Configurations of conjugate permutations

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Devoted to the memory of Valentin D. Belousov (1925-1988)

Abstract. We describe some configurations of conjugate permutations which may be used as a mathematical model of some genetical processes and crystal growth.

1. Introduction

Let $Q = \{1, 2, 3, \ldots, n\}$ be a finite set. The set of all permutations of Q will be denoted by $\mathbb{S}_n$. The multiplication (composition) of permutations φ and ψ of Q is defined as $\varphi\psi(x) = \varphi(\psi(x))$. Permutations will be written in the form of cycles and cycles will be separated by points, e.g.

$$\varphi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 1 & 2 & 5 & 4 & 6 \end{pmatrix} = (123.45.6.)$$

By a *type* of a permutation $\varphi \in \mathbb{S}_n$ we mean the sequence

$$C(\varphi) = \{l_1, l_2, \ldots, l_n\},$$

where l_i denotes the number of cycles of the length i. Obviously,

$$\sum_{i=1}^{n} i \cdot l_i = n\,.$$

For example, for $\varphi = (132.45.6.)$ we have $C(\varphi) = \{1, 1, 1, 0, 0, 0\}$; for $\psi = (123456.)$ we obtain $C(\psi) = \{0, 0, 0, 0, 0, 1\}$.

As is well-known, two permutations $\varphi, \psi \in \mathbb{S}_n$ are *conjugate* if there exists a permutation $\rho \in \mathbb{S}_n$ such that

$$\rho\varphi\rho^{-1} = \psi. \tag{1}$$

2000 Mathematics Subject Classification: 05B15; 20N05
Keywords: permutation, conjugate permutation, stem-permutation, symmetric group, flock, telomere, configuration.

Theorem 1. (Theorem 5.1.3 in [1]) *Two permutations are conjugated if and only if they have the same type.* □

In this short note we find all solutions of (1), i.e., for a given φ and ψ we find all permutations ρ satisfying this equation, and describe some graphs connected with these solutions.

2. Solutions of the equation (1)

Let's consider the equation (1). If $\varphi = \psi = \varepsilon$, then as ρ we can take any permutation from $\mathbb{S}_n$. So, in this case (1) has $n!$ solutions.

If permutations φ and ψ are cyclic, then without loss of generality, we can assume that

$$\varphi = (1\,\varphi(1)\,\varphi^2(1)\,\varphi^3(1)\,\ldots\,\varphi^{n-1}(1).),$$
$$\psi = (1\,\psi(1)\,\psi^2(1)\,\psi^3(1)\,\ldots\,\psi^{n-1}(1).),$$

where $\varphi^0(1) = \varphi^n(1) = 1$ and $\psi^0(1) = \psi^n(1) = 1$. In this case for ρ_0 defined by

$$\rho_0(\varphi^i(1)) = \psi^i(1) = x_i, \quad i = 0, 1, \ldots, n-1, \tag{2}$$

we have

$$\rho_0\varphi\rho_0^{-1}(x_i) = \rho_0\varphi\rho_0^{-1}(\psi^i(1)) = \rho_0\varphi^{i+1}(1) = \psi^{i+1}(1) = \psi(\psi^i(1)) = \psi(x_i),$$

which shows that ρ_0 satisfies (1). Moreover, as is not difficult to see, each permutation of the form

$$\rho = \rho_0\varphi^i, \quad i = 0, 1, \ldots, n-1 \tag{3}$$

also satisfies this equation. There are no other solutions. So, in this case we have n different solutions.

In the general case when φ and ψ are decomposed into cycles of the length $k_1, k_2, \ldots, k_r$, i.e.,

$$\varphi = (a_{11}\,a_{12}\ldots a_{1k_1})\ldots(a_{r1}\ldots a_{rk_r}),$$
$$\psi = (b_{11}\,b_{12}\ldots b_{1k_1})\ldots(b_{r1}\ldots b_{rk_r}),$$

the solution ρ, according to [1], has the form

$$\beta = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1k_1} & \ldots & a_{r1} & \ldots & a_{rk_r} \\ b_{11} & b_{12} & \ldots & b_{1k_1} & \ldots & b_{r1} & \ldots & b_{rk_r} \end{pmatrix}, \tag{4}$$

where the first row contains all elements of φ, the second – elements of ψ written in the same order as in decompositions of φ and ψ into cycles. Replacing in φ the cycle $(a_{11}\, a_{12} \ldots a_{1k_1})$ by $(a_{12}\, a_{13} \ldots a_{1k_1}\, a_{11})$ we save the permutation φ but we obtain a new ρ. Similar to arbitrary cycles of φ and ψ. In this way we obtain all ρ satisfying (1).

