Quasigroups and Related Systems **17** (2009), 39 – 54

Uniform topology and spectral topology on hyper MV-algebras

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Abstract. S-reflexive hyper MV-filters of a hyper MV-algebra, uniform structure and uniform topology are introduced, and it is proved that the uniform topology on an MV-algebra is discrete. Next, a (strong) prime hyper MV-filter and spectral topology are studied.

1. Introduction

MV-algebras are introduced by C. C. Chang in 1958 [1] to provide an algebraic proof of completeness theorem of infinite valued Łukasewicz propositional calculus. The hyper structure theory was introduced by Marty at 8th Congress of Scandinavian Mathematicians in 1934. Since then many researchers have worked on this area. Recently in [2] we applied the hyper structure to MV-algebras and introduced the notion of a hyper MV-algebra which is a generalization of MV-algebra and investigated some related results.

In the next section some preliminary theorems are stated from [2] and [3]. In section 3, we define the S-reflexive hyper MV-filter of a hyper MV-algebra and obtain some results. Then we define a uniform structure and a uniform topology. We show that each S-reflexive hyper MV-filter is a clopen subset and a uniform topology is discrete topology if and only if $\{1\}$ is a S-reflexive hyper MV-filter. In section 4, we define (strong) prime hyper MV-filters and prove some theorems. Then we define spectral topology and we show that it is a T_0 topology.

2000 Mathematics Subject Classification: 06D99, 54E15, 54B35
Keywords: Hyper MV-algebra, hyper MV-filter, uniformity, spectral topology.

2. Preliminaries

Definition 2.1. A *hyper MV-algebra* is a non-empty set M endowed with a hyper operation " $\oplus$ ", a unary operation " $*$ " and a constant 0 satisfying the following axioms:

(hMV1) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$,
(hMV2) $x \oplus y = y \oplus x$,
(hMV3) $(x^*)^* = x$,
(hMV4) $(x^* \oplus y)^* \oplus y = (y^* \oplus x)^* \oplus x$,
(hMV5) $0^* \in x \oplus 0^*$,
(hMV6) $0^* \in x \oplus x^*$,
(hMV7) if $x \ll y$ and $y \ll x$, then $x = y$, for all $x, y, z \in M$,

where $x \ll y$ is defined by $0^* \in x^* \oplus y$.

For every $A, B \subseteq M$, we define $A \ll B$ if and only if there exist $a \in A$ and $b \in B$ such that $a \ll b$. Also, we define $0^* := 1$ and $A^* = \{a^* : a \in A\}$.

Proposition 2.2. *Let $< M, \oplus, *, 0 >$ be a hyper MV-algebra. Then for all $x, y, z \in M$ and for all non-empty subset A, B and C of M the following hold:*

(1) $(A \oplus B) \oplus C = A \oplus (B \oplus C), \quad 0 \oplus 0 = 0,$
(2) $0 \ll x, \ \ x \ll x, \ \ x \ll 1, \ \ x \ll x \oplus y,$
(3) $x \ll y$ *implies* $y^* \ll x^*, \ \ A \ll B$ *implies* $B^* \ll A^*,$
(4) $A \ll A, \ \ A \ll A \oplus B, \ \ (A^*)^* = A,$
(5) $A \subseteq B$ *implies* $A \ll B,$
(6) $x \in x \oplus 0,$
(7) $y \in x \oplus 0$ *implies* $y \ll x, \ \ x \oplus 0 = y \oplus 0$ *implies* $y = x.$ □

In this paper a hyper MV-algebra $< M, \oplus, *, 0 >$ will be denoted by M. We will consider only *non-trivial* hyper MV-algebras, i.e., hyper MV-algebras such that $M \neq \{0\}$.

Definition 2.3. A non-empty subset F of a hyper MV-algebra M is called a *weak hyper MV-filter* of M, if

(whF1) $1 \in F$,
(whF2) if $F \subseteq x^* \oplus y$ and $x \in F$, then $y \in F$ for all $x, y \in M$.

