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Free topological acts over a topological monoid

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Abstract. First we present the free topological S-acts on sets, on topological spaces, and as well as on S-acts. Then, we give more concrete description of these free objects in some cases.

1. Introduction

The action of topological semigroups and their representations have a very wide usage in different branches of Mathematics like geometry, analysis, Lie groups or dynamical systems, and they are studied by many authors, see for example [4, 7, 20, 23, 24]. Furthermore, some notions are in fact topological S-acts with some extra properties, e.g., in analysis, S-flow is a compact topological S-act (see [5, 19]), or the representation of a discrete group G is in fact a topological G-act (see [2, 13, 17]). Also in geometry, flow is a smooth topological S-act, where S is $(\mathbb{R}, +)$ with its usual topology (see [7]). These kinds of topological S-acts are studied more and there are some works about their universal structures (for example see [15]). We note that, a space which a topological semigroup acts on it, sometimes has different names in different branches of Mathematics, e.g. in some text, it is called G-space where G is a topological group (e.g. see [12]), while in some others, it is called topological S-act (see for example [22]). In this note we use the latter terminology since we use theorems and terminology of [18]. Because of the importance of the universal structures and specially free structures, in this paper we study the notion of freeness which is a fruitful subject in the study of different categories (see for example [3, 8, 9, 16]). We present the free topological S-acts on sets, on topological spaces, and as well as on S-acts.

Let $(S, \cdot, \tau_S)$ be a topological monoid. In this note, we want to study different free topological S-acts. Note that since there are three forgetful functors from the category of topological S-acts to the category of topological spaces,

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the category of S-acts and the category of sets, we can define free topological S-acts on a topological space, on an S-act and on a set. In Section 2, we briefly study topological S-acts, semitopological S-acts and compare them. In Section 3, first, we introduce the free topological S-acts on a topological space, then we describe the topology of free topological S-acts more concretely and study some of its properties, like its behavior with separation axioms. Also we give a coarser and finer topology than the topology of the free topological S-act on a topological space (X, τ_X) according to the topology of topological space (X, τ_X) and the topology of the topological monoid $(S, \cdot, \tau_S)$. Finally in Section 3, we introduce the free topological S-acts on a set. In Section 4, we study the free topological S-act on an S-act and present it. Then by using the notion of free topological S-acts on S-acts, we present some method for studying universal objects in the category of topological S-acts, using the known universal structures in the category of S-acts. To illustrate this method, we apply it to characterize projective topological S-acts by using the characterization of projective S-acts.

Now we briefly recall some definitions about S-acts needed in the sequel. For more information see [11, 18].

Recall that, for a semigroup S, a set A is a left S-*act* (or S-*set*) if there is, so called, an *action* $\mu : S \times A \to A$ such that, denoting $\mu(s, a) := sa$, $(st)a = s(ta)$ and, if S is a monoid with 1, $1a = a$. Right S-acts are defined similarly. An S-act A is called *cyclic*, if there exists an $a \in A$ such that $A = Sa$.

Each semigroup S can be considered as an S-act with the action given by its multiplication.

The definitions of a *subact* A of B, written as $A \leq B$, and a *homomorphism* between S-acts are clear. In fact S-homomorphisms, or S-maps, are action-preserving maps: $f : A \to B$ with $f(sa) = sf(a)$, for $s \in S$, $a \in A$. We denote the category of S-acts with S-maps, by **S-Act**.

A topological space (X, τ_X) has *Alexandroff topology*, if the intersection of an arbitrary family of open sets in (X, τ_X) is open. An space with an Alexandroff topology is called an *Alexandroff space*.

The algebraic structure of the free topological S-act on a topological space can be characterized concretely, however, like free topological groups, the topology of free topological S-acts can not be described as concretely as its algebraic structure.

2. Topological S-acts

In this section, we briefly state the notions we need about topological S-acts. First recall the following

Definition 2.1. Let S be a semigroup and a topological space with topology τ_S. S with this topology is called a *topological semigroup* if multiplication $(s,t) \mapsto st : S \times S \to S$ is (jointly) continuous ([5, 10, 14]). We use Kelley's notation in [14], and denote a topological semigroup by $(S, \cdot, \tau_S)$

Despite the above convention, for simplicity, we denote a topological $(S, \cdot, \tau_S)$-act by topological S-act.

Definition 2.2. For a topological semigroup $(S, \cdot, \tau_S)$, a (left) topological *S-act* or a topological *S-act* is a left S-act A with a topology τ_A such that the action $S \times A \to A$ is (jointly) continuous. Similar to topological semigroup, we denote a topological S-act by (A, τ_A). We denote the category of all topological S-acts with continuous S-maps by **S-Top**.

Definition 2.3. We say that a topological semigroup $(S, \cdot, \tau_S)$ has a *left ideal topology*, if each of its open sets, including the empty one, is a left ideal (sub S-act) of S. Also, a topological S-act (A, τ_A) is said to have a *subact topology* if all of its open sets, including the empty one, are subacts of A.

We use the above definition of a left ideal topology which is more general than the definition in [22].

Definition 2.4. By weak topology on a set Z, with respect to a family of functions on Z, we mean the coarsest topology on Z which makes those functions continuous. In other words, given a set Z and an indexed family $(Y_i)_{i \in I}$ of topological spaces with functions $f_i : Z \to Y_i$, the weak topology on Z is generated by the sets of the form $f_i^{-1}(U)$, where U is an open set in Y_i.

NOTATION. For any two arbitrary topological spaces (X_1, τ_{X_1}) and (X_2, τ_{X_2}), by $\tau_{X_1 \times X_2}$ we mean the product topology on $X_1 \times X_2$. For any set Z, we denote Z with discrete topology by (Z, τ_{dis}). For any S-act A, by $|A|$ we mean the underlying set of A.

