ARO–quasigroups

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Abstract. In this paper the concept of ARO–quasigroup is introduced and some identities which are valid in a general ARO–quasigroup are proved. The "geometric" concepts of the midpoint, parallelogram and affine regular octagon are introduced in a general ARO–quasigroup. The geometric interpretation of some proved identities and introduced concepts is given in the quasigroup $\mathbb{C}\left(1+\frac{\sqrt{2}}{2}\right)$.

1. Definition and examples

A quasigroup $(Q,\cdot)$ will be called *ARO–quasigroup* if it satisfies the following identities of *idempotency* and *mediality*

$$aa = a, \tag{1}$$

$$ab\cdot cd = ac\cdot bd \tag{2}$$

and besides that the identity

$$ab\cdot b = ba\cdot a. \tag{3}$$

Example 1. Let $(G,+)$ be a commutative group in which there exists the automorphism φ which satisfies the identity

$$(\varphi\circ\varphi)(a)+(\varphi\circ\varphi)(a)-\varphi(a)-\varphi(a)-\varphi(a)-\varphi(a)+a=0,$$

which can be written in a simpler form

$$2(\varphi\circ\varphi)(a)-4\varphi(a)+a=0. \tag{4}$$

If the multiplication $\cdot$ on the set G is defined by the formula

$$ab = a+\varphi(b-a) \tag{5}$$

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we shall prove that $(G,\cdot)$ is ARO–quasigroup. For each $a,b\in G$ the equations $ax=b$ and $ya=b$, owing to (5), are equivalent to the equations

$$a+\varphi(x-a)=b, \qquad y+\varphi(a)-\varphi(y)=b. \tag{6}$$

The first equation has the unique solution $x=a+\varphi^{-1}(b-a)$, and out of the second equation it follows

$$\begin{aligned} 2(\varphi\circ\varphi)(y)-2\varphi(y) &= 2(\varphi\circ\varphi)(a)-2\varphi(b),\\ 2y-2\varphi(y) &= 2b-2\varphi(a). \end{aligned}$$

The addition of two last equations gives

$$2(\varphi\circ\varphi)(y)-4\varphi(y)+2y=2(\varphi\circ\varphi)(a)-2\varphi(a)-2\varphi(b)+2b,$$

i.e., owing to (4) the solution must have the form

$$y=2\varphi(a)-a-2\varphi(b)+2b. \tag{7}$$

Really, it is a solution of (6) because from (7), according to (4), we get

$$\begin{aligned} y-\varphi(y) &= 2\varphi(a)-a-2\varphi(b)+2b-\varphi(2\varphi(a)-a-2\varphi(b)+2b)\\ &= 2(\varphi\circ\varphi)(b)-4\varphi(b)+2b-(2(\varphi\circ\varphi)(a)-3\varphi(a)+a)=b-\varphi(a). \end{aligned}$$

We have proved that $(G,\cdot)$ is a quasigroup. Its idempotency is obvious by (5). According to (5) we also get

$$\begin{aligned} ab\cdot cd &= ab+\varphi(cd-ab)=a+\varphi(b-a)+\varphi(c+\varphi(d-c)-a-\varphi(b-a))\\ &= a-2\varphi(a)+(\varphi\circ\varphi)(a)+\varphi(b)-(\varphi\circ\varphi)(b)+\varphi(c)-(\varphi\circ\varphi)(c)+(\varphi\circ\varphi)(d). \end{aligned}$$

The symmetry of the obtained expression by b and c proves the mediality (2). By (5) it follows

$$\begin{aligned} ab\cdot b &= ab+\varphi(b-ab)=a+\varphi(b-a)+\varphi(b-a-\varphi(b-a))\\ &= (\varphi\circ\varphi)(a)-2\varphi(a)+a+2\varphi(b)-(\varphi\circ\varphi)(b), \end{aligned}$$

and analogously

$$ba\cdot a=2\varphi(a)-(\varphi\circ\varphi)(a)+(\varphi\circ\varphi)(b)-2\varphi(b)+b,$$

whence owing to (4)

$$ab\cdot b-ba\cdot a=2(\varphi\circ\varphi)(a)-4\varphi(a)+a-(2(\varphi\circ\varphi)(b)-4\varphi(b)+b)=0,$$

i.e., the identity (3) is valid. □

Example 2. Let $(F, +, \cdot)$ be a field. If the equation

$$2q^2 - 4q + 1 = 0 \tag{8}$$

has the solution q in F and if the operation $*$ on F is defined by the formula

$$a * b = (1 - q)a + qb. \tag{9}$$

then $\varphi(a) = qa$ obviously defines an automorphism of a commutative group $(F, +)$. As the equality (8) is valid it implies that the equality (4) holds for all $a \in F$. However, (9) can be also written in the form

$$a * b = a + \varphi(b - a)$$

and by Example 1, $(F, *)$ is ARO–quasigroup. □

Example 3. Let $(\mathbb{C}, +, \cdot)$ be a field of complex numbers and $*$ binary operation on $\mathbb{C}$ defined by (9), where q is the solution of the equation (8), i.e., $q = 1 + \frac{\sqrt{2}}{2}$ or $q = 1 - \frac{\sqrt{2}}{2}$. According to Example 2 $(\mathbb{C}, *)$ is ARO–quasigroup. For example, let $q = 1 + \frac{\sqrt{2}}{2}$. The obtained quasigroup has a nice geometric interpretation, which justifies the studying ARO–quasigroups and defining the geometric concepts in them. Let us consider the set $\mathbb{C}$ as the set of the points in the Euclidean plane. For the different points a and b the equality (9) can be written as

$$\frac{a * b - a}{b - a} = q$$

which means that the points a, b, $a * b$ determine the quotient ratio q. The operation $*$ is presented in the Figure 1 where, instead of $a * b$, we shall shortly write ab, and in the sequel we will use this notation in all figures.