Definition 2.4. A non-empty subset F of a hyper MV-algebra M is called a *hyper MV-filter* of M, if

(hF1) $1 \in F$,
(hF2) if $F \ll x^* \oplus y$ and $x \in F$, then $y \in F$ for all $x, y \in M$.

The smallest hyper MV-filter containing a non-empty subset S of M is denoted by $\langle S\rangle$.

Proposition 2.5. *$\{1\}$ is a hyper MV-filter of any hyper MV-algebra.* □

Proposition 2.6. *Let F be a hyper MV-filter of a hyper MV-algebra M. If $x \ll y$ and $x \in F$, then $y \in F$.* □

Proposition 2.7. *Let F be a hyper MV-filter of a hyper MV-algebra M. Then F is a weak hyper MV-filter of M.* □

Proposition 2.8. *Let $\{F_\alpha\}_{\alpha\in\Gamma}$ be a family of hyper MV-filters of a hyper MV-algebra M. Then $\bigcap\limits_{\alpha\in\Gamma} F_\alpha$ is a hyper MV-filter of M.* □

Definition 2.9. Let M_1 and M_2 be two hyper MV-algebras. A mapping $f : M_1 \to M_2$ is said to be a *homomorphism*, if $f(0) = 0,\ f(x\oplus y) = f(x)\oplus f(y)$ and $f(x^*) = (f(x))^*$.

Clearly if f is a homomorphism, then $f(1) = 1$.

Theorem 2.10. *Let $f : M_1 \to M_2$ be a homomorphism of hyper MV-algebras. Then*

(1) *if F is a (weak) hyper MV-filter of M_2, then $f^{-1}(F)$ is a (weak) hyper MV-filter of M_1,*
(2) $\ker f = \{x \in M_1 : f(x) = 1\}$ *is a hyper MV-filter of M_1, consequently* $\ker f$ *is a weak hyper MV-filter of M_1,*
(3) *f is one-to-one if and only if* $\ker f = \{1\}$,
(4) *if f is onto and F is a hyper MV-filter of M_1 which contains* $\ker f$, *then $f(F)$ is a hyper MV-filter of M_2.* □

3. Uniform topology on hyper MV-algebras

Proposition 3.1. Let $A, B \subseteq M$. If F is a hyper MV-filter of M such that $F \ll A^* \oplus B$. Then

(a) $F \cap (A^* \oplus B) \neq \emptyset$,
(b) if $A \subseteq F$, then $F \ll B$.

Proof. Since $F \ll A^* \oplus B = \bigcup\limits_{a\in A,\ b\in B} a^* \oplus b$, then there exist $a_1 \in A$ and $b_1 \in B$ such that $F \ll a_1^* \oplus b_1$. So, there exist $r \in F$ and $t \in a_1^* \oplus b_1$ such that $r \ll t$. Thus $t \in F$ by Proposition 2.6. Therefore $F \cap (A^* \oplus B) \neq \emptyset$. Moreover, since F is a hyper MV-filter of M and $A \subseteq F$, we have also $F \ll B$. □

Definition 3.2. A hyper MV-filter F of M is called *S-reflexive* if for all $x, y \in M$ $(x^* \oplus y) \cap F \neq \emptyset$ implies $(x^* \oplus y) \subseteq F$.

Example 3.3. Let $M = \{0, a, b, 1\}$. Consider the following tables:

$\oplus$	0	a	b	1
0	$\{0\}$	$\{0, a\}$	$\{b\}$	$\{b, 1\}$
a	$\{0, a\}$	$\{0, a\}$	$\{b, 1\}$	$\{b, 1\}$
b	$\{b\}$	$\{b, 1\}$	$\{b, 1\}$	$\{b, 1\}$
1	$\{b, 1\}$	$\{b, 1\}$	$\{b, 1\}$	$\{b, 1\}$

$*$	0	a	b	1
	1	b	a	0

Then $< M, \oplus, *, 0 >$ is a hyper MV-algebra and $F = \{b, 1\}$ is a S-reflexive hyper MV-filter of M. But $J = \{1\}$ is not a S-reflexive hyper MV-filter because $(b^* \oplus b) \cap J \neq \emptyset$ and $(b^* \oplus b) \nsubseteq F$.