Remark 2.5. Recall that for a semigroup S and an S-act A, the functions λ_s and ρ_a are defined for any $s \in S$ and $a \in A$ as follows

$$\lambda_s : A \to A, \quad y \mapsto sy \qquad \text{and} \qquad \rho_a : S \to A, \quad t \mapsto ta.$$

In the special case $A = S$, we use the notation $\lambda_s^{(S)} : S \to S$, to prevent misunderstanding.

Now if S has a topology τ_S for which its multiplication $S \times S \to S$ is (separately) continuous, that is, $\lambda_s^{(S)}$ and ρ_s are continuous for all $s \in S$, then S with topology τ_S is called a *semitopological semigroup*.

Similarly, one can define a *semitopological S-act* by taking $\lambda_s : A \to A$ and $\rho_a : S \to A$ to be continuous for each $s \in S$ and $a \in A$.

Clearly any topological S-act is a semitopological S-act, because every jointly continuous function is separately continuous. But, as the following example shows, for a topological semigroup $(S, \cdot, \tau_S)$, a semitopological S-act need not be a topological S-act. Note that clearly if S with a topology τ_S is a semitopological semigroup which is not a topological semigroup, then S with τ_S is a semitopological S-act which is not a topological S-act. However the following example shows that for a topological semigroup S, the joint continuity of the action of S-acts is independent from the joint continuity of the multiplication of S.

Example 2.6. Suppose that $S = [0, 1]$ and τ_S is the usual topology on $[0, 1]$ which is inherited from $\mathbb{R}$ by subspace topology. Define for each s and t in S, $s \cdot t = 0$. It is obvious that $(S, \cdot, \tau_S)$ is a topological semigroup. Again, consider $[0, 1]$ with topology which is inherited from $\mathbb{R}$. For any $s, t \in S$, define the action of S on $[0, 1]$ by

$$\mu(s, t) = \sum_{n=1}^{\infty} (\frac{1}{2})^n f_n(s, t),$$

where

$$\mathrm{f_n(s,t)} = \begin{cases} 0 & \text{if } s \leqslant s_n \text{ or } t \leqslant t_n \\ \frac{|(\mathbf{s}-\mathbf{s_n})(\mathbf{t}-\mathbf{t_n})|}{(\mathbf{s}-\mathbf{s_n})^2+(\mathbf{t}-\mathbf{t_n})^2} & \text{otherwise} \end{cases}$$

and $\{(s_n, t_n) | n = 1, 2, \ldots\}$ is any (non-void) subset of the product $[\frac{1}{2}, 1] \times [\frac{1}{2}, 1]$. If we take $T = [0, \frac{1}{2}]$, then by an straightforward checking, we can see that μ has the following properties:

1. $\mu((T \times [0, 1]) \cup (S \times T)) = \{0\}$,
2. $\mu(S, [0, 1]) \subseteq T = [0, \frac{1}{2}]$.

(For more details about the properties of the function μ, see [23, Example 5.14.]) So we have for all s, s' and t in S

$$\mu(st, s') = \mu(0, s') \in \mu(T \times [0, 1]) = \{0\},$$

$$\mu(s, \mu(t, s')) \in \mu(S \times T) = \{0\}.$$

Therefore, $[0,1]$ is an S-act with the action μ. Again, by direct checking, one can see that μ, the action of $(S, \cdot, \tau_S)$ on $[0,1]$, is not continuous but all the functions $\lambda_s(-) = \mu(s, -)$ and $\rho_a(-) = \mu(-, a)$, for each s and a in S, are continuous. Hence $[0,1]$ is not a topological S-act but it is a semitopological S-act.

Now we recall the definition of different free topological S-acts in the following definition. Since these definitions are very similar, we state them together.

Definition 2.7. A topological S-act (F, τ_F) with one-one S-map $\nu : B \to F$, (the embedding $\nu : (X, \tau_X) \to (F, \tau_F)$), (one-one function $\nu : Z \to F$) is the *free topological S-act over the S-act B* (*over the topological space X*) (*over the set Z*), if for every topological S-act (A, τ_A) and an S-map $f : B \to A$, (a continuous function $f : (X, \tau_X) \to (A, \tau_A)$), (a function $f : Z \to A$), there exists a unique continuous S-map $\widetilde{f} : (F, \tau_F) \to (A, \tau_A)$ such that $\widetilde{f} \circ \nu = f$ (for the general definition of the free objects in an arbitrary category, see [1, 6]).

The free topological space over a set Z is the set Z together with the discrete topology. The free S-act, for a monoid S, on a set Z is defined as follow. Consider the set $S \times Z$ with the action defined by $t(s, z) = (ts, z)$ for any $t, s \in S$ and $z \in Z$, and define $\nu : Z \to S \times Z$ as follows $\nu(z) = (1, z)$. It is a known fact that $S \times Z$ with this action is an S-act. From now on, for any set Z, by $F(Z)$ we mean this S-act which is defined on $S \times Z$. Furthermore, it is a known fact that $F(Z)$ is the free S-act over the set Z (it means that for any S-act A and a function $f : Z \to A$, there exists a unique S-map $\widetilde{f} : F(Z) \to A$ such that $\widetilde{f} \circ \nu = f$ (for more details see, [11, 18])).

3. Free topological S-act on a topological space

In this section, we present the free topological S-act over a topological space and then describe it more concretely in some special instances, e.g, when τ_S is Alexandroff. First note the following remark.