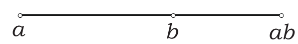

Figure 1.

The identity (3) is illustrated in the Figure 2.

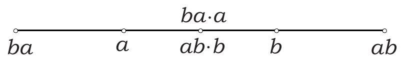

Figure 2.

□

2. The basic properties

The immediate consequences of the identities (1) and (2) are the identities of *elasticity*, *left* and *right distributivity*

$$ab \cdot a = a \cdot ba, \tag{10}$$

$$a \cdot bc = ab \cdot ac, \tag{11}$$

$$ab \cdot c = ac \cdot bc \tag{12}$$

Let us prove the following theorem.

Theorem 1. *In the ARO–quasigroup* $(Q, \cdot)$ *the following identities*

$$(ab \cdot b)a = (a \cdot ab)b, \tag{13}$$

$$(ab \cdot c)c = (c \cdot ba)a, \tag{14}$$

$$(ab \cdot b)b = (b \cdot ba)a, \tag{15}$$

$$(ab \cdot ba)c = (ac \cdot ca)b, \tag{16}$$

$$(ab \cdot ba)a = ab, \tag{17}$$

$$(ab \cdot ba)c \cdot c = cb \cdot a, \tag{18}$$

$$(ab \cdot ba)b \cdot b = ba, \tag{19}$$

$$(ab \cdot ba)b = ba \cdot ab, \tag{20}$$

$$(ab \cdot ba) \cdot ca = ac \cdot b \tag{21}$$

are valid.

Proof. Firstly we get

$$(ab \cdot b)a \overset{(12)}{=} (ab \cdot a) \cdot ba \overset{(10)}{=} (a \cdot ba) \cdot ba \overset{(2)}{=} ab \cdot (ba \cdot a) \overset{(3)}{=} ab \cdot (ab \cdot b) \overset{(12)}{=} (a \cdot ab)b,$$

$$(ab \cdot c)c \overset{(3)}{=} (c \cdot ab) \cdot ab \overset{(2)}{=} ca \cdot (ab \cdot b) \overset{(3)}{=} ca \cdot (ba \cdot a) \overset{(12)}{=} (c \cdot ba)a,$$

$$\begin{aligned}(ab\cdot ba)c &\overset{(12)}{=} (ab\cdot c)(ba\cdot c) \overset{(11)}{=} (ab\cdot c)(ba)\cdot(ab\cdot c)c \overset{(2)}{=} (ab\cdot b)(ca)\cdot(ab\cdot c)c\\ &\overset{(2)}{=} (ab\cdot b)(ab\cdot c)\cdot(ca\cdot c) \overset{(11)}{=} (ab\cdot bc)(ca\cdot c) \overset{(2)}{=} (ab\cdot ca)(bc\cdot c)\\ &\overset{(3)}{=} (ab\cdot ca)(cb\cdot b) \overset{(2)}{=} (ab\cdot cb)(ca\cdot b) \overset{(12)}{=} (ac\cdot b)(ca\cdot b) \overset{(12)}{=} (ac\cdot ca)b,\end{aligned}$$

so the identities (13), (14) and (16) hold. From (14) using $c=b$ the identity (15) follows, and using $c=a$ from (16) owing to (1) the identity (17) follows. Further we get

$$\begin{aligned}(ab{\cdot}ba)c{\cdot}c &\overset{(12)}{=} (ab\cdot c)c{\cdot}(ba\cdot c)c \overset{(3)}{=} (c\cdot ab)(ab){\cdot}(ba\cdot c)c \overset{(2)}{=} (ca)(ab\cdot b){\cdot}(ba\cdot c)c\\ &\overset{(3)}{=} (ca)(ba\cdot a){\cdot}(ba\cdot c)c \overset{(12)}{=} (c\cdot ba)a{\cdot}(ba\cdot c)c \overset{(2)}{=} (c\cdot ba)(ba\cdot c)\cdot ac\\ &\overset{(16)}{=} (c\cdot ac)(ac\cdot c)\cdot ba \overset{(3)}{=} (c\cdot ac)(ca\cdot a)\cdot ba \overset{(10)}{=} (ca\cdot c)(ca\cdot a)\cdot ba\\ &\overset{(11)}{=} (ca\cdot ca)\cdot ba \overset{(1)}{=} ca\cdot ba \overset{(12)}{=} cb\cdot a,\end{aligned}$$

i.e., the equality (18) is valid, wherefrom with $c=b$ because of (1) it follows (19). Finally, we obtain

$$(ab\cdot ba)b \overset{(12)}{=} (ab\cdot b)(ba\cdot b) \overset{(3)}{=} (ba\cdot a)(ba\cdot b) \overset{(11)}{=} ba\cdot ab,$$

$$(ab{\cdot}ba){\cdot}ca \overset{(2)}{=} (ab{\cdot}c)(ba{\cdot}a) \overset{(3)}{=} (ab{\cdot}c)(ab{\cdot}b) \overset{(11)}{=} ab{\cdot}cb \overset{(12)}{=} ac{\cdot}b. \qquad \square$$

3. Midpoints and parallelograms

Let $(Q,\cdot)$ be ARO–quasigroup. The elements of the set Q will be called *points.* The geometric presentation in the Figure 2 leads to the following definition. For any two points a and b the point c, given by the equalities

$$c = a*b = ab\cdot b \overset{(3)}{=} ba\cdot a, \tag{22}$$

will be called the *midpoint* of the points a and b.