Proposition 3.4. *The intersection of a family of S-reflexive hyper MV-filters is a S-reflexive hyper MV-filter.*

Proof. Use Proposition 2.8. □

Lemma 3.5. *Let $x, y \in M$. Then $(y^* \oplus z) \ll (x^* \oplus y)^* \oplus (x^* \oplus z)$.*

Proof. Use (hMV1), (hMV4) and Proposition 2.2. □

Definition 3.6. Let F be a S-reflexive hyper MV-filter of M. Then we define $U_F := \{(x, y) \in X \times X : x \sim_F y\}$, where

$$x \sim_F y \longleftrightarrow (F \ll x^* \oplus y \text{ and } F \ll y^* \oplus x).$$

Theorem 3.7. *$\sim_F$ is an equivalence relation on M.*

Proof. Since $1 \in x^* \oplus x$, $F << x^* \oplus x$. Hence $x \sim_F x$. Clearly $\sim_F$ is symmetric.

Let $x \sim_F y$ and $y \sim_F z$. Then $F \ll x^* \oplus y$, $F \ll y^* \oplus x$, $F \ll y^* \oplus z$ and $F \ll z^* \oplus y$. Since F is a S-reflexive hyper MV-filter of M, then we have $x^* \oplus y \subseteq F$, $y^* \oplus x \subseteq F$, $y^* \oplus z \subseteq F$ and $z^* \oplus y \subseteq F$. On the other hand by Lemma 3.5, we have $(y^* \oplus z) \ll (x^* \oplus y)^* \oplus (x^* \oplus z)$. Hence $F \ll (x^* \oplus y)^* \oplus (x^* \oplus z)$. Since $F \subseteq x^* \oplus y$, by Proposition 3.1, we have $F \ll x^* \oplus z$. Similarly $F \ll z^* \oplus x$. Therefore $x \sim_F z$. □

Let X be a non-empty set, U and V be subsets of $X \times X$. Then we define

$U \circ V = \{(x, y) \in X \times X : (x, z) \in U \text{ and } (z, y) \in V \text{ for some } z \in X\}$,

$U^{-1} = \{(x, y) \in X \times X : (y, x) \in U\}$,

$\Delta = \{(x, x) \in X \times X : x \in X\}$.

Definition 3.8. [4] A *uniformity* on X is a non-empty collection K of subsets of $X \times X$ satisfying the following conditions:

(U1) $\Delta \subseteq U$ for any $U \in K$,
(U2) if $U \in K$, then $U^{-1} \in K$,
(U3) if $U \in K$, then there exists a $V \in K$ such that $V \circ V \subseteq U$,
(U4) if $U, V \in K$, then $U \cap V \in K$,
(U5) if $U \in K$ and $U \subseteq V \subseteq X \times X$, then $V \in K$.

Then pair (X, K) is called a *uniform structure.*

Proposition 3.9. *Let $\{F_i\}_{i=1}^{n}$ be a finite family of S-reflexive hyper MV-filters of M. Then $U_{F_1} \cap \ldots \cap U_{F_n} = U_{F_1 \cap \ldots \cap F_n}$.*

Proof. Using induction and Proposition 3.1. □

Theorem 3.10. *Let F be a S-reflexive hyper MV-filter of M. If*

$$K^* = \{U_F : F \text{ is a } S-\text{reflexive hyper } MV-\text{filter of } M\},$$

then K^* satisfies conditions $(U1) - (U4)$.

