d = d(a, b)$ such that $c = a \cdot d$. Then

$$a \cdot (b \cdot R) = (a \cdot d) \cdot R. \tag{4}$$

So, for some $r_1 \in R$ we have $a \cdot (b \cdot r_1) = (a \cdot d) \cdot e = a \cdot d$. Thus, $b \cdot r_1 = d$, i.e., $d \in b \cdot R$. Therefore, $b \in R$ implies $d \in R$. Hence, for $b \in R$ from (4) it follows $a \cdot R = (a \cdot R) \cdot R$. The Lemma is proved. □

Lemma 2.7. *The following conditions are equivalent:*

1. *The left condition A is fulfilled in the loop L to its subloop R.*
2. *For every $a, b \in L$*

$$a \cdot (b \cdot R) = (a \cdot b) \cdot R. \tag{5}$$

Proof. $1 \Rightarrow 2$. Let the left condition A holds. Then for all $a, b \in L$ and all $r \in R$ there exist $c = c(a, b) \in L$ and $r_1 \in R$ such that $a \cdot (b \cdot r) = c \cdot r_1$. If $r = e$, then $a \cdot b = c \cdot r_1' \in c \cdot R$. Hence, according to Lemma 2.6,

$$(a \cdot b) \cdot R = (c \cdot R) \cdot R = c \cdot R,$$

which proves 2.

$2 \Rightarrow 1$. It is evident. □

Let us define (see [12]) for all $a, b \in L$ the *left inner mapping*

$$l_{a,b}(x) = (a \cdot b) \setminus (a \cdot (b \cdot x)), \qquad x \in L, \tag{6}$$

where "$\setminus$" is a left division in the loop $\langle L, \cdot, e \rangle$, and the *right inner mapping*

$$r_{a,b}(x) = ((x \cdot b) \cdot a) / (b \cdot a), \qquad x \in L, \tag{7}$$

where "$/$" is a right division in the loop $\langle L, \cdot, e \rangle$.

Lemma 2.8. *Let the left condition A in a loop L to its subloop R be satisfied. Then $l_{a,b}(R) = R$ for all $a, b \in L$.*

Proof. The proof is an evident corollary of Lemma 2.7. □

Lemma 2.9. *Let the right condition A in a loop L to its subloop R be satisfied. Then $r_{a,b}(R) = R$ for all $a, b \in L$.*

Proof. The proof is similar to the proof of a Lemma 2.8. □

Remark 2.10. It is known (see [12]) that the mappings $l_{a,b}$ generate the *left inner mappings group* $LI(\langle L, \cdot, e\rangle)$ of a loop L, and the mapings $r_{a,b}$ generate the *right inner mappings group* $RI(\langle L, \cdot, e\rangle)$ of a loop L. Therefore, if the left (right) condition A in a loop L to its subloop R is fulfilled, then the investigated class of loops satisfies a condition of an invariance of a subloop R relating to an action of the group $LI(\langle L, \cdot, e\rangle)$ (group $RI(\langle L, \cdot, e\rangle)$, respectively). So we can say that the subloop R is a *left* (*right*) *invariant subloop* of the loop L.

Remark 2.11. The condition (5) is called in [4] a *strong left coset decomposition of the loop L by its proper subloop R.*

Lemma 2.12. *Let the left condition A for a loop L and its subloop R is fulfilled. Then the following conditions hold:*

1. *Left cosets R_i form a left coset decomposition of the loop L;*
2. *If a loop L is finite, then the "Lagrange property" takes place: an order of the subloop R divides an order of the loop L.*

Proof. (see also [11]) 1. Let $R_i = aR$, $R_j = bR$. Assume that these cosets have a common element $c \in L$, i.e.,

$$c \in R_i \cap R_j = (aR) \cap (bR).$$

Then $c = a \cdot r_1 = b \cdot r_2$ for some $r_1, r_2 \in R$. So, $(a \cdot r_1) \cdot r = (b \cdot r_2) \cdot r$ for every $r \in R$. Let us show there exists an element $r_0 \in R$ such that

$$(a \cdot r_1) \cdot r_0 = a.$$

Indeed, if the left condition A for the loop L and its subloop R is fulfilled, then a subloop R is a left invariant subloop in the loop L. Hence $\forall a, b \in L$: $l_{a,b}(R) = R$. Let us take $r_0 = l_{a,r_1}(r_1 \backslash e)$. Then

$$r_0 = (a \cdot r_1) \backslash (a \cdot (r_1 \cdot (r_1 \backslash e))) = (a \cdot r_1) \backslash (a \cdot e) = (a \cdot r_1) \backslash a,$$

i.e., $(a \cdot r_1) \cdot r_0 = a$. So, by Lemma 2.6, we obtain

$$a = (a \cdot r_1) \cdot r_0 = (b \cdot r_2) \cdot r_0 = b \cdot r_2' \in b \cdot R.$$

Thus $a \cdot R = (b \cdot R) \cdot R = b \cdot R$. So, if $a \cdot R \neq b \cdot R$, then $(a \cdot R) \cap (b \cdot R) = \varnothing$.