Let's observe that the cycle $(a_{11}\, a_{12} \ldots a_{1k_1})$ gives k_1 possibilities for the construction ρ. From m cycles of the length k we can construct $m!\, k^m$ various ρ. So, in the case $C(\varphi) = C(\psi) = \{l_1, l_2, \ldots, l_n\}$ we can construct

$$N_\varphi = l_1! \cdot l_2! \cdot 2^{l_2} \cdot l_3! \cdot 3^{l_3} \cdot \ldots \cdot l_n! \cdot n^{l_n}$$

various ρ.

3. Configurations of conjugate permutations

As is well-known, any permutation φ of the set Q of order n can be decomposed into $r \leqslant n$ cycles of the length $k_1, k_2, \ldots, k_r$ with $k_1 + k_2 + \ldots + k_r = n$. We denote this fact by

$$Z = Z(\varphi) = [k_1, k_2, \ldots, k_r]$$

and assume that $k_1 \leqslant k_2 \leqslant \ldots \leqslant k_r$. $Z(\varphi)$ is called the *cyclic type* of φ. The set of all permutations of the set Q with the same cyclic type Z_i is denoted by F_i and is called a *flock*. Permutations belonging to the same flock are conjugate (Theorem 1). The number of flocks $F_i \subset \mathbb{S}_n$ is equal to the number of possible decompositions of n into a sum of natural numbers.

In each flock we select one permutation σ and call it a *stem-permutation*. For simplicity we can assume that elements of this permutation are written in the natural order.

Example 1. Let's consider the set $Q = \{1, 2, 3, 4, 5\}$. The number 5 has seven decompositions into a sum of natural numbers, so the set of all permutations of Q has seven flocks. Below we present these flocks and their stem-permutations.

$Z_1 : 5 = 5$	$\sigma = (12345.)$
$Z_2 : 5 = 1 + 4$	$\sigma = (1.2345.)$
$Z_3 : 5 = 2 + 3$	$\sigma = (12.345.)$
$Z_4 : 5 = 1 + 2 + 2$	$\sigma = (1.23.45.)$
$Z_5 : 5 = 1 + 1 + 3$	$\sigma = (1.2.345.)$
$Z_6 : 5 = 1 + 1 + 1 + 2$	$\sigma = (1.2.3.45.)$
$Z_7 : 5 = 1 + 1 + 1 + 1 = 1$	$\sigma = (1.2.3.4.5.) = \varepsilon$.

□

Let's consider an arbitrary flock $F_i \subset \mathbb{S}_n$ and its stem-permutation σ. For an arbitrary permutation $\varphi_0 \in F_i$ we define the sequence of permutations $\varphi_0, \varphi_1, \varphi_2, \ldots$ by putting

$$\varphi_{k+1} = \varphi_k \sigma \varphi_k^{-1}. \qquad (5)$$

Obviously all φ_k are in F_i. The set F_i is finite, so $\varphi_p = \varphi_s$ for some p and s.

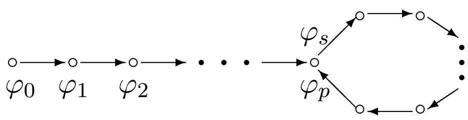


Fig. 1. The graph connected with the sequence (5).

The sequence $\varphi_1, \varphi_2, \varphi_3, \ldots$ can be initiated by various φ_0 because for fixed φ_1 and σ the equation $\varphi_1 = \varphi\sigma\varphi^{-1}$ has many solutions.

Let's denote by Φ_k the set of all possible solutions of the equation (5), where φ_{k+1} and σ are fixed. Let

$$\overline{\Phi}_k = \{\varphi \in \Phi_k \,:\, Z(\varphi) = Z(\sigma)\}.$$

In the case when $\overline{\Phi}_k$ has only one element the permutation φ_{k+1} is called *simple*. If $\overline{\Phi}_k$ is the empty set, then φ_{k+1} is called a *telomere* and is denoted by $\hat{\varphi}_{k+1}$. In the corresponding oriented graph a telomere is a vertex which is not preceded by another vertex.

The following theorem is obvious.