Proof. Since $x \sim_F x$ for all S-reflexive hyper MV-filter F of M and for all $x \in M$, then $\Delta \subseteq U_F$ for all $U_F \in K^*$. Clearly $U_F^{-1} = U_F$. Let $U_F \in K^*$. Then the transitivity of $\sim_F$ implies that $U_F \circ U_F \subseteq U_F$. Let $U_F, U_J \in K^*$. By Proposition 3.10 we have $U_F \cap U_J = U_{F \cap J}$. □

Corollary 3.11. *$K = \{U \subseteq M \times M : U_F \subseteq U \; for \; some \; U_F \in K^*\}$ satisfies the uniformity conditions on M and the pair (M, K) is a uniform structure.*

Proof. The proof easily follows from Theorem 3.10. □

Let $U[x] := \{y \in M : (x, y) \in U\}$, where $x \in M$ and $U \in K$.

Theorem 3.12. *$\tau = \{A \subseteq M : \forall x \in A, \; \exists U \in K \; (U[x] \subseteq A)\}$ is a topology on M called the uniform topology on M induced by K.*

Proof. Clearly $\emptyset, M \in \tau$ and τ is closed under an arbitrary union. Let$A, B \in \tau$. Then there exist $U, V \in K$ such that $U[x] \subseteq A$ and $V[x] \subseteq B$. Let $W = U \cap V$. Then $W \in K$ by (U4) and $W[x] = U[x] \cap V[x]$. Hence $W[x] \subseteq A \cap B$ and $A \cap B \in \tau$. Thus τ is a topology on M. □

Remark 3.13. $U[x]$ is an open neighborhood of x.

Lemma 3.14. *If* $U_{\{1\}} \in K^*$, *then* $U_{\{1\}}[x] = \{x\}$.

Proof. $U_{\{1\}}[x] = \{y \in M : x \sim_{\{1\}} y\} = \{y \in M : \{1\} \ll x^* \oplus y, \ \{1\} \ll y^* \oplus x\}$ $= \{y \in M : x^* \oplus y \subseteq \{1\}, \ y^* \oplus x \subseteq \{1\}\} = \{y \in M : x = y\} = \{x\}$. □

Proposition 3.15. *If* F *and* $\{1\}$ *are two S-reflexive hyper MV-filters of* M *and* $U_{\{1\}} = U_F$, *then* $F = \{1\}$.

Proof. Let $F \neq \{1\}$, then there exists $1 \neq z \in F$. We have $z \in 1^* \oplus z$ and $1 \in z^* \oplus 1$ by Proposition 2.2 and (hMV5). Thus $F \ll 1^* \oplus z$ and $F \ll z^* \oplus 1$ then $(z, 1) \in U_F$. On the other hand since we have $\{1\} \not\ll 1^* \oplus z$, then $(z, 1) \notin U_{\{1\}}$. Hence $U_{\{1\}} \neq U_F$. □

Lemma 3.16. *Let* F *be a S-reflexive hyper MV-filter of* M. *If* $\Delta = U_F$, *then* $F = \{1\}$.

Proof. Let $F \neq \{1\}$, then there exists $1 \neq z \in F$. Similar to Proposition 3.13, we can show that $(z, 1) \in U_F$ but $(z, 1) \notin \Delta$. Hence $\Delta \neq U_F$. □

Theorem 3.17. $U_{\{1\}} \in K^*$ *if and only if* τ *is a discrete topology.*

Proof. Let $U_{\{1\}} \in K^*$. By using Lemma 3.14, we have $\{x\} = U_{\{1\}}[x] \in \tau$ for all $x \in M$. Hence τ is a discrete topology.