Since $c \in (c \cdot R)$, for any element $c \in L$, we have $\bigcup\limits_{c \in L} (c \cdot R) = L$. So, left cosets R_i form a left coset decomposition of the loop L.

2. Let L be finite. Let us show that the number of elements in any left coset R_i is equal to the number of elements in R. Because L is a loop then

$$r_1 \neq r_2 \quad \Leftrightarrow \quad a \cdot r_1 \neq a \cdot r_2 \quad \forall r_1, r_2 \in R.$$

So, the left translation $L_a(r) = a \cdot r$ is an injection. Since L is finite, then the translation L_a is a surjection, i.e., it is a bijection. So, R and $a \cdot R$ have the same order for any $a \in R$.

Then, by 1, we have $L = \bigcup\limits_{c \in L} (c \cdot R)$, and consequently

$$|L| = \sum_{c_i \in L} |c_i \cdot R| = m \cdot |R|.$$

The Lemma is completely proved. □

Now we give two examples of loops and its proper subloops, where the left condition A is fulfilled.

Example 2.13. A loop L and its normal subloop R.

It is well known (see [2]), that if a subloop R is normal in a loop L, then an action of the left and right inner permutations $l_{a,b}$ and $r_{a,b}$ is an invariant relation $\forall a, b \in L$. Therefore both left and right conditions A are fulfilled in this case.

Example 2.14. A loop of pairs $L = \langle E \times E \backslash \{\Delta\}, *, \langle 0, 1 \rangle \rangle$ of an arbitrary DK-ternar $\langle E, (x, t, y), 0, 1 \rangle$ and its subloop $R = \{\langle 0, x \rangle \, | x \in E \backslash \{0\}\}$.

As it is known (see [7]), in a loop of pairs $L = \langle E \times E \backslash \{\Delta\}, *, \langle 0, 1 \rangle \rangle$ the operation "$*$" is defined through the ternary operation (x, t, y) of the DK-ternar $\langle E, (x, t, y), 0, 1 \rangle$ by the following way:

$$\langle x, y \rangle * \langle u, v \rangle \overset{def}{=} \langle (x, u, y), (x, v, y) \rangle.$$

The elements $\langle 0, x \rangle$ (where $x \in E \backslash \{0\}$) form a subloop R with the operation "$*$". Then for $a = \langle x, y \rangle \in L$, $b = \langle u, v \rangle \in L$ and $r = \langle 0, z \rangle \in R$ we have:

$$\begin{aligned} a * (b * r) &= \langle x, y \rangle * (\langle u, v \rangle * \langle 0, z \rangle) = \langle x, y \rangle * \langle u, (u, z, v) \rangle \\ &= \langle (x, u, y), (x, (u, z, v), y) \rangle = \langle \alpha_{x,y}(u), \alpha_{x,y} \alpha_{u,v}(z) \rangle. \end{aligned}$$

On the other hand, for $r_1 = \langle 0, z_1 \rangle$, we have:

$$\begin{aligned}(a * b) * r_1 &= (\langle x, y\rangle * \langle u, v\rangle) * \langle 0, z_1\rangle = \langle (x, u, y), (x, v, y)\rangle * \langle 0, z_1\rangle \\ &= \langle (x, u, y), ((x, u, y), z_1, (x, v, y))\rangle \\ &= \left\langle \alpha_{x,y}(u), \alpha_{\alpha_{x,y}(u),\alpha_{x,y}(v)}(z_1)\right\rangle .\end{aligned}$$

If elements $x, y, u, v \in E$ are given, then for every $z \in E\backslash\{0\}$ there exists $z_1 \in E\backslash\{0\}$ such that

$$\alpha_{x,y}\alpha_{u,v}(z) = \alpha_{\alpha_{x,y}(u),\alpha_{x,y}(v)}(z_1),$$

namely,

$$z_1 = \alpha^{-1}_{\alpha_{x,y}(u),\alpha_{x,y}(v)}\alpha_{x,y}\alpha_{u,v}(z).$$

Thus $a * (b * R) = (a * b) * R$. Hence the left condition A is fulfilled.

3. A transversal in a loop to its subloop.

Definition 3.1 (see [9]). Let $\langle R, \cdot, e\rangle$ be a subloop of the loop $\langle L, \cdot, e\rangle$ and let the left (right) condition A be satisfied. If $\{R_x\}_{x\in E}$ is the set of all left (right) cosets on L determined by R, then the set $T = \{t_x\}_{x\in E} \subset L$ is called the *left* (*right*) *transversal* in L if for every $x \in E$ there exists a unique element $t_x \in T$ such that $t_x \in R_x$. If $T = \{t_x\}_{x\in E}$ is both left and right transversal in L simultaneously, then it is called the *two-sided transversal*.

