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Decompositions of an Abel-Grassmann's groupoid

Madad Khan

Abstract. In this paper we have decomposed AG-groupoids. We have proved that if S is an AG*-groupoid, then S/ρ is isomorphic to S/σ, for $n, m \geqslant 2$, where ρ and σ are congruence relations. Further it has shown that S/η is a separative semilattice homomorphic image of an AG-groupoid S with left identity, where η is a congruence relation.

1. Introduction

An *Abel-Grassmann's groupoid* [5], abbreviated as an *AG-groupoid*, is a groupoid S whose elements satisfy the invertive law:

$$(ab)c = (cb)a, \qquad \text{for all } a, b, c \in S. \tag{1}$$

It is also called a *left almost semigroup* [3, 4]. In [1], the same structure is called a *left invertive groupoid*. It is a useful non-associative algebraic structure, midway between a groupoid and a commutative semigroup, with wide applications in the theory of flocks.

An AG-groupoid S is *medial* [3], that is,

$$(ab)(cd) = (ac)(bd), \qquad \text{for all } a, b, c, d, \in S. \tag{2}$$

If an AG-groupoid satisfies the following property, then it is called an *AG^*-groupoid* [5].

$$(ab)c = b(ca), \qquad \text{for all } a, b, c \in S. \tag{3}$$

Then also

$$(ab)c = b(ac), \qquad \text{for all } a, b, c \in S. \tag{4}$$

It is easy to see that the conditions (3) and (4) are equivalent. In an AG*-groupoid S holds all permutation identities of a next type [6],

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$$(x_1x_2)(x_3x_4) = (x_{\pi(1)}x_{\pi(2)})(x_{\pi(3)}x_{\pi(4)}), \tag{5}$$

where $\{\pi(1), \pi(2), \pi(3), \pi(4)\}$ means any permutation of the set $\{1, 2, 3, 4\}$. It means that if $S = S^2$, then S becomes a commutative semigroup. Many characteristics of a non-associative AG*-groupoid are similar to a commutative semigroup.

As a consequence of (5), we would have $(x_1x_2x_3)^m = (x_{p(1)}x_{p(2)}x_{p(3)})^m$, where $\{p(1), p(2), p(3)\}$ means any permutation of the set $\{1, 2, 3\}$ and $m \geqslant 2$. The result can be generalized for finite numbers of elements of S.

2. The smallest separative congruences

In an AG*-groupoid S, $(ab)c = b(ac)$ holds for all $a, b, c \in S$. This leads us to $(aa)a = a(aa)$ which implies that $a^2a = aa^2$. Hence it is easy to note that $a^{n+1}a = aa^{n+1}$, $a^ma^n = a^{m+n}$, $(a^m)^n = a^{mn}$, $(ab)^n = a^nb^n$, for all a, b and positive integers m and n.

We define a relation ρ on an AG-groupoid S as follows: $a\rho b$ if and only if there exists a positive integer n such that $ab^n = b^{n+1}$ and $ba^n = a^{n+1}$.

We define a relation σ on an AG-groupoid S as follows: $a\sigma b$ if and only if there exists a positive integer n such that $a^nb = a^{n+1}$ and $b^na = b^{n+1}$.

A relation ρ on an AG-groupoid S is called separative if $ab\rho a^2$ and $ab\rho b^2$ imply that $a\rho b$.

The following lemma has been proved in [6].

Lemma 1. *Let $\acute{\sigma}$ be a separative congruence on an AG*-groupoid S, then for all a, $b \in S$ it follows that $ab\acute{\sigma}ba$.*

In the following two lemmas we have proved that the relations ρ and σ are commutative without using separativity.

Lemma 2. *If S is an AG*-groupoid, then $ab\rho ba$ for all a, b in S.*

Proof. By using (5) and (2), we have, $(ab)(ba)^m = (ab)(b^ma^m) = (ab)(a^mb^m)$ $= (aa^m)(bb^m) = (bb^m)(aa^m) = b^{m+1}a^{m+1} = (ba)^{m+1}$. Similarly $(ba)(ab)^m =$ $(ab)^{m+1}$. Hence $ab\rho ba$. □

Lemma 3. *If S is an AG*-groupoid, then $ab\sigma ba$ for all a, b in S.*

Proof. By using (5), we have, $(ba)^n(ab) = (b^na^n)(ab) = (b^nb)(a^na) =$ $b^{n+1}a^{n+1} = (ba)^{n+1}$. Similarly $(ab)^n(ba) = (ab)^{n+1}$. Hence $ab\sigma ba$. □

The proofs of the following theorems are available in [6] and [5].