Conversely, let τ be a discrete topology. Hence $\{x\}$ is open for all $x \in M$. By definition there exists a S-reflexive hyper MV-filter F of M such that $U_F[x] \subseteq \{x\}$. Therefore $U_F[x] = \{x\}$ for all $x \in M$. Hence $\Delta = U_F$ and then we have $F = \{1\}$ by Lemma 3.16. □

Remark 3.18. Let $< M, +, *, 0 >$ be an MV-algebra. Then $< M, \oplus, *, 0 >$ is a hyper MV-algebra where $x \oplus y = \{x + y\}$ for all $x, y \in M$. We can show that every filter of $< M, +, *, 0 >$ is a S-reflexive hyper MV-filter of $< M, \oplus, *, 0 >$. So, by the above theorem, uniform topology on M is a discrete topology.

Theorem 3.19. *Let* F *be a S-reflexive hyper MV-filter of* M. *Then* $U_F[x]$ *is clopen set in the topological space* (M, τ).

Proof. Let F be an arbitrary S-reflexive hyper MV-filter of M and x be an element of M. It is clear that $U_F[x]$ is an open set in (M, τ). We will show that $(U_F[x])^c$ is an open set in (M, τ). Let $y \in (U_F[x])^c$. We claim that $U_F[y] \subseteq (U_F[x])^c$. Suppose that $z \in U_F[y]$. Then we have $F \ll y^* \oplus z$ and

$F \ll z^* \oplus y$. Since F is a S-reflexive hyper MV-filter of M, then $y^* \oplus z \subseteq F$ and $z^* \oplus y \subseteq F$. If we have $x \sim_F z$, then $x^* \oplus z \subseteq F$ and $z^* \oplus x \subseteq F$. By Lemma 3.5, we have $z^* \oplus y \ll (x^* \oplus z)^* \oplus (x^* \oplus y)$. Using Proposition 3.1, we have $F \ll x^* \oplus y$. Similarly $F \ll y^* \oplus x$. Therefore $x \sim_F y$, i.e., $y \in U_F[x]$ which is a contradiction. Hence $z \in (U_F[x])^c$. Thus $U_F[y] \subseteq (U_F[x])^c$ and then $U_F[x]$ is closed. □

Theorem 3.20. [7] *If (X, τ) a is uniform space, then the corresponding topological space is completely regular.* □

Corollary 3.21. *The topological space (M, τ) is completely regular.* □

Example 3.22. Consider Example 3.3 and S-reflexive hyper MV-filter $F = \{b, 1\}$. We have

$$K^* = \{U_F\} = \{(0\,,0),\ (a,\ a),\ (b,\ b),\ (1,\ 1),\ (0,\ a),\ (a,\ 0)\}.$$

We can show that (M, K) is a uniform structure, where

$$K = \{U \subseteq M \times M : U_F \subseteq U\}.$$

Moreover the open neighborhoods are

$$U_F[0] = \{0, a\} = U_F[a], \quad U_F[b] = \{b, 1\} = U_F[1].$$

We have $\tau = \{\emptyset,\ \{b\},\ \{1\},\ \{0, a\},\ \{0, a, b, 1\}\}$. Then (M, τ) is a uniform topology space which is not discrete. Also, we can show that (M, τ) is not Hausdorff.

Remember that a topological space X is connected if and only if the only subsets of X that are both open and closed in X are empty set and X itself.

Corollary 3.23. *The topological space (M, τ) is connected if and only if M is the only S-reflexive hyper MV-filter of M.* □

Notation: $U_F[A] = \bigcup_{a \in A} U_F[a]$.

Theorem 3.24. *Let $A \subseteq M$. Then $\overline{A} = \bigcap\{U_F[A] : U_F \in K^*\}$, where $\overline{A}$ is closure of A in the topological space (M, τ).*

Proof. Let $b \in \overline{A}$ and $U_F \in K^*$. Since $U_F[b]$ is an open neighborhood of b, then $U_F[b] \cap A \neq \emptyset$. Hence there exists $a \in A$ such that $a \in U_F[b]$, i.e., $F \ll a^* \oplus b$ and $F \ll b^* \oplus a$. Thus $b \in U_F[a] \subseteq U_F[A]$. Therefore $b \in \bigcap\{U_F[A] : U_F \in K^*\}$.