Remark 3.2. Analogously as in groups we assume that $t_1 = e$. If this assumption is not fulfilled then we have the so-called *non-reducible left* (*right*) *transversals*.

On E we define the following *transversal operations*:

$$x \overset{(T)}{\cdot} y = z \overset{def}{\Leftrightarrow} t_x \cdot t_y = t_z \cdot r, \tag{8}$$

where $t_x, t_y, t_z \in T$ are left transversals L to R and $r \in R$,

$$x \overset{(T)}{\circ} y = z \quad \overset{def}{\Leftrightarrow} \quad t_x \cdot t_y = r \cdot t_z, \tag{9}$$

where $t_x, t_y, t_z \in T$ are right transversals L to R.

Also we can define the operation on the set of left transversal by putting

$$t_x \overset{(T)}{\cdot} t_y = t_z \quad \overset{def}{\Leftrightarrow} \quad t_x \cdot t_y = t_z \cdot r \tag{10}$$

for $t_x, t_y, t_z \in T$ and $r \in R$. Similarly for the right transversal.

Lemma 3.3. $\langle E, \overset{(T)}{\cdot}, 1\rangle$ *is isomorphic to* $\langle T, \overset{(T)}{\cdot}, t_1\rangle$.

Proof. The proof follows easily from (8) and (10). The isomorphism has the form $\varphi : E \to T$, $\varphi(x) = t_x$. □

Lemma 3.4. $\langle E, \overset{(T)}{\cdot}, 1\rangle$ *is a left loop with the two-sided unit* 1.

Proof. Since $t_1 = e \in R$, for ever $x \in E$ we have

$$x \overset{(T)}{\cdot} 1 = u \Leftrightarrow t_x \cdot e = t_u \cdot r \Leftrightarrow t_x = t_u \cdot r_1 \Leftrightarrow t_x \in t_u \cdot R \Leftrightarrow u = x.$$

Hence $x \overset{(T)}{\cdot} 1 = x$. On the other sided

$$1 \overset{(T)}{\cdot} x = v \Leftrightarrow e \cdot t_x = t_v \cdot r \Leftrightarrow t_x = t_v \cdot r_1 \Leftrightarrow t_x \in t_v \cdot R \Leftrightarrow v = x.$$

Thus $1 \overset{(T)}{\cdot} x = x$. So, $1 \in E$ is a two-sided unit in $\langle E, \overset{(T)}{\cdot}, 1\rangle$.

Let $a \overset{(T)}{\cdot} x = b$ for some $a, b \in E$. Then $t_a \cdot t_x = t_b \cdot r$. Hence

$$t_x = t_a \backslash (t_b \cdot r) = t_c \cdot r' \quad \text{for some } c \in E \;\; \Leftrightarrow \;\; x = c.$$

So, there exists an element $c \in E$ such that $a \overset{(T)}{\cdot} c = b$. This means that the equation $a \overset{(T)}{\cdot} x = b$ has a solution. If this solution is not uniquely determined, then $a \overset{(T)}{\cdot} x_1 = b = a \overset{(T)}{\cdot} x_2$ for some $x_1, x_2 \in E$, $x_1 \neq x_2$. Then

$$\begin{cases} t_a \cdot t_{x_1} = t_b \cdot r_1, \\ t_a \cdot t_{x_2} = t_b \cdot r_2. \end{cases}$$

Hence, by Lemmas 2.6 and 2.7 we obtain

$$t_a \cdot (t_{x_1} R) = (t_a \cdot t_{x_1}) \cdot R = (t_b \cdot r_1) \cdot R = t_b R,$$
$$t_a \cdot (t_{x_2} R) = (t_a \cdot t_{x_2}) \cdot R = (t_b \cdot r_2) \cdot R = t_b R.$$

So, for every $r' \in R$ there exists $r'' \in R$ such that

$$t_a \cdot \left(t_{x_1} \cdot r'\right) = t_b \cdot r^* = t_a \cdot \left(t_{x_2} \cdot r''\right).$$

This implies $t_{x_1} \cdot r' = t_{x_2} \cdot r''$, and consequently $x_1 = x_2$, which is a contradiction. So, $\langle E, \overset{(T)}{\cdot}, 1\rangle$ is a left loop. □

In the same way we can prove

Lemma 3.5. $\langle E, \overset{(T)}{\circ}, 1\rangle$ *is a right loop with the two-sided unit* 1. □

If $\langle E, \overset{(T)}{\cdot}, 1\rangle$ (resp. $\langle E, \overset{(T)}{\circ}, 1\rangle$) is a loop, then the transversal T is called a *left* (*right*) *loop transversal* in L to R.