Remark 3.1. Let $\{(A, \tau_i)\}_{i \in I}$ be a family of topological S-acts. Let τ_A be the topology generated by the subbasis $\cup_{i \in I} \tau_i$ on A. Then we show that (A, τ_A) is a topological S-act. Let $s \in S$, $a \in A$, and $U \in \tau_A$ such that $sa \in U$ and $U \in \tau_A$. As we have in section 2.18 of [21], we can and will suppose that U is an element of the subbasis $\cup_{i \in I} \tau_i$. So there is some $i \in I$ such that $U \in \tau_i$.

Since (A, τ_i) is a topological S-act, there exist open sets $W \in \tau_i$ and $V \in \tau_S$ which contain a and s, respectively such that $V \cdot W \subseteq U$. Since $\tau_i \subseteq \tau_A$, (A, τ_A) is a topological S-act.

Proposition 3.2. *For any topological monoid $(S, \cdot, \tau_S)$, the free topological S-act on a topological space (X, τ_X) is $F(X)$ with the topology τ_X^* which is generated by the union of all topologies τ_i on $|F(X)| = S \times X$ which makes $F(X)$ to a topological S-act and furthermore $\nu : (X, \tau_X) \longrightarrow (S \times X, \tau_i)$ is a topological embedding.*

Proof. Let (X, τ_X) be a topological space. We first show that if τ_X^* is the topology generated by the union of all topologies τ_i on $|F(X)| = S \times X$ where $(F(X), \tau_i)$ satisfies the following conditions

(a) the map $\nu : X \to (F(X), \tau_i)$ defined by $\nu(x) = (1, x)$ is a topological embedding.

(b) $(F(X), \tau_i)$ is a topological S-act.

Then $(F(X), \tau_X^*)$ satisfies conditions (a) and (b).

Define

$$\Gamma_{(X,\tau_X)} := \{\tau | \tau \text{ is a topology on } |F(X)| = S \times X \text{ satisfying (a) and (b)}\}.$$

We show that τ_X^* belongs to $\Gamma_{(X,\tau_X)}$ and $(F(X), \tau_X^*)$ is the desired free topological S-act. (One can easily check that $\tau_{S\times X} \in \Gamma_{(X,\tau_X)}$ and so $\Gamma_{(X,\tau_X)} \neq \emptyset$.)

Since τ_X^* is finer than each $\tau_i \in \Gamma_{(X,\tau_X)}$, so ν^{-1} is continuous and since τ_X^* is generated by all $\tau_i \in \Gamma_{(X,\tau_X)}$, so ν is continuous, therefore τ_X^* satisfies condition (a). By Remark 3.1, τ_X^* satisfies condition (b), too. Thus, $\tau_X^* \in \Gamma_{(X,\tau_X)}$. Therefore $(F(X), \tau_X^*)$ is a topological S-act.

Finally, to prove that $(F(X), \tau_X^*)$ is actually the free topological S-act on X, let $g : (X, \tau_X) \to (A, \tau_A)$ be a continuous function into a topological S-act (A, τ_A). We claim that the function $\tilde{g} : F(X) \to A$, defined by $\tilde{g}((s, x)) := sg(x)$, is the unique continuous S-map with $\tilde{g}\nu = g$. Clearly, $\tilde{g}$ is an S-map. Since $\tau_{S\times X} \subseteq \tau_X^*$, $(id_S, g) : (S \times X, \tau_X^*) \to (S \times A, \tau_{S\times A})$ is continuous and since the action $S \times A \to A$ is also continuous, $\tilde{g}$ is continuous.

For the uniqueness of $\tilde{g}$, let $\tilde{g} \circ \nu = h \circ \nu$. Therefore $h((1, x)) = \tilde{g}((1, x))$, and so $\tilde{g} = h$. Hence, the S-act $F(X)$ with τ_X^* is the free topological S-act on the topological space (X, τ_X). □

Before we begin to describe the topology τ_X^* more concretely, we need some definitions and results which are presented in the following

Remark 3.3. Suppose that we are given a topological space (X, τ_X) and a topological monoid $(S, \cdot, \tau_S)$. We define $\tau(S, X)$ as follows: $O \in \tau(S, X)$ if there exist open sets $Y \in \tau_X$ and $T \in \tau_S$ such that $\pi_1(O) = T$ and $\pi_2(O) = Y$ and for any $(s, x) \in O$, there exist an open set $V(O, x) \in \tau_S$ and an open set $W(O, s) \in \tau_X$ which contain s and x, respectively such that

$$\pi_1(O \cap (S \times \{x\})) = V(O, x) \text{ and } \pi_2(O \cap (\{s\} \times X)) = W(O, s).$$

One can obviously see that

$$V(O, x) = \{s \in S | (s, x) \in O\} \text{ and } W(O, s) = \{x \in X | (s, x) \in O\}. \qquad \text{(I)}$$

(where π_1 and π_2 are the usual projections of O onto its first and second factors, respectively). Note that for each $O \in \tau(S, X)$ and the corresponding open sets $\{V(O, x)\}_{x \in Y} \subseteq \tau_S$ and $\{W(O, s)\}_{s \in T} \subseteq \tau_X$ which are obtained by the definition of $\tau(S, X)$, we have

$$O = \bigcup_{x \in Y} (V(O, x) \times \{x\}) \text{ and } O = \bigcup_{s \in T} (\{s\} \times W(O, s)). \qquad \text{(II)}$$