Conversely, let $b \in \bigcap\{U_F[A] : U_F \in K^*\}$, then $b \in U_F[A]$ for each $U_F \in K^*$. Hence there exists $a \in A$ such that $a \in U_F[b]$ and then $U_F[b] \cap A \neq \emptyset$ for each $U_F \in K^*$. Therefore $b \in \overline{A}$. □

Corollary 3.25. *Each S-reflexive hyper MV-filter F of M is clopen in the topological space (M, τ).*

Proof. Let F be a S-reflexive hyper MV-filter of M. First we will show that $U_F[F] = F$. Let $x \in U_F[F]$. Then there exists $a \in F$ such that $F \ll a^* \oplus x$ and $F \ll x^* \oplus a$. Hence $x \in F$ and then $U_F[F] \subseteq F$.

Conversely, let $x \in F$ then $x \in U_F[x] \subseteq U_F[F]$. Hence $F \subseteq U_F[F]$. By the above theorem, we get that $F \subseteq \overline{F} \subseteq U_F[F] = F$ and then $F = \overline{F}$. It is clear that F is an open subset of M. □

Theorem 3.26. *Let A be a compact subset of M and O be an open set containing A. Then there exists a S-reflexive hyper MV-filterF of M such that $A \subseteq U_F[A] \subseteq O$.*

Proof. Since O is an open set containing A, for $a \in A$ there exists a S-reflexive hyper MV-filter F_a of M such that $U_{F_a}[a] \subseteq O$. Hence $A \subseteq \bigcup\limits_{a \in A} U_{F_a}[a]$. Since A is a compact subset of M, then there exist $a_1, a_2, \ldots, a_n \in A$ such that

$$A \subseteq U_{F_1}[a_1] \cup \ldots \cup U_{F_n}[a_n].$$

Put $F = \bigcap\limits_{i=1}^{n} F_i$. Then by Proposition 3.9, we have $U_F = U_{F_1} \cap \ldots \cap U_{F_n}$. We claim that $U_F[a] \subseteq O$ for any $a \in A$. Let $a \in A$. Then there exists $1 \leqslant i \leqslant n$ such that $a \in U_{F_i}[a_i]$ and hence $a \sim_{F_i} a_i$. Let $y \in U_F[a]$, then $y \sim_F a$. Therefore we have $y \sim_{F_i} a_i$ and hence $y \in U_{F_i}[a_i] \subseteq O$. It shows that $U_F[a] \subseteq O$ for any $a \in A$. Thus $A \subseteq U_F[A] \subseteq O$. □

Theorem 3.27. *Let K be a compact subset of M and C be a closed subset of M. If $K \cap C = \emptyset$, then there exists a S-reflexive hyper MV-filter F of M such that $U_F[K] \cap U_F[C] = \emptyset$.*

Proof. Since $K \cap C = \emptyset$ and C is closed, $M \backslash C$ is an open set containing K. Then there exists a S-reflexive hyper MV-filter F of M such that $K \subseteq U_F[K] \subseteq M \backslash C$ by the above theorem. We claim that $U_F[K] \cap U_F[C] = \emptyset$. Suppose that $U_F[K] \cap U_F[C] \neq \emptyset$, then there exists $y \in M$ such that $y \in U_F[a]$ and $y \in U_F[b]$ for some $a \in K$ and $b \in C$, respectively. Hence $a \sim_F b$ and then $b \in U_F[a] \subseteq U_F[K]$. This contradicts to the fact that $U_F[K] \subseteq M \backslash C$. Hence $U_F[K] \cap U_F[C] = \emptyset$. □

Let F_0 be the intersection of all S-reflexive hyper MV-filters of M. Then F_0 is a S-reflexive hyper MV-filter of M by Proposition 3.4. Define $K^*_{F_0} = \{U_{F_0}\}$ and $K_{F_0} = \{U \subseteq M \times M : U_{F_0} \subseteq U\}$. Then we can show that (M, K_{F_0}) is a uniform structure. The uniform topology induced by K_{F_0} is denoted by τ_{F_0}.