4. Representation of loops by cosets

Let $\langle R,\cdot,e\rangle$ be a subloop of the loop $\langle L,\cdot,e\rangle$ and let the left condition A be satisfied in $\langle L,\cdot,e\rangle$. Using the left transversal L to R we define the *left action* of L on E as the map $f : L\times E\to E$, $(g,x)\to y=\hat{g}\,(x)$ such that

$$\hat{g}\,(x)=y\ \overset{def}{\Leftrightarrow}\ g\cdot(t_x\cdot R)=t_y\cdot R. \tag{11}$$

Lemma 4.1. *$\hat{g}$ is a permutation on E.*

Proof. Let g be an arbitrary element of L. Then for every $y\in E$, every $r'\in R$ and some $x\in E$ we have

$$g\backslash\left(t_y\cdot r'\right)=g'\in t_x\cdot R.$$

So, $g\cdot(t_x\cdot R)=t_y\cdot R$, i.e., $\hat{g}\,(x)=y$. Hence $\hat{g}$ is a surjective map.

Now, if $\hat{g}\,(x_1)=y=\hat{g}\,(x_2)$ for some $x_1,x_2\in E$, then, according to (11), we have:

$$g\cdot(t_{x_1}\cdot R)=g\cdot(t_{x_2}\cdot R)\,.$$

Hence, for every $r_1\in R$ there exists $r_2\in R$ such that

$$g\cdot(t_{x_1}\cdot r_1)=g\cdot(t_{x_2}\cdot r_2)\,.$$

Thus, $t_{x_1}\cdot r_1=t_{x_2}\cdot r_2$, which implies $t_{x_1}\cdot R=t_{x_2}\cdot R$, and consequently $x_1=x_2$. Therefore $\hat{g}$ is a permutation on E. □

In this way we obtain a permutation representation of a loop $\langle L,\cdot,e\rangle$ by $\varphi : L\to\hat{L}\subset S_E$, where $\varphi : g\to\hat{g}$. The multiplication of permutations from $\hat{L}$ is defined by

$$\hat{g}_1*\hat{g}_2=\hat{g}_3\quad\overset{def}{\Leftrightarrow}\quad g_1\cdot g_2=g_3\quad\text{in a loop }\langle L,\cdot,e\rangle.$$

Since $\varphi(g_1)*\varphi(g_2)=\hat{g}_1*\hat{g}_2=\hat{g}_3=\widehat{g_1\cdot g_2}=\varphi(g_1\cdot g_2)$, φ is a homomorphism from $\langle L,\cdot,e\rangle$ to $\left\langle\hat{L},*,id\right\rangle$.

Lemma 4.2. *The kernel of the homomorphism φ is a subloop R^* of a loop L such that $R^*\subseteq R$ and*

$$R^*=\bigcap_{u\in L}R_u^{-1}L_u(R).$$

Proof. The kernel of this homomorphism is the set

$$R^* = \{g \in L |\, \hat{g}(x) = x \quad \forall\, x \in E\}.$$

By Lemmas 2.6 and 2.7 for every $x \in E$ we have

$$\hat{g}(x) = x \;\Leftrightarrow\; g \cdot (t_x \cdot R) = t_x \cdot R \;\;\Leftrightarrow\; g \cdot ((t_x \cdot r) \cdot R) = (t_x \cdot r) \cdot R.$$

Thus

$$\hat{g}(x) = x \; \forall x \in E \;\Leftrightarrow\; g \cdot (u \cdot R) = u \cdot R \; \forall u \in L \;\Leftrightarrow\; (g \cdot u) \cdot R = u \cdot R \; \forall u \in L.$$

The last is equivalent to the fact that $g \in (u \cdot R)/u \; \forall u \in L$, i.e., $g \in R_u^{-1} L_u(R) \; \forall u \in L$. Hence $R^* = \bigcap\limits_{u \in L} R_u^{-1} L_u(R)$.

For $u = e$ we have $g \in R$. Thus $R^* \subseteq R$. □

Obviously, R^* is a normal subloop of L and has the form

$$R^* = \{r \in R \,|\, L_u^{-1} R_u(r) \in R \quad \forall u \in L\}.$$

Further R^* will be denoted as $Core_L(R)$ and will be called the *core* of R in L.

Lemma 4.3. *The following statements are true:*

1) *$Core_L(R)$ is a maximal subloop among the all normal subloops of L contained in R.*
2) *Let $L' = L/Core_{L(R)}$. If $T = \{t_x\}_{x \in E}$ is a left transversal in L to R and $\psi : L \to L'$ is a natural homomorphism, then:*
   a) The set $T' = \{\psi(t_x) | x \in E\}$ is a left transversal in L' to $R' = \psi(R) = R/Core_L(R)$;
   b) $\langle E, \overset{(T')}{\cdot}, 1\rangle \equiv \langle E, \overset{(T)}{\cdot}, 1\rangle$.
3) *$Core_{L'}(R') = \{e\}$.*

Proof. 1) Let N be any normal subloop of L contained in R. Since N is normal, it is invariant by any middle inner permutation of the loop L, i.e., $L_u^{-1} R_u(N) = N$ for all $u \in L$. Then $R_u^{-1} L_u(N) = N$ for every $u \in L$.