Therefore if we define for an open set $Y \in \tau_X$ and an open set $T \in \tau_S$,

$$\begin{aligned} \tau_1(T, Y) := \quad & \{O \subseteq T \times Y | \forall (s, x) \in O, \exists V(O, x) \in \tau_S : s \in V(O, x) \text{ and} \\ & \pi_1(O \cap (S \times \{x\})) = V(O, x)\} \\ \tau_2(T, Y) := \quad & \{O \subseteq T \times Y | \forall (s, x) \in O, \exists W(O, s) \in \tau_X : x \in W(O, s) \text{ and} \\ & \pi_2(O \cap (\{s\} \times X)) = W(O, s)\} \end{aligned}$$

and

$$\tau_1(S, X) := \bigcup_{T \in \tau_S,\ Y \in \tau_X} \tau_1(T, Y) \quad \text{and} \quad \tau_2(S, X) := \bigcup_{T \in \tau_S,\ Y \in \tau_X} \tau_2(T, Y),$$

then by the definition of $\tau(S, X)$, one can easily see that

$$\tau(S, X) = \tau_1(S, X) \cap \tau_2(S, X).$$

By an easy check, one can see that $\tau_1(S, X)$ and $\tau_2(S, X)$ are two topologies on $|F(X)| = S \times X$ (Note that each element of $\tau_1(S, X)$ satisfies the right side of Relation (II) and each element of $\tau_2(S, X)$ satisfies the left side of Relation (II)), so $\tau(S, X)$ is a topology on $F(X)$, too. (Since the intersection of any two topologies on a space is a topology on it.)

Lemma 3.4. *Let $(S, \cdot, \tau_S)$ be a topological semigroup and (X, τ_X) be a topological space. Then $(F(X), \tau(S, X))$ is a semitopological S-act.*

Proof. We prove that for any $s \in S$ and $(t, x) \in F(X)$, the functions $\lambda_s : F(X) \to F(X)$ and $\rho_{(t,x)} : S \to F(X)$ are continuous. First, we show that the function λ_s is continuous. Suppose that we are given $U \in \tau(S, X)$. We show that $\lambda_s^{-1}(U)$ is an open set in $F(X)$. By the definition of $\tau(S, X)$ there exist open sets $T \in \tau_S$ and $Y \in \tau_X$ such that $U \subseteq T \times Y$ and for any $t' \in T$ and $x' \in Y$ such that $(t', x') \in U$, there exist open sets $V(U, x')$ and $W(U, t')$ which contain t' and x', respectively, such that

$$\pi_1(U \cap (S \times \{x'\})) = V(U, x') \text{ and } \pi_2(U \cap (\{t'\} \times X)) = W(U, t').$$

Note that since $(S, \cdot, \tau_S)$ is a topological monoid, the function $\lambda_s^{(S)} : S \to S$ is continuous. Now by the definition of the action of $F(X)$, we have

$$\lambda_s^{-1}(U) = \bigcup_{y \in Y} [(\lambda_s^{(S)})^{-1}(V(U, y)) \times \{y\}].$$

To prove $\lambda_s^{-1}(U)$ is in $\tau(S, X)$, we show that it is equal to an open set which belongs to $\tau(S, X)$. Define $V_1 := (\lambda_s^{(S)})^{-1}(T)$ and $U' := \cup_{t' \in V_1}(\{t'\} \times W(U, st'))$ where $W(U, st')$ is the open set which is found for the element $(st', y) \in U$ for some $y \in X$, by the assumption $U \in \tau(S, X)$. (Note that since we have $\pi_2(U \cap (\{st'\} \times X)) = W(U, st')$, $W(U, st')$ does not depend on the choice of $y \in X$.) We show that $\lambda_s^{-1}(U)$ equals U', and U' belongs to $\tau_1(S, X)$, since it is easy to see that $U' \in \tau_2(V_1, Y) \subseteq \tau_2(S, X)$. (Note that $U \in \tau_2(S, X)$ and recall Relation (I).) By the definition of the action of $F(X)$, we have obviously $\lambda_s(U') \subseteq U$. Suppose that $(t_1, y) \in \lambda_s^{-1}(U)$ for some $t_1 \in S$ and $y \in X$, so we have $(st_1, y) \in U$. Therefore we have $\{st_1\} \times W(U, st_1) \subseteq U$ which by the definition of the action $F(X)$, implies that $(t_1, y) \in \{t_1\} \times W(U, st_1)$. But $\{t_1\} \times W(U, st_1)$ is a subset of U', hence $(t_1, y) \in U'$. Therefore $U' = \lambda_s^{-1}(U)$ which implies that $\lambda_s^{-1}(U) \in \tau(S, X)$.

Now, we show the continuity of $\rho_{(t,x)}$. Consider U like the above and suppose that we are given $s' \in S$ such that $s' \in \rho_{(t,x)}^{-1}(U)$. Again note that since $(S, \cdot, \tau_S)$ is a topological monoid, the function $\rho_t : S \to S$ is continuous. Since $U \in \tau(S, X)$, there exists open set $V(U, x)$ in τ_S which contains $s't$ and $V(U, x) \times \{x\} \subseteq U$. Therefore $s' \in \rho_t^{-1}(V(U, x)) \in \tau_S$. We have $\rho_{(t,x)}(s') \in \rho_{(t,x)}(\rho_t^{-1}(V(U, x))) \subseteq V(U, x) \times \{x\} \subseteq U$. So $\rho_t^{-1}(V(U, x)) \subseteq \rho_{(t,x)}^{-1}(U)$. Hence $\rho_{(t,x)}^{-1}(U) \in \tau_S$. □

The following result shows a characterization of $\tau(S,X)$.