Theorem 1. *S/ρ is a maximal separative commutative image of an AG^*-groupoid S.*

Theorem 2. *S/σ is a maximal separative commutative image of an AG^*-groupoid S.*

Lemma 4. *ρ is equivalent to σ for $m, n \geqslant 2$, on an AG^*-groupoid S.*

Proof. Let $a\rho b$, then there exists a positive integer n such that $ab^n = b^{n+1}$ and $ba^n = a^{n+1}$. Now multiply b on both sides of $ab^n = b^{n+1}$, then using (1), we get $b^{n+1}b = (ab^n)b = b^{n+1}a$.

Similarly $ba^n = a^{n+1}$ implies that $a^{n+1}b = a^{n+2}$. Hence $a\sigma b$.

Conversely, assume that $a\sigma b$, then there exists a positive integer m such that $b^m a = b^{m+1}$ and $a^m b = a^{m+1}$. Assume that $m \geqslant 2$. Now multiply b on both sides of $b^m a = b^{m+1}$, then, using (3) and (5), we get

$$bb^{m+1} = b(b^m a) = (ab)b^m = (ab)(b^{m-1}b) = (ba)(b^{m-1}b) = a(b^m b) = ab^{m+1}.$$

Similarly $a^m b = a^{m+1}$ implies that $ba^{m+1} = a^{m+2}$. Hence $a\rho b$. □

Theorem 3. *If S is an AG^*-groupoid, then S/ρ is isomorphic to S/σ, for $m, n \geqslant 2$.*

Proof. It follows from Lemma 4. □

Remark 1. *S/ρ is not isomorphic to S/σ for $n = m = 1$.*

If S is an AG-groupoid then $(ab)c = a(bc)$, is not generally true for all $a, b, c \in S$, that is $(Sx)S \neq S(xS)$, for some x in S.

The relations γ and δ be defined in S as follows:

$a\gamma b$ if and only if there exists a positive integer n such that $b^n \in S(aS)$ and $a^n \in S(bS)$ for all a and b in S

$a\delta b$ if and only if there exists a positive integer m such that $b^m \in (Sa)S$ and $a^m \in (Sb)S$ for all a and b in S.

Lemma 5. *δ is equivalent to γ on an AG^*-groupoid S.*

Proof. Let $a^n \in S(bS)$, then using (3) and (1), we get

$$\begin{aligned} a^{n+2} &\in (S(bS))a^2 = ((bS)S)a^2 = (a((bS)S))a = (a(S^2 b))a \\ &= ((S^2 a)b)a \subseteq (Sb)S. \end{aligned}$$

Similarly $b^n \in S(aS)$ implies that $b^{n+2} \in (Sa)S$.

Conversely, assume that $a^n \in (Sb)S$, using (1) and (5), we get,

$$a^{n+1} \in ((Sb)S)a = (aS)(Sb) = (aS)(bS) \subseteq S(bS).$$

Similarly $b^n \in (Sa)S$ implies that $b^{n+1} \in S(aS)$. □

3. The semilattice decomposition

In an AG-groupoid S with left identity we have,

$$a(bc) = b(ac), \quad \text{for all } a, b, c \in S. \tag{6}$$

The following law holds for an AG-groupoid with left identity,

$$(ab)(cd) = (dc)(ba), \quad \text{for all } a, b, c, d \in S. \tag{7}$$

Also it is easy to see that if an AG-groupoid S contains left identity e, then $SS = S$ and $Se = S = eS$.

In [2] the power of elements in an AG-groupoid has been defined as follows: $a^m = (...(((aa)a)a)...)a$, ($m$-times).

Here we begin with an example of an AG-groupoid.

Example 1. Let $S = \{1, 2, 3, 4\}$ and the binary operation "$\cdot$" be defined on S as follows:

$\cdot$	1	2	3	4
1	3	4	1	2
2	2	3	4	1
3	1	2	3	4
4	4	1	2	3

Then clearly $(S, \cdot)$ is an AG-groupoid with left identity 3. □

From now, by S, we shall mean an AG-groupoid with left identity e.

The following Lemma 6 and Theorems $4 - 8$ are available in [2].

Lemma 6. *If $a \in S$, then for every positive integer m,*

(i) $a^m = a^{m-1}a = a^{m-3}a^3 = a^{m-5}a^5 = a^{m-7}a^7 = ...,$

(ii) $a^m = a^2a^{m-2} = a^4a^{m-4} = a^6a^{m-6} = ...$

Theorem 4. *If $a \in S$, then $a^m a^{2n-1} = a^{m+2n-1}$, for all positive integers m and n.*

Theorem 5. *If $a \in S$, then $a^{2n}a^m = a^{2n+m}$, for all positive integers m and n.*

Theorem 6. *If $a \in S$, then $a^{2n} = a^{2n}e$, for every positive integer n.*

Theorem 7. *If $a \in S$, then $(a^m)^n = a^{mn}$, for all positive integers m and n.*

Theorem 8. *If each $a \in S$, then $(ab)^n = a^n b^n$, for every positive integer n.*

Define a relation η on S as follows: $x\eta y$ if and only if there exists n such that $(xa)^n \in (ya)S$ and $(ya)^n \in (xa)S$.