Since $N \subseteq R$, for all $u \in L$ we have $N = R_u^{-1} L_u(N) \subseteq R_u^{-1} L_u(R)$, and consequently

$$N = \bigcap_{u \in L} N = \bigcap_{u \in L} R_u^{-1} L_u(N) \subseteq R_u^{-1} L_u(R) = R^*.$$

2) Let $T = \{t_x\}_{x\in E}$ be a left transversal in L to R and

$$\psi: \; L \to L' = L\backslash Core_{L(R)}$$

be a natural homomorphism. Let us denote:

$$R' = \psi(R), \quad t'_x = \psi(t_x) \quad \forall x \in E.$$

a) Let us show that $T' = \{\psi(t_x)|x \in E\}$ is a left transversal in a loop L' to its subloop R'. Firstly, because $a\cdot(b\cdot R) = (a\cdot b)\cdot R$ for all $a, b \in L$, then $\psi(a\cdot(b\cdot R)) = \psi((a\cdot b)\cdot R)$, i.e., $\psi(a)\cdot(\psi(b)\cdot\psi(R)) = (\psi(a)\cdot\psi(b))\cdot\psi(R)$. Thus $a'\cdot(b'\cdot R') = (a'\cdot b')\cdot R'$ for all $a', b' \in L'$, which shows that the left condition A is fulfilled for a loop L' and its subloop R'.

Secondly, for every $g' \in L'$ there exists $g \in L$ such that $g' = \psi(g)$. Since for any $g \in L$ we have a representation $g = t_u \cdot r$, $t_u \in T$, $r \in R$, we obtain

$$g' = \psi(g) = \psi(t_u \cdot r) = \psi(t_u) * \psi(r) = t'_u * r',$$

where $t'_u \in T'$, $r' \in R'$. This means that each $g' \in L'$ may be represented in the form $g' = t'_u \cdot r'$, where $t'_u \in T'$, $r' \in R'$.

Finally, let $t'_y = t'_x * r'_1$ for some $x, y \in E$ and $r'_1 \in R'$. Then, for $r'_1 = \psi(r_1)$ we have $\psi(t_y) = \psi(t_x) * \psi(r_1) = \psi(t_x \cdot r_1)$. From this we obtain $t_y \cdot Core_L(R) = (t_x \cdot r_1)\cdot Core_L(R)$.

Since $R^* = Core_L(R) \subseteq R$, then $t_y \cdot r_1^* = (t_x \cdot r_1)\cdot r_2^*$, where r_1^*, r_2^* are in $R^* \subseteq R$. Thus

$$t_y \cdot R = (t_y \cdot r_1^*)\cdot R = ((t_x \cdot r_1)\cdot r_2^*)\cdot R = (t_x \cdot r_1)\cdot R = t_x \cdot R.$$

So $x = y$, since T is a left transversal in L to R. Therefore T' is a left transversal in L' on R'.

b) We have

$$x \overset{(T)}{\cdot} y = z \;\Leftrightarrow\; t_x \cdot t_y = t_z \cdot r \text{ (where } t_x, t_y, t_z \in T,\; r \in R) \;\Leftrightarrow$$
$$\psi(t_x \cdot t_y) = \psi(t_z \cdot r) \;\Leftrightarrow\; \psi(t_x) * \psi(t_y) = \psi(t_z) * \psi(r) \;\Leftrightarrow$$
$$t'_x \cdot t'_y = t'_z \cdot r' \text{ (where } t'_x, t'_y, t'_z \in T',\; r' \in R') \;\Leftrightarrow\; x \overset{(T')}{\cdot} y = z.$$

Thus $x \overset{(T)}{\cdot} y = z = x \overset{(T')}{\cdot} y$. So, $\langle E, \overset{(T)}{\circ}, 1\rangle$ and $\langle E, \overset{(T')}{\circ}, 1\rangle$ are isomorphic.

3) Let $Core_{L'}(R') = M_0 \neq \{e\}$. Since M_0 is a normal subloop of L', the preimage

$$M_1 = \psi^{-1}(M_0) = \{g \in L \,|\, \psi(g) \in M_0\}$$

is a subloop in L. Further,

$$\begin{array}{rcl} e \in M_0 & \Rightarrow & Core_L(R) = Ker\,\psi = \psi^{-1}(e) \subset \psi^{-1}(M_0) = M_1, \\ M_0 \subseteq R' & \Rightarrow & M_1 = \psi^{-1}(M_0) \subseteq \psi^{-1}(R') = R. \end{array}$$

Since a homomorphism ψ transforms any inner permutation from L to an inner permutation from L', then M_1 should be a normal subloop in L. So, $M_1 \subset R$ and $Core_L(R) \subset M_1$. This contradicts to the previous condition of this Lemma. □

Remark 4.4. According to the above lemma, the study of left transversals in loops may be reduced to the case, when $Core_L(H) = \{e\}$. In this case $\langle E, *, id\rangle \equiv \hat{L} \cong L = \langle E, \cdot, e\rangle$.