Proposition 3.5. *Let (X,τ_X) be a topological space and $(S,\cdot,\tau_S)$ be a topological monoid. Then $\tau(S,X)$ is the finest topology on $F(X)$ such that $F(X)$ is a semitopological S-act and $\nu:(X,\tau_X)\to(F(X),\tau(S,X)), x\rightsquigarrow(1,x)$, is continuous.*

Proof. By the above proposition and the definition of $\tau(S,X)$, $\tau(S,X)$ has the above properties. Let τ be a topology on $|F(X)|=S\times X$ with the above properties. First note that if $s(1,x)=(s,x)\in U$ and $U\in\tau$, then by the continuity of $\rho_{(1,x)}$, λ_s and ν we can conclude that

$$s\in\rho^{-1}_{(1,x)}(U) \text{ and } x\in\nu^{-1}(\lambda_s^{-1}(U)),$$

where $\rho^{-1}_{(1,x)}(U)\in\tau_S$ and $\nu^{-1}(\lambda_s^{-1}(U))\in\tau_X$. Furthermore we have obviously

$$\pi_1(U\cap(S\times\{x\}))=\rho^{-1}_{(1,x)}(U)\in\tau_S$$

and also

$$\pi_2(U\cap(\{s\}\times X))=\nu^{-1}(\lambda_s^{-1}(U))\in\tau_X$$

Hence, $U\in\tau(S,X)=\tau_1(S,X)\cap\tau_2(S,X)$. Therefore $\tau\subseteq\tau(S,X)$ □

By the above proposition, we can explain the topology τ_X^* in another way and we can present a coarser and finer topology than it, according to the topologies τ_S and τ_X (note that any topological S-act is a semitopological S-act and note that τ_X^* satisfies condition (b) in the proof of Proposition 3.2).

Corollary 3.6. *Let $(S,\cdot,\tau_S)$ be a topological monoid and (X,τ_X) be a topological space. Then, $\tau_{S\times X}\subseteq\tau_X^*\subseteq\tau(S,X)$ and τ_X^* is the finest topology which is coarser than $\tau(S,X)$ and it makes $F(X)$ a topological S-act.* □

Proposition 3.7. *For any Alexandroff topological monoid $(S,\cdot,\tau_S)$ and any topological space (X,τ_X), the topology τ_X^* is the product topology on $|F(X)|=S\times X$. In fact we have $\tau_X^*=\tau_{S\times X}=\tau(S,X)$.*

Proof. We first show that, in this case, τ_X^* equals to $\tau(S,X)$ and then we show that $\tau(S,X)$ equals to the product topology $\tau_{S\times X}$. Note that by Corollary 3.6, we have $\tau_X^*\subseteq\tau(S,X)$. On the other hand, since $\tau(S,X)$ obviously satisfies condition (a) by Relation (I) in Remark 3.3, to complete our proof, it is enough to prove that $(F(X),\tau(S,X))$ is a topological S-act. Suppose $t(s,x)=(ts,x)\in U$ and $U\in\tau(S,X)$. Hence there exists open set $W(U,ts) \in \tau_X$ with $x \in W(U,ts)$ such that $\{ts\} \times W(U,ts) \subseteq U$. But for any $y \in W(U,ts)$, since again $U \in \tau(S,X)$, there exists open set $V(U,y) \in \tau_S$ such that $V(U,y) \times \{y\} \subseteq U$ and $ts \in V(U,y)$. Now define $V := \cap_{y \in V(U,y)} V(U,y) \in \tau_S$, because τ_S is Alexandroff, V contains ts and we have:

$$V \times W(U,ts) \subseteq \bigcup_{y \in W(U,ts)} (V(U,y) \times \{y\}) \subseteq U. \quad (*)$$

Now since $(S,\cdot,\tau_S)$ is a topological monoid, there exist open sets V_s and V_t which contain s and t, respectively and satisfy the relation $V_t \cdot V_s \subseteq V$. By Corollary 3.6, if we define $W := V_s \times W(U,ts)$, then $W \in \tau_{S\times X} \subseteq \tau(S,X)$ which contains (s,x) such that

$$t(s,x) \in V_t \cdot W = (V_t \cdot V_s) \times W(U,ts) \subseteq V \times W(U,ts) \subseteq U.$$

So $(F(X), \tau(S,X))$ is a topological S-act. Now suppose that $U \in \tau(S,X)$. If U is a non-empty open subset of $|F(X)| = S \times X$, then consider an arbitrary element (t,x) in U. We have clearly $t(1,x) \in U$, so by the above discussion, there exists an open set $V \in \tau_S$ which contains t such that $(t,x) = t(1,x) \in V \times W(U,t) \subseteq U$. (Recall Relation (*) with $s = 1$.) Since $V \times W(U,t)$ belongs to the product topology on $|F(X)| = S \times X$, $\tau_{S\times X}$ is finer than $\tau(S,X)$. Therefore by Corollary 3.6 we have $\tau_X^* = \tau(S,X) = \tau_{S\times X}$. □

Proposition 3.8. *Suppose that $(S,\cdot,\tau_S)$ is a topological monoid. For each Alexandroff topological space (X,τ_X), the topology τ_X^* is the product topology on $S \times X$ and more precisely $\tau_X^* = \tau_{S\times X} = \tau(S,X)$.*

Proof. τ_X^* satisfies conditions (a) and (b) in Proposition 3.2 so $\tau_{S\times X} \subseteq \tau(S,X)$. Suppose that we are given $(ts,x) \in U$ for some $t,s \in S$, $x \in X$ and an open set $U \in \tau(S,X)$. Since $U \in \tau(S,X)$, we can choose for $(ts,x) \in U$, the open set $V(U,x)$ such that $V(U,x) \times \{x\} \subseteq U$ and $ts \in V(U,x)$. Choose for any $s' \in V(U,x)$, an open set $W(U,s')$ such that $\{s'\} \times W(U,s') \subseteq U$ and $x \in W(U,s')$. Define $W := \cap_{s' \in V(U,x)} W(U,s')$. Now, by a similar argument as in the proof of Proposition 3.7, we can get the result. □

Since every discrete topological space is Alexandroff, as an immediate consequence of the above proposition and Proposition 3.5, we have

Proposition 3.9. (Free topological S-act on a set) *Let $(S,\cdot,\tau_S)$ be a topological monoid and Z be a set. Then the free topological S-act on the set Z is $F(Z)$ with the topology $\tau_{S\times Z}$ where τ_Z in the definition of $\tau_{S\times Z}$ is the discrete topology.*

Now we discuss the properties of the free topological S-act on a topological space which satisfies some of the separation axiom, (for more details about the separation axioms, see [21].)