Theorem 2. *If the operation $*$ on the set Q is defined by the formula* (22), *then $(Q,*)$ is idempotent medial commutative quasigroup.*

Proof. The equations $a * x = b$ and $y * a = b$, which according to (22) can be written as $xa \cdot a = b$ and $ya \cdot a = b$, are uniquely solvable for x and y for each $a, b \in Q$. Commutativity and idempotency of the operation $*$ are obvious, and mediality follows by means of (2) like this:

$$\begin{aligned}(a * b) * (c * d) &= (ab \cdot b)(cd \cdot d) \cdot (cd \cdot d) = (ab \cdot cd)(bd) \cdot (cd \cdot d) \\ &= (ac \cdot bd)(cd) \cdot (bd \cdot d) = (ac \cdot c)(bd \cdot d) \cdot (bd \cdot d) \\ &= (a * c) * (b * d). \qquad \square\end{aligned}$$

We shall say that the points a, b, c, d are the *vertices of a parallelogram* and we shall write $Par(a, b, c, d)$ if $a * c = b * d$. If $a * c = b * d = o$, we shall say that the point o is the *center* of that parallelogram and write $Par_o(a, b, c, d)$.

Theorem 3. *(Q, Par) is a parallelogram space, i.e., the following properties are valid:*

$(P1)$ *For any points a, b, c there is the unique point d such that $Par(a, b, c, d)$.*

$(P2)$ *For any cyclic permutation (e, f, g, h) of (a, b, c, d) or of (d, c, b, a) from $Par(a, b, c, d)$ follows $Par(e, f, g, h)$.*

$(P3)$ *From $Par(a, b, c, d)$ and $Par(c, d, e, f)$ follows $Par(a, b, f, e)$.*

Proof. The statement $Par(a, b, c, d)$ is according to (22) equivalent to the equality $ac \cdot c = db \cdot b$, which is unique solvable by d, so the property (P1) is valid. The property (P2) is the consequence of the commutativity of the operation $*$. It remains to prove the property (P3). From $Par(a, b, c, d)$ and $Par(c, d, e, f)$ it follows $a * c = b * d$ and $c * e = d * f$. By means of the mediality and commutativity of the operation $*$ we get

$$\begin{aligned}(a * f) * (c * f) &= (a * c) * (f * f) = (b * d) * (f * f) = (b * f) * (d * f) \\ &= (b * f) * (c * e) = (b * f) * (e * c) = (b * e) * (f * c) \\ &= (b * e) * (c * f),\end{aligned}$$

wherefrom we get $a * f = b * e$, i.e., $Par(a, b, f, e)$. $\square$

4. Affine–regular octagon

Now we are going to introduce the concept of the affine regular octagon in a general ARO–quasigroup. Firstly, we will prove the theorem which will lead to the definition of the mentioned concept.

Theorem 4. *In a cyclical sequence from eight equalities* $a_i a_{i+1} = a_{i+3} a_{i+2}$ $(i = 1,2,3,4,\ 5,6,7,8)$*, where indexes are taken modulo* 8 *from the set* $\{1,2,3,4,\ 5,\ 6,7,8\}$*, each five adjacent equalities imply the remaining three equalities.*

Proof. It is sufficient to prove that the equalities

$$a_1a_2 = a_4a_3, \tag{23}$$

$$a_2a_3 = a_5a_4, \tag{24}$$

$$a_3a_4 = a_6a_5, \tag{25}$$

$$a_4a_5 = a_7a_6, \tag{26}$$

$$a_5a_6 = a_8a_7, \tag{27}$$

imply the equality

$$a_6a_7 = a_1a_8. \tag{28}$$

Firstly, let us prove that from the equality (23)–(25) the equality

$$a_1a_3 = a_6a_4, \tag{29}$$

follows, and in the same manner (by the substitution $i \to i+2$) from equalities (25) – (27) the equality

$$a_3a_5 = a_8a_6 \tag{30}$$

follows. Really, we get successively

$$\begin{aligned}(a_1a_3\cdot a_5)a_4 &\overset{(12)}{=} (a_1a_5\cdot a_3a_5)a_4 \overset{(12)}{=} (a_1a_4\cdot a_5a_4)(a_3a_4\cdot a_5a_4)\\ &\overset{(24)}{=} (a_1a_4\cdot a_2a_3)(a_3a_4\cdot a_5a_4) \overset{(2)}{=} (a_1a_2\cdot a_4a_3)(a_3a_4\cdot a_5a_4)\\ &\overset{(23)}{=} (a_4a_3\cdot a_4a_3)(a_3a_4\cdot a_5a_4) \overset{(1)}{=} a_4a_3\cdot(a_3a_4\cdot a_5a_4)\\ &\overset{(2)}{=} (a_4\cdot a_3a_4)(a_3\cdot a_5a_4) \overset{(10)}{=} (a_4a_3\cdot a_4)(a_3\cdot a_5a_4)\\ &\overset{(2)}{=} (a_4a_3\cdot a_3)(a_4\cdot a_5a_4) \overset{(3)}{=} (a_3a_4\cdot a_4)(a_4\cdot a_5a_4)\\ &\overset{(10)}{=} (a_3a_4\cdot a_4)(a_4a_5\cdot a_4) \overset{(12)}{=} (a_3a_4\cdot a_4a_5)a_4\\ &\overset{(25)}{=} (a_6a_5\cdot a_4a_5)a_4 \overset{(12)}{=} (a_6a_4\cdot a_5)a_4,\end{aligned}$$

wherefrom the equality (29) follows. Now, we can also prove the equality (28), which follows from

$$a_1a_8 \cdot a_6 \overset{(12)}{=} a_1a_6 \cdot a_8a_6 \overset{(30)}{=} a_1a_6 \cdot a_3a_5 \overset{(2)}{=} a_1a_3 \cdot a_6a_5 \overset{(29)}{=} a_6a_4 \cdot a_6a_5$$
$$\overset{(11)}{=} a_6 \cdot a_4a_5 \overset{(26)}{=} a_6 \cdot a_7a_6 \overset{(10)}{=} a_6a_7 \cdot a_6. \qquad \square$$

We shall say that a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8 are the *vertices of an affine–regular octagon* and we shall write $\mathrm{ARO}(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8)$ if any five adjacent, and then all eight, equalities from eight equalities $a_ia_{i+1} = a_{i+3}a_{i+2}$ $(i = 1, 2, 3, 4, 5, 6, 7, 8)$ are valid (Figure 3).