Theorem 2. *Let σ be a stem-permutation of a flock F_i. If $\varphi \in F_i$ is a telomere, then also $\psi = \sigma\varphi\sigma^{-1}$ is a telomere.* □

Two permutations $\varphi, \psi \in F_i \subset \mathbb{S}_n$ have the same *configuration* K if $\varphi_p = \psi_q$ for some natural p and q, where

$$\varphi_p = \varphi_{p-1}\sigma\varphi_{p-1}^{-1}, \ \ldots, \ \varphi_1 = \varphi\sigma\varphi^{-1},$$

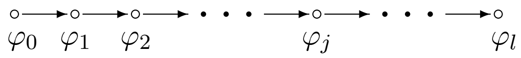
$$\psi_q = \psi_{q-1}\sigma\psi_{q-1}^{-1}, \ \ldots, \ \psi_1 = \psi\sigma\psi^{-1}$$

and σ is a stem-permutation from F_i.

4. A simple algorithm for determining configurations

1. In a given flock F_i we select a stem-permutation σ and one permutation $\varphi_0 \neq \sigma$. Using these two permutations and (5) we construct the sequence

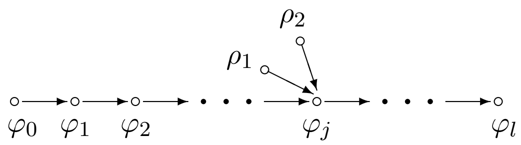
$\varphi_0, \varphi_1, \ldots, \varphi_l$, where $\varphi_l \neq \varphi_s$ for all $0 \leqslant s < l$ and $\varphi_{l+1} = \varphi_t$ for some $0 \leqslant t < l$. In this way we obtain the graph



2. For each φ_j from the above sequence, from all solutions of the equation

$$\rho\sigma\rho^{-1} = \varphi_j$$

we select these solutions $\rho \neq \varphi_{j-1}$ which are in F_i and attach them to the previous solutions as immediately preceding φ_j. In this way we obtain the configuration $K = \{\varphi_0, \varphi_1, \ldots, \varphi_l, \rho_1, \rho_2, \ldots\}$ and the graph



Next, for all new ρ_k attached to K we solve the equation $\rho\sigma\rho^{-1} = \rho_k$ and attach to K these solutions $\rho' \neq \rho_k$ which are in F_i. For this new ρ' we solve the equation $\rho\sigma\rho^{-1} = \rho'$ and so on. Since F_i is finite after some steps we obtain a telomere which completes this procedure.

5. Examples

Now we give some examples. We will consider the set $Q = \{1, 2, 3, 4, 5, 6\}$ and its permutations. For simplicity we consider the flock F_1 containing all cyclic permutations of Q and select $\sigma = (123456.)$ as a stem-permutation of F_1.

Example 2. If we choose $\varphi_0 = (125634.)$, then, according to (5), we obtain

$$\begin{aligned} \varphi_1 &= \varphi_0\sigma\varphi_0^{-1} = (163254.), \\ \varphi_2 &= \varphi_1\sigma\varphi_1^{-1} = (143625.), \\ \varphi_3 &= \varphi_2\sigma\varphi_2^{-1} = (163254.) = \varphi_1\,. \end{aligned}$$

Thus, the first step of our algorithm gives the configuration $K = \{\varphi_0, \varphi_1, \varphi_2\}$.

Now, for each $\varphi_i \in K$ we solve the equation $\rho\sigma\rho^{-1} = \varphi_i$ and add to K all solutions belonging to F_1.

The equation $\rho\sigma\rho^{-1} = \varphi_0$ is satisfied by the permutation $\rho_0 = (1.2.35.46.)$. So, according to (3), other solutions of this equation have the form

$$\begin{aligned}
\varphi_{01} &= \rho_0\sigma\ = (1.2.35.46.)(123456.)\ = (125436.),\\
\varphi_{02} &= \rho_0\sigma^2 = (1.2.35.46.)(135.246.) = (15.26.3.4.),\\
\varphi_{03} &= \rho_0\sigma^3 = (1.2.35.46.)(14.25.36.) = (165234.),\\
\varphi_{04} &= \rho_0\sigma^4 = (1.2.35.46.)(153.264.) = (13.24.5.6.),\\
\varphi_{05} &= \rho_0\sigma^5 = (1.2.35.46.)(165432.)\ = (145632.).
\end{aligned}$$

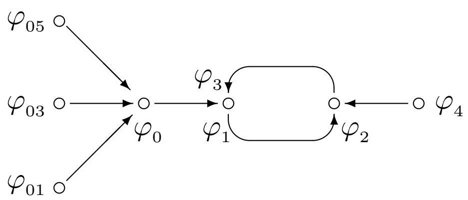
From these solutions only $\varphi_{01}, \varphi_{03}, \varphi_{05}$ are in F_1. We attach these solutions to K as the immediately preceding φ_0.