Proposition 3.10. *Let $(S, \cdot, \tau_S)$ be a topological monoid with left ideal topology. Suppose that (X, τ_X) satisfies one of the separation axioms T_i for $i = 0, 1, 2, 3, 3\frac{1}{2}$. Then, the free topological S-act on (X, τ_X) satisfies that separation axiom if and only if $S = \{1\}$.*

Proof. For the non-trivial part, let (X, τ_X) be a T_i space for some i. Then, by assumption, the free topological S-act on (X, τ_X) is a T_i space. Note that if a topological S-act (A, τ_A) which has subact topology, satisfies T_i, then for any $a \in A$, $Sa = \{a\}$. For, if there exist $s \in S$ and $a \in A$ such that $sa \neq a$, then any open set in the subact topology τ_A containing a, also contains sa. Thus, we have $S(s, x) = \{(s, x)\}$ for each $(s, x) \in F(X)$. In particular, $S(1, x) = \{(1, x)\}$. Therefore $S = S1 = \{1\}$. □

Although Proposition 3.10 shows that for any non-trivial topological monoid $(S, \cdot, \tau_S)$ with left ideal topology, the free topological S-act on a T_i space does not satisfy any of the separation axioms T_i, but the following proposition shows that if $(S, \cdot, \tau_S)$ itself satisfies any T_i, $i = 0, 1, 2$ then the free topological S-act on a topological space which satisfies that T_i, satisfies that separation axiom, too.

First, note that if (X_1, τ_{X_1}) and (X_2, τ_{X_2}) are two topological spaces which satisfy T_i for some $i = 0, 1, 2$, then their product space satisfies that T_i, too (for more details, see [10] or [21]).

Proposition 3.11. *Let $(S, \cdot, \tau_S)$ be a topological monoid which satisfies T_i for some $0 \leq i < 3$. Then, the free topological S-act on a topological space which satisfies that T_i, satisfies that separation axiom, too.*

Proof. suppose that the topological space (X, τ_X) satisfies T_i. Clearly $S \times X$ with product topology also satisfies T_i, too and since for any topological space (X, τ_X), we have $\tau_{S \times X} \subseteq \tau_X^*$, then $(F(X), \tau_X^*)$ satisfies T_i. □

Remark 3.12. About the preservation of $T_{3\frac{1}{2}}$, first, we prove that if we define $\Gamma'_{(X,\tau_X)}$ as follows,

$$\{\tau | \tau \text{ is a completely regular topology on } |F(X)| \text{ satisfing (a) and (b)}\}$$

and let τ'_X be defined to be the generated topology by $\cup_{\tau_i \in \Gamma_{(X,\tau_X)}} \tau_i$, then $(F(X), \tau'_X)$ is a completely regular topological S-act. Then we give a condition such that the completely regularity is preserved. For our assertion, we just need to show the completely regularity of $(F(X), \tau'_X)$, since it is straightforward to see that τ'_X satisfies conditions (a) and (b). For this purpose, we show that the generated topology by a family of topologies $(\tau_i)_{i\in I}$ on a set C such that each τ_i is completely regular for any $i \in I$, is a completely regular topology on C. Let $(\tau_i)_{i\in I}$ be a family of completely regular topologies on a set C. Let τ be the generated topology by $\cup_{i\in I}\tau_i$. Let K be a closed set in C with the topology τ and $c \in C \setminus K$. Since $O = C \setminus K$ belongs to τ, there exists a family of open sets $\{O_j\}_{j\in J} \subseteq \cup_{i\in I}\tau_i$ such that O is equal to a union of their finite intersections of O_i's. Therefore we can assume that there exists $O_1 \cap \ldots \cap O_n$ such that $K = C \setminus O \subseteq C \setminus (O_1 \cap \ldots \cap O_n)$ and $c \in O_1 \cap \ldots \cap O_n$. Since for any i, O_i is open in τ_{n_i} and since τ_{n_i} is completely regular, for closed set $C \setminus O_i$ and c, there exists a continuous real valued function $f_i : C \to \mathbb{R}$ such that $f_i(C \setminus O_i) = 1$ and $f_i(c) = 0$. Since τ is the generated topology by τ_i, all the functions f_i are continuous real valued function from C with the topology τ to $\mathbb{R}$ such that $f_i(C \setminus O_i) = 1$ and $f_i(c) = 0$. Let f be defined by $f(x) := max\{f_1(x), \ldots, f_n(x)\}$, for any $x \in C$. Therefore τ is completely regular, since f is a continuous function from C with topology τ to $\mathbb{R}$ such that f is continuous and $f(K) = 1$ and $f(c) = 0$. Therefore, since τ'_X is the generated topology by $\cup_{\tau_i\in\Gamma'_{(X,\tau_X)}}\tau_i$, and since for each $\tau_i \in \Gamma'_{(X,\tau_X)}$, τ_i is completely regular, τ'_X is completely regular. Hence $(F(X), \tau'_X)$ is a completely regular topological S-act.