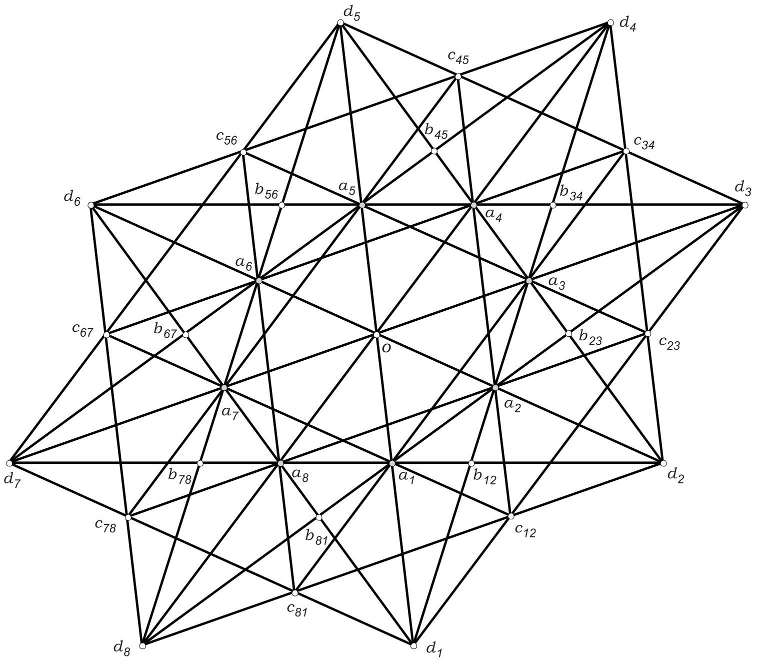

Figure 3.

Corollary 1. *If* $(i_1, i_2, i_3, i_4, i_5, i_6, i_7, i_8)$ *is any cyclic permutation of* $(1, 2, 3, 4, 5, 6, 7, 8)$ *or of* $(8, 7, 6, 5, 4, 3, 2, 1)$, *then* $ARO(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8)$ *implies* $ARO(a_{i_1}, a_{i_2}, a_{i_3}, a_{i_4}, a_{i_5}, a_{i_6}, a_{i_7}, a_{i_8})$.

Corollary 2. *If the statement* $ARO(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8)$ *holds, then for each* $i \in \{1, 2, 3, 4, 5, 6, 7, 8\}$ *the statement* $a_ia_{i+2} = a_{i+5}a_{i+3}$ *also holds.*

Corollary 3. *Affine–regular octagon is uniquely determined by any three adjacent vertices.* □

Theorem 5. *If the statement* $ARO(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8)$ *is valid, then for each* $i \in \{1,2,3,4,5,6,7,8\}$ *we have*

$$a_{i+1}a_{i+2} \cdot a_{i+2}a_{i+1} = a_{i+4}a_i, \quad a_{i+3}a_{i+2} \cdot a_{i+2}a_{i+3} = a_i a_{i+4}, \tag{31}$$

$$a_{i+4}a_i \cdot a_{i+1} = a_{i+1}a_{i+2} = a_{i+4}a_{i+3}, \quad a_i a_{i+4} \cdot a_{i+3} = a_{i+3}a_{i+2} = a_i a_{i+1}. \tag{32}$$

Proof. The proof of the second equality (31) follows from the proof of the first one (31) by the substitution of indexes $i \leftrightarrow i+4$, $i+1 \leftrightarrow i+3$. Because of Corollary 1 it is sufficient to prove, for example, the equality $a_2a_3 \cdot a_3a_2 = a_5a_1$. We get successively

$$\begin{aligned}(a_2a_3 \cdot a_3a_2)a_2 &\stackrel{(17)}{=} a_2a_3 \stackrel{(1)}{=} a_2a_3 \cdot a_2a_3 \stackrel{(24)}{=} a_5a_4 \cdot a_2a_3 \\ &\stackrel{(2)}{=} a_5a_2 \cdot a_4a_3 \stackrel{(23)}{=} a_5a_2 \cdot a_1a_2 \stackrel{(12)}{=} a_5a_1 \cdot a_2,\end{aligned}$$

so $a_2a_3 \cdot a_3a_2 = a_5a_1$ follows. The first equalities in (32) are obtained by multiplication the equalities (31) with a_{i+1} respectively a_{i+3} because of the identity (17), and other equalities are taken from the definition of the relation ARO. □

Theorem 6. *Let the statement* $ARO(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8)$ *be valid. There is the point* o *such that for each* $i \in \{1,2,3,4,5,6,7,8\}$ *the equalities*

$$(a_{i+1}a_i \cdot a_ia_{i+1})\, a_{i+2} = o, \quad (a_{i+1}a_{i+2} \cdot a_{i+2}a_{i+1})\, a_i = o \tag{33}$$

are valid, where indexes are taken modulo 8.

Proof. By (16) the mutual equivalence of the equalities (33) hold.