In the case when $\langle R, \cdot, e\rangle$ is as subloop of $\langle L, \cdot, e\rangle$ and the right condition A is satisfied we obtain analogical results. Namely, if $T = \{t_x\}_{x\in E}$ is a right transversal in L to R, then $f : L \times E \to E$, $f : (g, x) \to y = \check{g}(x)$ defined by

$$\check{g}(x) = y \overset{def}{\Leftrightarrow} (R \cdot t_x) \cdot g = R \cdot t_y.$$

is a right action of L on E. Consequently, the following lemmas are true.

Lemma 4.5. *$\check{g}$ is a permutation on E.* □

So, $\varphi' : L \to \check{L} \subset S_E$, $\varphi' :\ g \to \check{g}$ is another permutation representation of a loop L.

Lemma 4.6. *The kernel $R^{\circledast}$ of the homomorphism φ' is a subloop L such that $R^{\circledast} \subseteq R$ and $R^{\circledast} = \bigcap_{u\in L} L_u^{-1}R_u(R)$.* □

Lemma 4.7. *The following statements are true:*

1) *$R^{\circledast}$ is a maximal subloop among the all normal subloops of the loop L contained in R.*
2) *Let $L'' = L/R^{\circledast}$. If $T = \{t_x\}_{x\in E}$ is a right transversal in L to R and $\psi : L \to L''$ is a natural homomorphism, then:*
   a) $T'' = \{\psi(t_x) | x \in E\}$ is a right transversal in L'' to $R'' = \psi(R) = R/R^{\circledast}$;
   b) $\langle E, \overset{(T'')}{\cdot}, 1\rangle \equiv \langle E, \overset{(T)}{\cdot}, 1\rangle$.
3) $\bigcap_{u\in L''} L_u^{-1}R_u(R'') = \{e\}$. □

Remark 4.8. According to the last Lemma a research of right transversals in loops may be reduced to a case when $\bigcap\limits_{u\in L''} L_u^{-1}R_u(R'') = \{e\}$. In this case
$$\langle \breve{L}, *, id\rangle \equiv \hat{L} \cong L = \langle L, \cdot, e\rangle\,.$$

Lemma 4.9. *If $T = \{t_x\}_{x\in E}$ is a two-sided transversal in a loop L to its subloop R and two-sided conditions A is satisfied, then*

$$R^{\circledast} = \bigcap_{u\in L} L_u^{-1}R_u(R) = R^* = \bigcap_{u\in L} R_u^{-1}L_u(R) = Core_L(R).$$

Proof. It is a consequence of Lemmas 4.3 and 4.7. □

Definition 4.10. [3] A loop $\langle L, \cdot, e\rangle$ is called a *permutation loop* on a set E, if there exists a map $f : L\times E \to E$, $f(g, x) = \hat{g}(x)$ satisfying the following conditions:

(1) $\hat{e}(x) = x$ for all $x \in E$, where e is a unit of the loop L,

(2) if $b \in N(\langle L, \cdot, e\rangle)$, where N is a kernel of L, then

$$\widehat{(a\cdot b)}(x) = \hat{a}(\hat{b}(x))$$

for every $a \in L$ and $x \in E$,

(3) there exists an element $x_0 \in E$ such that

$$R_{x_0} \stackrel{def}{=} \{g \in L |\, \hat{g}(x_0) = x_0\}$$

is a subloop of L and the following conditions are fulfilled:

(a) $(\widehat{b\cdot a})(x_0) = \hat{b}(\hat{a}(x_0))$ for $b \in R_{x_0}$ and $a \in L$,

(b) $(\widehat{g_2\cdot g_1})(x_0) \neq \hat{g}_2(x_0)$ for $g_1, g_2 \in L$ and $\hat{g}_1(x_0) \neq x_0$,

(c) $(\widehat{g_2\cdot g_1})(x_0) \neq \hat{g}_1(x_0)$ for $g_2 \notin R_{\hat{g}_1(x_0)}$.

Let us show that a permutational representation $\hat{L}$ defined by (11) satisfies all conditions of Definition 4.10.