Now if for a topological semigroup $(S, \cdot, \tau_S)$ and a topological space (X, τ_X), we have $\tau'_X = \tau^*_X$ or more specially, if $\Gamma'_{(X,\tau_X)} = \Gamma_{(X,\tau_X)}$, then the separation axiom $T_{3\frac{1}{2}}$ is preserved. For an example of a topological semigroup $(S, \cdot, \tau_S)$ and a topological space (X, τ_X) with this property, let $(S, \cdot, \tau_{dis})$ be a topological monoid. Then for any completely regular space (X, τ_X), clearly, by Proposition 3.7, $\tau^*_X = \tau_{S\times X} = \tau'_X$. Therefore for a topological semigroup which has discrete topology, the separation axiom $T_{3\frac{1}{2}}$ is preserved.

4. The free topological S-act on an S-act

The category **S-Act** is a very well-known category and its universal structures are studied comprehensively by many authors. In this section we want to present a very useful and effective tool which enables us to study **S-Top** by using the studies in **S-Act**. First, in this section, we present the free topological S-act on an S-act, then to illustrate the application of this result, we characterize the projective topological S-acts. In fact, we show that the projective topological S-acts are exactly the free topological S-acts on projective S-acts.

Now we discuss the free topological S-act on an S-act. One might naturally expect that an S-act A with discrete topology to be the free topological S-act on A, but, as Proposition 4.1 shows, A with this topology may not be a topological S-act and if it happens to be so, then it is indeed the free topological S-act on A.

Since by the definition of topological S-acts, the proof of the following result is straightforward, we state it without proof.

Proposition 4.1. *An S-act A with the discrete topology is a topological S-act if and only if for any $a \in A$ and $s \in S$, $(sa : a) := \{t \in S | ta = sa\} \in \tau_S$.* □

Proposition 4.2. *If $(S, \cdot, \tau_S)$ is a topological semigroup with a right identity, then the following statements are equivalent*

(1) *All the S-acts with discrete topology are topological S-acts.*

(2) *τ_S is the discrete topology.*

(3) *If we define* G *from category* **S-Act** *to category* **S-Top** *as follows, $A \mapsto (A, \tau_{dis})$, then* G *is the free functor.*

Proof. Since (1) and (3) are equivalent, for the non-trivial part of the proof, by Proposition 4.1, we just need to show (1) $\Rightarrow$(2). Since S with the discrete topology is a topological S-act, if e is the right identity of S, then the function $id_S = \rho_e : (S, \tau_S) \to (S, \tau_{dis})$ is continuous and hence $\tau_S = \tau_{dis}$. □

Now, we discuss about the free topological S-act on an S-act in general.

Proposition 4.3. *For any topological semigroup $(S, \cdot, \tau_S)$, the free topological S-act on an S-act A is defined as follows*

$$(A, \tau_{*A}), \qquad (A \in \mathbf{S-Act})$$

in which τ_{*A} is the topology generated on A by the union of all τ_i on A, where (A, τ_i) is a topological S-act.

Proof. Let A be an arbitrary S-act and define

$$\Sigma_A := \{\tau |\ (A, \tau) \text{ is a topological } S\text{-act}\}.$$

(Note that every S-act is a topological S-act with trivial topology, so Σ_A is not empty.)

Similar to the proof of Proposition 3.2, we can show that τ_{*A} which is the topology generated by the union of all τ_i where $\tau_i \in \Sigma_A$, makes A a topological S-act.

To prove that (A, τ_{*A}) with $id_A : A \to (A, \tau_{*A})$ is the free topological S-act on A, let $f : A \to (B, \tau_B)$ be an S-map into a topological S-act (B, τ_B). Then, the same function $f : (A, \tau_{*A}) \to (B, \tau_B)$ is claimed to be a continuous S-map.

Let $\tau_f := \{f^{-1}(U)\}_{U \in \tau_B}$. To prove the claim, first we show that (A, τ_f) is a topological S-act. Let $U \in \tau_B$, $sa \in f^{-1}(U)$ for some $a \in A$ and $s \in S$. Since $f(sa) = sf(a) \in U$ and (B, τ_B) is a topological S-act, there exists $V_s \in \tau_S$ and $W_{f(a)} \in \tau_B$ such that $s \in V_s$ and $f(a) \in W_{f(a)}$ and

$$sf(a) \in V_s \cdot W_{f(a)} \subseteq U.$$

Thus, $sa \in V_s \cdot f^{-1}(W_{f(a)}) \subseteq f^{-1}(U)$, and so (A, τ_f) is a topological S-act.

Now, since $\{f^{-1}(U)\}_{U \in \tau_B}$ belongs to Σ_A, by the definition of τ_{*A}, we have

$$\tau_f = \{f^{-1}(U)\}_{U \in \tau_B} \subseteq \tau_{*A}.$$

So $f : (A, \tau_{*A}) \to (B, \tau_B)$ is continuous.

The rest of the proof is trivial. □

Now using the concept of weak topology and the above proposition and its proof, we can explain τ_{*A} in these ways.

Proposition 4.4.

(i) *τ_{*A} is the weak topology which is induced on $|A|$ with respect to the family of S-homomorphisms $id : A \to (A, \tau_i)$ where (A, τ_i) is a topological S-act.*

(ii) *τ_{*A} is the weak topology on $|A|$ with respect to the family of all S-homomorphisms from A to other topological S-acts.* □

Note that, for a topological space (X, τ_X) and any topological monoid $(S, \cdot, \tau_S)$, since $(F(X), \tau_X^*)$ is a topological S-act, it is obvious that τ_X^* on $|F(X)| = S \times X$ is coarser than $\tau_{*F(X)}$. (See the definitions of $\Gamma_{(X,\tau_X)}$ and $\Sigma_{F(X)}$ in the proof of Propositions 3.2 and 4.3.)

But, the following example shows that τ_X^* can be a proper subset of $\tau_{*F(X)}$.