If $o = (a_2a_3 \cdot a_3a_2)a_1$, then $o = (a_2a_1 \cdot a_1a_2)a_3$. By Corollary 1 it is sufficient to prove the equality $o = (a_3a_4 \cdot a_4a_3)a_2$. We get

$$\begin{aligned}(a_3a_4 \cdot a_4a_3)a_2 &\stackrel{(23)}{=} (a_3a_4 \cdot a_1a_2)a_2 \stackrel{(12)}{=} (a_3a_4 \cdot a_2)(a_1a_2 \cdot a_2) \\ &\stackrel{(3)}{=} (a_3a_4 \cdot a_2)(a_2a_1 \cdot a_1) \stackrel{(2)}{=} (a_3a_4 \cdot a_2a_1) \cdot a_2a_1 \\ &\stackrel{(3)}{=} (a_2a_1 \cdot a_3a_4) \cdot a_3a_4 \stackrel{(2)}{=} (a_2a_1 \cdot a_3)(a_3a_4 \cdot a_4) \\ &\stackrel{(3)}{=} (a_2a_1 \cdot a_3)(a_4a_3 \cdot a_3) \stackrel{(2)}{=} (a_2a_1 \cdot a_4a_3)a_3 \\ &\stackrel{(23)}{=} (a_2a_1 \cdot a_1a_2)a_3 = o.\end{aligned}$$

□

The point o from Theorem 6 will be called the *center* of the affine–regular octagon $(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8)$ and it will be written in the form $\mathrm{ARO}_o(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8)$.

Theorem 7. *With hypotheses of Theorem* 6 *for each* $i \in \{1, 2, 3, 4, 5, 6, 7, 8\}$ *the equalities*

$$o = a_i * a_{i+4} = a_i a_{i+4} \cdot a_{i+4}, \tag{34}$$

$$a_{i+1}a_{i+2} \cdot a_i = o \cdot a_{i+1}a_i, \quad a_{i+1}a_i \cdot a_{i+2} = o \cdot a_{i+1}a_{i+2}, \tag{35}$$

$$oa_i = a_i a_{i+2} \cdot a_{i+1}, \quad oa_{i+2} = a_{i+2}a_i \cdot a_{i+1} \tag{36}$$

are valid.

Proof. We get

$$a_i * a_{i+4} \overset{(22)}{=} a_{i+4}a_i \cdot a_i \overset{(31)}{=} (a_{i+1}a_{i+2} \cdot a_{i+2}a_{i+1})a_i \overset{(33)}{=} o,$$

$$\begin{aligned} a_{i+1}a_{i+2} \cdot a_i &\overset{(17)}{=} (a_{i+1}a_{i+2} \cdot a_{i+2}a_{i+1})a_{i+1} \cdot a_i \\ &\overset{(12)}{=} (a_{i+1}a_{i+2} \cdot a_{i+2}a_{i+1})a_i \cdot a_{i+1}a_i \overset{(33)}{=} o \cdot a_{i+1}a_i, \end{aligned}$$

$$oa_i \overset{(33)}{=} (a_{i+1}a_{i+2} \cdot a_{i+2}a_{i+1})a_i \cdot a_i \overset{(18)}{=} a_i a_{i+2} \cdot a_{i+1}. \qquad \square$$

In the previous proof the equivalence of the equations (33) and (34) is proved, therefore the center of an affine–regular octagon can be also characterized by (34).

5. The determination of the affine–regular octagon

The statements of the unique determination of the affine regular octagon will be proved in this chapter.

Theorem 8. *Affine–regular octagon is uniquely determined by any three of its vertices.*

Proof. By Corollary 1 and 3 it is sufficient to prove only the following statements

(i) The vertices a_1, a_2, a_4 uniquely determine the vertex a_3. This statement is obvious from the equalities (23).

(ii) The vertices a_1, a_2, a_5 or a_1, a_3, a_5 uniquely determine the vertex a_3, respectively a_2. Indeed, let o is the point such that $o = a_5a_1 \cdot a_1$, and then a_3 respectively a_2 the point such that $oa_1 = a_1a_3 \cdot a_2$, and a_4 the point such that $a_1a_2 = a_4a_3$. It should be proved the equality $a_2a_3 = a_5a_4$. It is the consequence of the following consideration:

$$\begin{aligned}
(a_2a_3 \cdot a_2a_3)(a_4a_3) \cdot a_4a_3 &\overset{(1)}{=} (a_2a_3 \cdot a_4a_3) \cdot a_4a_3 \overset{(3)}{=} (a_4a_3 \cdot a_2a_3) \cdot a_2a_3 \\
&\overset{(2)}{=} (a_4a_3 \cdot a_2)(a_2a_3 \cdot a_3) \overset{(3)}{=} (a_4a_3 \cdot a_2)(a_3a_2 \cdot a_2) \\
&\overset{(12)}{=} (a_4a_3 \cdot a_3a_2)a_2 = (a_1a_2 \cdot a_3a_2)a_2 \overset{(12)}{=} (a_1a_3 \cdot a_2)a_2 \\
&= oa_1 \cdot a_2 = (a_5a_1 \cdot a_1)a_1 \cdot a_2 \\
&\overset{(12)}{=} (a_5a_2 \cdot a_1a_2)(a_1a_2) \cdot (a_1a_2) \\
&= (a_5a_2 \cdot a_4a_3)(a_4a_3) \cdot (a_4a_3) \\
&\overset{(2)}{=} (a_5a_4 \cdot a_2a_3)(a_4a_3) \cdot (a_4a_3).
\end{aligned}$$