Lemma 4.11. *Let the left condition A for a loop $\langle L, \cdot, e\rangle$ to its subloop $\langle R, \cdot, e\rangle$ be satisfied. If a permutation representation $\left\langle \hat{L}, \cdot, \hat{e}\right\rangle$ of the loop L is defined by* (11)*, then $\left\langle \hat{L}, \cdot, \hat{e}\right\rangle$ is a loop of permutations in the sense of Definition* 4.10*.*

Proof. If a representation is defined by (11), then

$$\hat{e}\,(x) = u \;\Leftrightarrow\; e \cdot (t_x \cdot R) = t_u \cdot R \;\Leftrightarrow\; t_x \cdot R = t_u \cdot R \;\Leftrightarrow\; u = x,$$

which shows that in this case $\hat{e}\,(x) = x$ for all $x \in E$. This verifies the first condition of Definition 4.10.

Now, if $b \in N\,(\langle L, \cdot, e\rangle)$, then for $u, v \in L$ we have $(b \cdot u) \cdot v = b \cdot (u \cdot v)$. Thus $(u \cdot v) \cdot b = u \cdot (v \cdot b)$, and consequently $(u \cdot b) \cdot v = u \cdot (b \cdot v)$. This means that for every $a \in L$ and every $x \in E$ we have $\widehat{(a \cdot b)}\,(x) = y$. Therefore $(a \cdot b) \cdot (t_x \cdot R) = t_y \cdot R$, which means that $(a \cdot b) \cdot t_x = t_y \cdot r'$ for some $r' \in R$. But $a \cdot (b \cdot t_x) = t_y \cdot r'$, $b \cdot t_x = t_z \cdot r''$ and $a \cdot (t_z \cdot r'') = t_y \cdot r'$ imply $\hat{b}(x) = z$ and $\hat{a}(z) = y$. Hence $\hat{a}(\hat{b}(x)) = y$. Consequently $\widehat{(a \cdot b)}\,(x) = \hat{a}(\hat{b}(x))$. This verifies the second condition of Definition 4.10.

Now we prove that the third condition of Definition 4.10 is satisfied for $x_0 = 1$. First we prove that for all g_1, $g_2 \in L$ we have

$$(\widehat{g_1 \cdot g_2})(1) = \hat{g}_1(\hat{g}_2(1)). \tag{12}$$

Indeed, by Lemma 2.7, $(\widehat{g_1 \cdot g_2})(1) = u$, i.e., $(g_1 \cdot g_2)(e \cdot R) = t_u \cdot R$. Thus $(g_1 \cdot g_2) \cdot R = t_u \cdot R$. But $g_1 \cdot (g_2 \cdot R) = t_u \cdot R$, $g_2 \cdot R = t_z \cdot R$ and $g_1 \cdot (t_z \cdot R) = t_u \cdot R$ imply $\hat{g}_2(1) = z$ and $\hat{g}_1(z) = u$. Hence $\hat{g}_1(\hat{g}_2(1)) = u$. This completes the proof of (12). From (12) the condition (a) follows automatically.

Further, let g_1, $g_2 \in L$ and $\hat{g}_1(1) = u_0 \neq 1$. Then by (12) we have

$$(\widehat{g_2 \cdot g_1})(1) = \hat{g}_2 \cdot (\hat{g}_1(1)) = \hat{g}_2(u_0) \neq \hat{g}_2(1),$$

since $\hat{g}_2$ is a permutation. This proves (b).

Finally, let

$$g_2 \notin R_{\hat{g}_1(1)} = \{g \in L |\, \hat{g}(\hat{g}_1(1)) = \hat{g}_1(1)\}\,.$$

Then, by (12), we obtain $(\widehat{g_2 \cdot g_1})(1) = \hat{g}_2 \cdot (\hat{g}_1(1)) \neq \hat{g}_1(1)$, since $g_2 \notin R_{\hat{g}_1(1)}$. This proves (c). □

Lemma 4.12. *For an arbitrary left transversal $T = \{t_x\}_{x \in E}$ in a loop $L = \langle L, \cdot, e\rangle$ to its subloop $R = \langle R, \cdot, e\rangle$ the following statements are true:*

1) $\hat{r}(1) = 1$ *for all $r \in R$,*

2) $\hat{t}_x(y) = x \overset{(T)}{\cdot} y$, $\hat{t}_x^{-1}(y) = x \backslash y$ *for all $x, y \in E$,*
where $\hat{t}_x^{-1}$ is an inverse permutation to a permutation $\hat{t}_x$ in S_E, and "$\backslash$" is a left division in a left loop $\langle E, \overset{(T)}{\cdot}, 1\rangle$. Moreover,

$$\hat{t}_x(1) = x, \quad \hat{t}_1(x) = x, \quad \hat{t}_x^{-1}(1) = x\backslash 1, \quad \hat{t}_x^{-1}(x) = 1.$$

Proof. 1) Let $\hat{r}(1) = u$. Then $r \cdot (e \cdot R) = t_u \cdot R$, i.e., $R = t_u \cdot R$. Thus $t_u = e = t_1$. Consequently, $u = 1$. This proves $\hat{r}(1) = 1$.