Lemma 7. *If $a, b \in S$, then $a^2b^2 = b^2a^2$.*

Theorem 9. *η is a semilattice congruence on S.*

Proof. It is reflexive and symmetric. For transitivity let us suppose that $x\eta y$ and $y\eta z$, then there exist positive integers m, n such that $(xa)^n \in (ya)S$, $(ya)^n \in (xa)S$ and $(ya)^m \in (za)S$, $(za)^m \in (ya)S$. More specifically, there exist t_1, $t_2 \in S$, such that $(xa)^n = (ya)t_1$ and $(za)^m = (ya)t_2$. Now using Theorems 7, 8, (1) and (6), we have,

$$\begin{aligned} (xa)^{2mn} &= ((xa)^n)^{2m} = ((ya)t_1)^{2m} = ((ya)^m)^2 t_1^{2m} \in ((za)S)^2 S, \quad \text{but} \\ ((za)S)^2 S &= ((za)S)(za)S))S = (S((za)S))((za)S) \\ &= (za)(S((za)S))S) = (za)S. \end{aligned}$$

Therefore $(xa)^{2mn} \in (za)S$. Similarly $(za)^{2mn} \in (xa)S$. Hence η is transitive.

To show compatibility, let $x\eta y$ then there exists a positive integer m such that $(xa)^m \in (ya)S$ and $(ya)^m \in (xa)S$. Hence there exists t_3 and t_4 such that $(xa)^m = (ya)t_3$ and $(ya)^m = (xa)t_4$. Now using Theorem 8, Lemma 7, (2), (7) and (6), we get

$$\begin{aligned} ((xz)a)^{2m} &= ((xz)^2a^2)^m = ((xz)^2(a^2e))^m = ((xa)^2z^2)^m = ((xa)z)^2)^m \\ &= ((xa)z)^m)^2 = ((xa)^m z^m)^2 = (((ya)t_3)z^m)^2 = ((ya)^2 z^{2m})t_3^2 \\ &= ((yz^m)^2a^2)t_3^2 = (\left(y^2\left(z^{2m-1}z\right)\right)a^2)t_3^2 = (((yz^{2m-1})(yz))a^2)t_3^2 \\ &= (((yz^{2m-1})a)((yz)\,a))t_3^2 = t_3^2(((yz)a)((yz^{2m-1})a)) \\ &= ((yz)a)\left(t_3^2((yz^{2m-1})a)\right) \in ((yz)\,a)\,S. \end{aligned}$$

Similarly we can show that $((yz)a)^{2m} \in ((xz)\,a)\,S$. Therefore $(xz)\eta(yz)$. Similarly we can show that η is left compatable. Hence η is a congruence relation.

Next we shall show that η is a band congruence, by using Theorem 8, Lemma 7 and (1), we have $(xa)^2 = x^2a^2 = a^2x^2 = (aa)\,x^2 = \left(x^2a\right)a \in \left(x^2a\right)S$. Also using (6), (1), (2) and (7) we get $\left(x^2a\right)^2 = \left(x^2a\right)\left(x^2a\right) = x^2(\left(x^2a\right)a) = x^2(a^2x^2) = x^2((ax)(ax)) = x^2((xa)(xa)) = (xa)(x^2(xa)) \in (xa)S$. Therefore $x\eta x^2$, that is, $x_\eta^2 = x_\eta$. Hence S/η is idempotent. Now let $x\eta y$ which implies that $x\eta x^2\eta xy$, therefore $x\eta xy$.

Let $x\eta y$ and using Lemma 7, we have

$$((xy)\,a)^2 = ((yx)\,a)^2 = ((yx)\,a)\,((yx)\,a) \in ((yx)\,a)\,S.$$

Similarly $((yx)\,a)^2 \in ((xy)a)\,S$. Therefore $xy\eta yx$, that is, $x_\eta y_\eta = y_\eta x_\eta$. Hence S/η is a commutative AG-groupoid and so is commutative semigroup of idempotents. □

Theorem 10. *η is separative on S.*

Proof. Let $x^2\eta xy$ and $xy\eta y^2$. Then we have $x^2\eta y^2$, but, $x^2\eta x$ and $y^2\eta y$. So, $x\eta x^2\eta y^2\eta y$. Therefore, $x\eta y$. Hence η is separative. □

Theorem 11. *S/η is a separative semilattice homomorphic image of S.*

Proof. It follows from Theorems 9 and 10. □

Remark 2. *If every congruence on S is left zero, i.e., $ax\tau a$, then S/η is a maximal separative semilattice homomorphic image of S.*

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Department of Mathematics, COMSATS Institute of Information Technology, Abbottabad, Pakistan.
E-mail: madadmath@yahoo.com