(iii) The vertices a_1, a_3, a_6 uniquely determine the vertex a_2. Really, let a_4 be a point such that $a_1a_3 = a_6a_4$, then a_2 be a point such that $a_1a_2 = a_4a_3$, and a_5 the point such that $a_2a_3 = a_5a_4$. It should be proved the equality $a_3a_4 = a_6a_5$, which follows from this:

$$\begin{aligned}
(a_3a_4 \cdot a_4a_5)a_4 &\overset{(12)}{=} (a_3a_4 \cdot a_4)(a_4a_5 \cdot a_4) \overset{(10)}{=} (a_3a_4 \cdot a_4)(a_4 \cdot a_5a_4) \\
&\overset{(3)}{=} (a_4a_3 \cdot a_3)(a_4 \cdot a_5a_4) \overset{(2)}{=} (a_4a_3 \cdot a_4)(a_3 \cdot a_5a_4) \\
&\overset{(10)}{=} (a_4 \cdot a_3a_4)(a_3 \cdot a_5a_4) \overset{(2)}{=} a_4a_3 \cdot (a_3a_4 \cdot a_5a_4) \\
&\overset{(12)}{=} a_4a_3 \cdot (a_3a_5 \cdot a_4) \overset{(1)}{=} (a_4a_3 \cdot a_4a_3)(a_3a_5 \cdot a_4) \\
&= (a_1a_2 \cdot a_4a_3)(a_3a_5 \cdot a_4) \overset{(2)}{=} (a_1a_4 \cdot a_2a_3)(a_3a_5 \cdot a_4) \\
&= (a_1a_4 \cdot a_5a_4)(a_3a_5 \cdot a_4) \overset{(12)}{=} (a_1a_5 \cdot a_4)(a_3a_5 \cdot a_4) \\
&\overset{(12)}{=} (a_1a_5 \cdot a_3a_5)a_4 \overset{(12)}{=} (a_1a_3 \cdot a_5)a_4 \\
&= (a_6a_4 \cdot a_5)a_4 \overset{(12)}{=} (a_6a_5 \cdot a_4a_5)a_4.
\end{aligned}$$

□

If the statement $\mathrm{ARO}(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8)$ hold, then two vertices of the form a_i and a_{i+4} are said to be *opposite vertices* of the considered affine–regular octagon.

Theorem 9. *Affine–regular octagon is uniquely determined by its center and by any two of its vertices which are not opposite.*

Proof. (i) The center o and vertices a_1, a_2 respectively the vertices a_1, a_3 uniquely determine the remaining vertices. Let a_3 respectively a_2 be a point such that $oa_1 = a_1a_3 \cdot a_2$, then a_4 be a point such that $a_1a_2 = a_4a_3$, and a_5 be a point such that $o = a_5a_1 \cdot a_1$. It should be proved $a_2a_3 = a_5a_4$, and the proof is the same as the proof of the part (ii) of the proof Theorem 8.

(ii) The center o and the vertices a_2, a_5 uniquely determine the remaining vertices. Let a_1 be a point such that $o = a_1a_5 \cdot a_5$, and a_3 point such that $oa_1 = a_1a_3 \cdot a_2$, and a_4 point such that $a_1a_2 = a_4a_3$. Further the proof is the same as in a previous case. □

5. Some new associated affine–regular octagons

In this chapter we are going to consider some new octagons whose vertices can be obtained by means of the vertices of the initial octagon.

Equal products from the definition of the affine–regular octagon will be labelled like this

$$a_ia_{i+1} = b_{i+1,i+2} = a_{i+3}a_{i+2}, \tag{37}$$

where the indexes will be always taken mod 8 from the set $\{1, 2, 3, 4, 5, 6, 7, 8\}$. On the base of the proof of Theorem 4 according to Corollary 1 it follows that there exists the point $c_{i+2,i+3}$ such that

$$a_ia_{i+2} = c_{i+2,i+3} = a_{i+5}a_{i+3}. \tag{38}$$

Besides that, let

$$d_i = a_{i+4}a_i. \tag{39}$$

With these labels the equalities (31) and (32) can be written in the form

$$b_{i+2,i+3}\, b_{i,i+1} = d_i, \quad b_{i,i+1}\, b_{i+2,i+3} = d_{i+3}, \tag{40}$$

$$d_i\, a_{i+1} = b_{i+2,i+3}, \quad d_{i+3}\, a_{i+2} = b_{i,i+1}, \tag{41}$$

where the indexes in the second equalities in (40) and (41) are reduced for 1. The equalities (31) can also be written in the form

$$d_i = a_{i+1}\, a_{i+2} \cdot a_{i+2}\, a_{i+1}, \quad d_{i+2} = a_{i+1}\, a_i \cdot a_i\, a_{i+1}, \tag{42}$$

and the equalities (33) can be written in this shortened form:

$$d_ia_i = o. \tag{43}$$

The equalities (35) and (36) can also be written as the equalities

$$b_{i+2,i+3}\,a_i = o\,b_{i-1,i}, \quad b_{i-1,i}\,a_{i+2} = o\,b_{i+2,i+3}, \tag{44}$$

$$o\,a_i = c_{i+2,i+3}\,a_{i+1}, \quad o\,a_{i+2} = c_{i-1,i}\,a_{i+1}. \tag{45}$$