2) Let $\hat{t}_x(y) = u$. Then $t_x \cdot (t_y \cdot R) = t_u \cdot R$, and consequently

$$t_u \cdot R = (t_x \cdot t_y) \cdot R = (t_{x \cdot y} \cdot r') \cdot R = t_{x \cdot y} \cdot R.$$

Thus $u = x \cdot y$ and $\hat{t}_x(y) = x \cdot y$.

Further,

$$\hat{t}_x^{-1}(y) = z \ \Leftrightarrow \ y = \hat{t}_x(z) = x \cdot z \ \Leftrightarrow \ z = x \backslash y,$$

so, $\hat{t}_x^{-1}(y) = x \backslash y$. The rest follows from just proved identities. □

Lemma 4.13. *The following conditions are equivalent:*

1) $T = \{t_x\}_{x \in E}$ *is a left loop transversal in a loop L to its subloop R;*

2) $\hat{T} = \{\hat{t}_x\}_{x \in E}$ *is a sharply transitive set of permutations in S_E.*

Proof. The proof is based on the following sequence of the equivalent statements:

- $T = \{t_x\}_{x \in E}$ is a left loop transversal in a loop L to its subloop R,
- $\langle E, \overset{(T)}{\cdot}, 1 \rangle$ is a loop with the unit 1,
- $x \overset{(T)}{\cdot} a = b$ has a unique solution in E for every $a, b \in E$,
- $\hat{t}_x(a) = b$ has a unique solution in E for every $a, b \in E$,
- $\hat{T} = \{\hat{t}_x\}_{x \in E}$ is a sharply transitive set of permutations in S_E. □

The proof of the following two lemmas about is analogous to the proof of Lemmas 4.12 and 4.13.

Lemma 4.14. *For an arbitrary right transversal $T = \{t_x\}_{x \in E}$ in a loop $L = \langle L, \cdot, e \rangle$ to its subloop $R = \langle R, \cdot, e \rangle$ the following statements are true:*

1) $\check{r}(1) = 1$ *for all $r \in R$,*

2) $\check{t}_x(y) = y \overset{(T)}{\circ} x, \quad \check{t}_x^{-1}(y) = x/y$ *for all $x, y \in E$,*
where $\check{t}_x^{-1}$ is an inverse permutation to a permutation $\check{t}_x$ in S_E, and "/" is a right division in a right loop $\langle E, \overset{(T)}{\circ}, 1 \rangle$. Moreover,

$$\check{t}_x(1) = x, \quad \check{t}_1(x) = x, \quad \check{t}_x^{-1}(1) = x/1, \quad \check{t}_x^{-1}(x) = 1.$$

Lemma 4.15. *The following conditions are equivalent:*

1) $T = \{t_x\}_{x \in E}$ *is a right loop transversal in a loop L to its subloop R;*

2) $\check{T} = \{\check{t}_x\}_{x \in E}$ *is a sharply transitive set of permutations in S_E.*

References

[1] **P. Baer**, **Nets and groups**. Trans. Amer. Math. Soc. **46** (1939), 110 – 141.

[2] **V. D. Belousov**, *Foundations of quasigroup and loop theory*, (Russian), Moscow, "Nauka", 1967.

[3] **F. Bonetti, G. Lunardon and K. Strambach**, *Cappi di permutazioni.* Rend. Math., **12**(1979), No. 3-4, 383 – 395.

[4] **T. Foguel and L. C. Kappe**, *On loops covered by subloops*, Expositiones Math. **23** (2005), 255 – 270.

[5] **K. W. Johnson**, *S-rings over loops, right mapping groups and transversals in permutation groups*, Math. Proc. Camb. Phil. Soc. **89** (1981), 433 – 443.

[6] **E. A. Kuznetsov**, *Transversals in groups.* 1. *Elementary properties*, Quasigroups and Related Systems **1** (1994), 22 – 42.

[7] **E. A. Kuznetsov**, *About some algebraic systems related with projective planes*, Quasigroups and Related Systems **2** (1995), 6 – 33.

[8] **E. A. Kuznetsov**, *Transversals in groups. Semidirect product of a transversal operation and subgroup*, Quasigroups and Related Systems **8** (2001), 37 – 44.

[9] **E. A. Kuznetsov**, *Transversals in loops*, Abstracts of International Conference "Loops-03", Prague, August 10-17, 2003, 18 – 20.

[10] **M. Niemenmaa and T. Kepka**, *On multiplication groups of loops*, J. Algebra **135** (1990), 112 – 122.

[11] **H. Pflugfelder**, *Quasigroups and loops: Introduction.* Sigma Series in Pure Math., 7, Helderman Verlag, New York, 1972.

[12] **L. V. Sabinin and O. T. Mikheev**, *Quasigroups and differential geometry*, in "Quasigroups and loops: theory and applications, Helderman-Verlag, Berlin 1990, 357 – 430.

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