Next, we consider the equation $\rho\sigma\rho^{-1} = \varphi_1$. This equation has only one solution belonging to F_1. Since this solution coincides with ρ, we do not obtain permutations which should be added to K.

The equation $\rho\sigma\rho^{-1} = \varphi_2$ has only one solution $\rho = (145236.) \neq \varphi_1$ belonging to F_1. We denote it by φ_4 and add to K as the solution immediately preceding φ_2. At this instant we have the configuration (uncomplete)

$$K = \{\varphi_0, \varphi_1, \varphi_2, \varphi_{01}, \varphi_{03}, \varphi_{05}, \varphi_4\}$$

and the graph



Further we will work with the permutations φ_{01}, φ_{03}, φ_{05}, φ_4. Equations $\rho\sigma\rho^{-1} = \varphi_{0i}$, $i = 1, 3, 5$, do not have solutions belonging to F_i. So, φ_{01}, φ_{03}, φ_{05} are telomeres. We denote them by $\hat{\varphi}_{01}$, $\hat{\varphi}_{03}$, $\hat{\varphi}_{05}$.

The equation $\rho\sigma\rho^{-1} = \varphi_4$ has three solutions belonging to F_1. Namely,

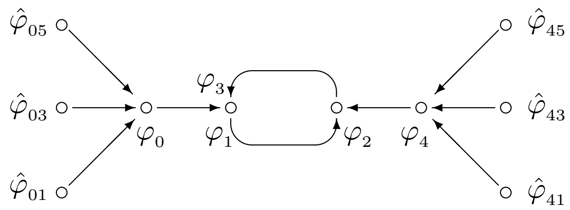
$$\begin{aligned}
\varphi_{41} &= \rho'\sigma\ = (1.6.24.35.)(123456.)\ = (143256.),\\
\varphi_{43} &= \rho'\sigma^3 = (1.6.24.35.)(14.25.36.) = (123654.),\\
\varphi_{45} &= \rho'\sigma^5 = (1.6.24.35.)(165432.)\ = (163452.).
\end{aligned}$$

Since equations $\rho\sigma\rho^{-1} = \varphi_{4j}$, $j = 1, 3, 5$, do not have solutions belonging to F_1, φ_{41}, φ_{43}, φ_{45} are telomeres.

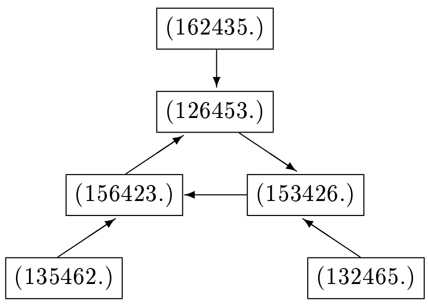
Summarizing the above we obtain the configuration

$$K = \{\varphi_0, \varphi_1, \varphi_2, \hat{\varphi}_{01}, \hat{\varphi}_{03}, \hat{\varphi}_{05}, \varphi_4, \hat{\varphi}_{41}, \hat{\varphi}_{43}, \hat{\varphi}_{45}\}$$

and the graph



Example 3. Using the same flock F_1 and the same σ but selecting another φ_0 we can obtain another configuration. For example by selecting $\varphi_0 = (162435.)$ we obtain the configuration K_2 presented by the following graph:



Remark. The flock F_1 has six configurations:

- K_1 and K_2 are described in the above examples,
- K_3 induced by $\varphi_0 = (125643.)$ contains 18 permutations,
- K_4 induced by $\varphi_0 = (135624.)$ contains 42 permutations,
- K_5 induced by $\varphi_0 = (136245.)$ contains 42 permutations,
- K_6 has only two permutations: σ and σ^{-1}.

Flocks K_4 and K_5 are isomorphic as graphs.

The set $\mathbb{S}_6$ is divided into 11 flocks.

The author does'nt know a general method that would allow to determine the number of configurations in each flock. Neither does he know how to quickly find a telomere using stem-permutations. It is also unknown how to check if two telomeres belong to the same configuration.

6. Conclusions

The results shown were inspired by some research in genetics. Some terminology (stem-permutation, telomere) was also drawn from genetics. The author thinks that the described method of configuration can be effectively used in chemistry in researching growth of crystals.

References

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Received May 8, 2010

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