Example 4.5. Let $(S, \cdot, \tau_{dis})$ be a topological monoid and let (X, τ_X) be a non-discrete topological space. Then $\tau_X^* \subsetneq \tau_{*F(X)}$. Because, by Proposition 4.2, $\tau_{*F(X)}$ is discrete. On the contrary, suppose that $\tau_{*F(X)}$ equals to τ_X^*. Since ν is an embedding, and since $\{1\} \times X$ with the subspace topology is the discrete topology (because $\tau_{*F(X)}$ is discrete), (X, τ_X) is a discrete space, which is impossible. So we have the result.

For all universal objects in category **S-Top**, we can use the free topological S-acts on S-acts to change any given diagrams in **S-Act** to a given diagram in **S-Top**. Therefore, we can study the algebraic structure of universal structures by using the known universal objects in **S-Act**. To illustrate this method, we apply it in the next proposition to characterize the projective topological S-acts.

Proposition 4.6. *Let $(S, \cdot, \tau_S)$ be a topological monoid. Then the projective topological S-acts are the free topological S-acts on the S-acts $\sqcup_{i\in I} Se_i$, where e_i's are idempotents in S, I is a set and $\sqcup_{i\in I} Se_i$ denote the coproduct of Se_i's.*

Proof. Let (P, τ_P) be a projective S-act. First, we show that (P, τ_P) is the free topological S-act on S-act P. For this purpose, we show that topology τ_P is the finest topology which makes P a topological S-act. Let (P, τ) be a topological S-act. We show that τ is coarser than τ_P. Consider the generated topology by the union of τ and τ_P, and denote it by τ'. Consider the identity maps $id_P : (P, \tau_P) \to (P, \tau_P)$ and $id_P : (P, \tau') \to (P, \tau_P)$. Since (P, τ_P) is a projective topological S-act, the identity map $id_P : (P, \tau_P) \to (P, \tau')$ is continuous. Therefore τ' is coarser than τ_P and therefore $\tau \subseteq \tau_P$. Now, to complete the proof, we show that P is a projective S-act and then we use [18, Theorem 1.5.10], to characterize the algebraic structure of (P, τ_P). Suppose that $f : A \to B$ be a surjective S-map, where A and B are S-acts and let $g : P \to B$ be an S-map. Since the epimorphisms in category **S-Act** are exactly onto S-maps (see [18]), it is straightforward to see that $f : (A, \tau_{*A}) \to (B, \tau_{*B})$ is an epimorphism in **S-Top** and $g : (P, \tau_P) \to (B, \tau_{*B})$ is continuous (note that if C is an S-act, (D, τ_D) is a topological S-act and $h : C \to (D, \tau_D)$ is an S-map, then $\tau_1 = \{V \subseteq C | V = f^{-1}(U)$, where U is an open set in $(D, \tau_D)\}$ is a topology on C such that (C, τ_1) is a topological S-act). Since (P, τ_P) is a projective topological S-act, there exists a continuous S-map $h : (P, \tau_P) \to (A, \tau_{*A})$ such that $f \circ h = g$. Since h is an S-map, P is a projective S-act. Therefore by [18, Theorem 1.5.10], there exists a family $\{e_i\}_{i\in I}$ of idempotents in S such that P is algebraically isomorphic to $\sqcup_{i\in I} Se_i$, where $\sqcup$ denotes the coproduct of Se_i's in **S-Act**. Therefore, P is the projective S-act which is a coproduct of cyclic S-acts in **S-Act** and (P, τ_P) is the free topological S-act on S-act P. □

Finally in this paper we show that the free topological S-act on the set

For a non-empty family of S-acts, like $\{A_i\}_{i\in I}$, the coproduct of A_i's in **S-Act** is the disjoint union of A_i's with its natural action (see [18]).

Z is the free topological S-act on the S-act $F(Z)$. (So if we define the free topological S-act on a set Z in this way, then the result will be the same.)

Proposition 4.7. *let $(S,\cdot,\tau_S)$ be a topological monoid. The free topological S-act on the set Z equals to the free topological S-act on the S-act $F(Z)$.*

Proof. Since a discrete topological space (Z,τ_{dis}) is Alexandroff, by Proposition 3.8 we have $\tau^*_Z = \tau_{S\times Z}$. We show that the topology τ_* on $F(Z)$ equals to τ^*_Z. For this purpose, we show that $\Sigma_{F(Z)} = \Gamma_{(Z,\tau_{dis})}$. Since obviously, $\tau^*_Z \in \Sigma_{F(Z)}$, it is enough to show that $\tau_{*F(Z)}$ belongs to $\Gamma_{(Z,\tau_{dis})}$. Clearly, $\tau_{*F(Z)}$ on $F(Z)$ satisfies condition (a). Since $\tau_{S\times Z} = \tau^*_Z \subseteq \tau_{*F(Z)}$ and Z is a discrete space, then $\{U \cap (\{1\}\times Z) | U \in \tau_{*F(Z)}\}$ is the discrete topology on $\{1\}\times Z$. Since $\nu : Z \to \{1\}\times Z$ is a one to one, onto function from a discrete topological space to another discrete topological space, it is an embedding. Therefore $\tau_{*F(Z)}$ satisfies conditions (a) and (b) in Proposition 3.2 and hence $\tau_{*F(Z)} \in \Gamma_{(Z,\tau_{dis})}$. □

In fact, the proof of the above proposition shows that:

Corollary 4.8. *Let $(S,\cdot,\tau_S)$ be a topological monoid. Then for each set Z, we have $\tau_{*F(Z)}$ is the product topology $\tau_{S\times Z}$ on $S\times Z$, where τ_Z in the definition of $\tau_{S\times Z}$ is the discrete topology on Z.* □

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