Let us prove some more similar equalities. We get for example:

$$d_1a_3 \overset{(42)}{=} (a_2a_3 \cdot a_3a_2)a_3 \overset{(20)}{=} a_3a_2 \cdot a_2a_3 \overset{(42)}{=} d_4,$$

and generally the equalities

$$d_i\,a_{i+2} = d_{i+3}, \quad d_i\,a_{i-2} = d_{i-3} \tag{46}$$

are valid. Due to the example

$$d_1a_2 \overset{(42)}{=} (a_2a_3 \cdot a_3a_2)a_2 \overset{(17)}{=} a_2a_3 \overset{(37)}{=} b_{34},$$

the general equalities

$$d_i\,a_{i+1} = b_{i+2,i+3}, \quad d_i\,a_{i-1} = b_{i-3,i-2} \tag{47}$$

hold. Let us prove for example

$$c_{12}c_{23} \overset{(38)}{=} a_4a_2 \cdot a_5a_3 \overset{(2)}{=} a_4a_5 \cdot a_2a_3 \overset{(37)}{=} a_4a_5 \cdot a_5a_4 \overset{(42)}{=} d_3$$

and generally,

$$c_{i,i+1}\,c_{i+1,i+2} = d_{i+2}, \quad c_{i+1,i+2}\,c_{i,i+1} = d_i. \tag{48}$$

On the base of the equalities (37) and (48) we get for example

$$b_{12}b_{23} \overset{(37)}{=} a_3a_2 \cdot a_4a_3 \overset{(2)}{=} a_3a_4 \cdot a_2a_3 \overset{(37)}{=} b_{45}b_{34},$$

$$c_{12}c_{23} \overset{(48)}{=} d_3 = c_{45}c_{34},$$

i.e., generally we have $b_{i,i+1}b_{i+1,i+2} = b_{i+3,i+4}b_{i+2,i+3}$ and $c_{i,i+1}c_{i+1,i+2} = c_{i+3,i+4}c_{i+2,i+3}$, which proves the statements

$$ARO(b_{12}, b_{23}, b_{34}, b_{45}, b_{56}, b_{67}, b_{78}, b_{81}), \tag{49}$$

$$ARO(c_{12}, c_{23}, c_{34}, c_{45}, c_{56}, c_{67}, c_{78}, c_{81}). \tag{50}$$

The proof of the statement

$$ARO(d_1, d_2, d_3, d_4, d_5, d_6, d_7, d_8) \tag{51}$$

is more complicated. We get for example

$$\begin{aligned} d_1d_2 &\overset{(46)}{=} d_4a_2 \cdot d_5a_3 \overset{(2)}{=} d_4d_5 \cdot a_2a_3 \overset{(37)}{=} d_4d_5 \cdot a_5a_4 \overset{(2)}{=} d_4a_5 \cdot d_5a_4 \overset{(47)}{=} b_{67}b_{23} \\ &\overset{(47)}{=} d_1a_8 \cdot d_8a_1 \overset{(2)}{=} d_1d_8 \cdot a_8a_1 \overset{(37)}{=} d_1d_8 \cdot a_3a_2 \overset{(2)}{=} d_1a_3 \cdot d_8a_2 \overset{(46)}{=} d_4d_3. \end{aligned}$$

All three affine–regular octagons (52)–(54) have the center o because we get for example

$$\begin{aligned} b_{12} * b_{56} &= b_{12}b_{56} \cdot b_{56} \overset{(37)}{=} (a_3a_2 \cdot a_7a_6) \cdot a_7a_6 \overset{(2)}{=} (a_3a_7 \cdot a_2a_6) \cdot a_7a_6 \\ &\overset{(2)}{=} (a_3a_7 \cdot a_7)(a_2a_6 \cdot a_6) = (a_3 * a_7)(a_2 * a_6) \overset{(34)}{=} o\,o \overset{(1)}{=} o, \end{aligned}$$

$$\begin{aligned} c_{12} * c_{56} &= c_{12}c_{56} \cdot c_{56} \overset{(38)}{=} (a_4a_2 \cdot a_8a_6) \cdot a_8a_6 \overset{(2)}{=} (a_4a_8 \cdot a_2a_6) \cdot a_8a_6 \\ &\overset{(2)}{=} (a_4a_8 \cdot a_8)(a_2a_6 \cdot a_6) = (a_4 * a_8)(a_2 * a_6) \overset{(34)}{=} o\,o \overset{(1)}{=} o, \end{aligned}$$

$$\begin{aligned} d_1 * d_5 &= d_1d_5 \cdot d_5 \overset{(42)}{=} (a_2a_3 \cdot a_3a_2)(a_6a_7 \cdot a_7a_6) \cdot (a_6a_7 \cdot a_7a_6) \\ &\overset{(2)}{=} (a_2a_3 \cdot a_6a_7)(a_3a_2 \cdot a_7a_6) \cdot (a_6a_7 \cdot a_7a_6) \\ &\overset{(2)}{=} (a_2a_6 \cdot a_3a_7)(a_3a_7 \cdot a_2a_6) \cdot (a_6a_7 \cdot a_7a_6) \\ &\overset{(2)}{=} (a_2a_6 \cdot a_3a_7)(a_6a_7) \cdot (a_3a_7 \cdot a_2a_6)(a_7a_6) \\ &\overset{(2)}{=} (a_2a_6 \cdot a_6)(a_3a_7 \cdot a_7) \cdot (a_3a_7 \cdot a_7)(a_2a_6 \cdot a_6) \\ &= (a_2 * a_6)(a_3 * a_7) \cdot (a_3 * a_7)(a_2 * a_6) \overset{(34)}{=} o\,o \cdot o\,o \overset{(1)}{=} o. \end{aligned}$$

A numerous parallelograms are related to the affine–regular octagon. So, for example we get the equalities

$$\begin{aligned} a_1 * a_2 &= a_2a_1 \cdot a_1 \overset{(21)}{=} (a_2a_1 \cdot a_1a_2) \cdot a_1a_2 \overset{(37)}{=} (a_2a_1 \cdot a_4a_3) \cdot a_4a_3 \\ &\overset{(2)}{=} (a_2a_1 \cdot a_4)(a_4a_3 \cdot a_3) \overset{(3)}{=} (a_2a_1 \cdot a_4)(a_3a_4 \cdot a_4) \overset{(12)}{=} (a_2a_1 \cdot a_3a_4)a_4 \\ &\overset{(2)}{=} (a_2a_3 \cdot a_1a_4)a_4 \overset{(37)}{=} (a_5a_4 \cdot a_1a_4)a_4 \overset{(12)}{=} (a_5a_1 \cdot a_4)a_4 \overset{(39)}{=} d_1a_4 \cdot a_4 \\ &= a_4 * d_1, \end{aligned}$$

$$\begin{aligned} a_1 * b_{34} &= a_1b_{34}\cdot b_{34} \overset{(37)}{=} (a_1\cdot a_2a_3)\cdot a_2a_3 \overset{(2)}{=} a_1a_2\cdot(a_2a_3\cdot a_3) \\ &\overset{(3)}{=} a_1a_2\cdot(a_3a_2\cdot a_2) \overset{(37)}{=} a_4a_3\cdot(a_3a_2\cdot a_2) \overset{(2)}{=} (a_4\cdot a_3a_2)\cdot a_3a_2 \\ &\overset{(37)}{=} a_4b_{12}\cdot b_{12} = b_{12} * a_4, \end{aligned}$$

$$\begin{aligned} a_1 * d_1 &= a_1d_1\cdot d_1 \overset{(39)}{=} (a_1\cdot a_5a_1)\cdot a_5a_1 \overset{(10)}{=} (a_1a_5\cdot a_1)\cdot a_5a_1 \overset{(12)}{=} (a_1a_5\cdot a_5)a_1 \\ &= (a_1 * a_5)a_1 \overset{(34)}{=} oa_1 \overset{(45)}{=} c_{34}a_2 \overset{(38)}{=} a_1a_3\cdot a_2 \overset{(12)}{=} a_1a_2\cdot a_3a_2 \\ &\overset{(32)}{=} (a_1a_5\cdot a_4)(a_3a_7\cdot a_8) \overset{(2)}{=} (a_1a_5\cdot a_3a_7)\cdot a_4a_8 \overset{(2)}{=} (a_1a_3\cdot a_5a_7)\cdot a_4a_8 \\ &\overset{(38)}{=} (a_6a_4\cdot a_2a_8)\cdot a_4a_8 \overset{(2)}{=} (a_6a_2\cdot a_4a_8)\cdot a_4a_8 \overset{(39)}{=} d_2d_8\cdot d_8 = d_2 * d_8, \end{aligned}$$

$$\begin{aligned} b_{12} * d_3 &= b_{12}d_3\cdot d_3 \overset{(37),(39)}{=} (a_3a_2\cdot a_7a_3)\cdot a_7a_3 \overset{(2)}{=} (a_3a_7\cdot a_2a_3)\cdot a_7a_3 \\ &\overset{(2)}{=} (a_3a_7\cdot a_7)(a_2a_3\cdot a_3) \overset{(34),(3)}{=} o(a_3a_2\cdot a_2) \overset{(34)}{=} (a_2a_6\cdot a_6)(a_3a_2\cdot a_2) \\ &\overset{(2)}{=} (a_2a_6\cdot a_3a_2)\cdot a_6a_2 \overset{(2)}{=} (a_2a_3\cdot a_6a_2)\cdot a_6a_2 \overset{(37),(39)}{=} b_{34}d_2\cdot d_2 = d_2 * b_{34}, \end{aligned}$$

$$\begin{aligned} o * c_{34} &= oc_{34}\cdot c_{34} \overset{(33),(38)}{=} ((a_2a_1\cdot a_1a_2)a_3\cdot a_1a_3)\cdot a_1a_3 \\ &\overset{(12)}{=} ((a_2a_1\cdot a_1a_2)a_1\cdot a_1)a_3 \overset{(19)}{=} a_1a_2\cdot a_3 \overset{(37)}{=} a_4a_3\cdot a_3 = a_3 * a_4, \end{aligned}$$

and we get the statements $Par(a_1, a_4, a_2, d_1)$, $Par(a_1, b_{12}, b_{34}, a_4)$, $Par(a_1, d_2, d_1, d_8)$, $Par(b_{12}, d_2, d_3, b_{34})$, $Par(o, a_3, c_{34}, a_4)$ or more general statements

$$Par(a_i, a_{i+3}, a_{i+1}, d_i), \quad Par(a_i, a_{i-3}, a_{i-1}, d_i), \tag{52}$$

$$Par(a_i, b_{i,i+1}, b_{i+2,i+3}, a_{i+3}), \tag{53}$$

$$Par(a_i, d_{i+1}, d_i, d_{i-1}), \tag{54}$$

$$Par(b_{i,i+1}, d_{i+1}, d_{i+2}, b_{i+2,i+3}), \tag{55}$$

$$Par(o, a_i, c_{i,i+1}, a_{i+1}). \tag{56}$$

We have proved:

Theorem 10. *Let the statement* $ARO_o(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8)$ *holds. Then there are the points* $b_{i,i+1}$, $c_{i,i+1}$, d_i *such that the statements* (37)–(48) *and* (52) – (56) *hold, where the indexes are taken modulo* 8 *from the set* $\{1, 2, 3, 4, 5, 6, 7, 8\}$, *and the statements* $ARO_o(b_{12}, b_{23}, b_{34}, b_{45}, b_{56}, b_{67}, b_{78}, b_{81})$, $ARO_o(c_{12}, c_{23}, c_{34}, c_{45}, c_{56}, c_{67}, c_{78}, c_{81})$ *and* $ARO_o(d_1, d_2, d_3, d_4, d_5, d_6, d_7, d_8)$ *are also valid.* □

All results from the Theorems 5, 6, 7 and 10 can be illustrated in the Figure 3.

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