Theorem 3.28. $\tau = \tau_{F_0}$.

Proof. Let $A \in \tau$. Then for all $x \in A$, there exists $U \in K$ such that $U[x] \subseteq A$. So there exists a S-reflexive hyper MV-filter F of M such that $U_F \subseteq U$. Since $F_0 \subseteq F$, we have $U_{F_0} \subseteq U_F \subseteq U$ and $U[x] \subseteq A$. Hence $A \in \tau_{F_0}$ and then $\tau \subseteq \tau_{F_0}$.

Conversely, let $O \in \tau_{F_0}$. Then for all $x \in O$, there exists $U \in K_{F_0}$ such that $U[x] \subseteq O$ and $U_{F_0} \subseteq U$. Since F_0 is a S-reflexive hyper MV-filter of M, we have $U \in K$ and hence $O \in \tau$. Therefore $\tau_{F_0} \subseteq \tau$. □

Theorem 3.29. *F_0 and $U_{F_0}[x]$ are compact sets in the topological space (M, τ).*

Proof. Let $F_0 \subseteq \bigcup_{\alpha \in \Gamma} O_\alpha$, where O_α is an open set in M for each $\alpha \in \Gamma$. Since $1 \in F_0$, there exists $\alpha \in \Gamma$ such that $1 \in O_\alpha$. Since O_α is an open set, there exists $U \in K_{F_0}$ such that $U[1] \subseteq O_\alpha$ and $U_{F_0} \subseteq U$. We show that $F_0 = U_{F_0}[1]$. Let $x \in F_0$. Since $x \in 1^* \oplus x$ and $1 \in x^* \oplus 1$, we have $F_0 \ll x^* \oplus 1$ and $F_0 \ll 1^* \oplus x$. Hence $x \in U_{F_0}[1]$. So $F_0 \subseteq U_{F_0}[1]$. Conversely, let $x \in U_{F_0}[1]$. Then $F_0 \ll x^* \oplus 1$ and $F_0 \ll 1^* \oplus x$. Since $F_0 \ll 1^* \oplus x$, $1 \in F_0$ and F_0 is a hyper MV-filter of M, we get $x \in F_0$. So $U_{F_0}[1] \subseteq F_0$. Hence $F_0 \subseteq O_\alpha$ and then F_0 is compact.

Let $U_{F_0}[x] = \bigcup_{\alpha \in \Gamma} O_\alpha$, where O_α is an open set in M for each $\alpha \in \Gamma$. Since $x \in U_{F_0}[x]$, there exists $\alpha \in \Gamma$ such that $x \in O_\alpha$. Since O_α is an open set, there exists $U \in K_{F_0}$ such that $U[x] \subseteq O_\alpha$ and $U_{F_0} \subseteq U$. Hence $U_{F_0}[x] \subseteq O_\alpha$ and then $U_{F_0}[x]$ is compact set. □

Corollary 3.30. *The topological space (M, τ) is locally compact.*

Proof. It follows from Theorem 3.19, Theorem 3.28 and Theorem 3.29. □

Theorem 3.31. *The topological space (M, τ) is compact if and only if There exists $X = \{x_1, x_2, \ldots, x_n\} \subseteq M$ such that for each $a \in M$ there exists $x_i \in X$ which $F_0 \ll x_i^* \oplus a$ and $F_0 \ll a^* \oplus x_i$.*

Proof. Let the topological space (M,τ) be compact. Then $M \subseteq \bigcup\limits_{x\in M} U_{F_0}[x]$, where $U_{F_0}[x]$ is an open set for each $x \in M$ by Theorem 3.19. Therefore there exist $x_1, x_2, \ldots, x_n \in M$ such that $M \subseteq \bigcup\limits_{i=1}^{n} U_{F_0}[x_i]$. Now, let $a \in M$ be arbitrary, then there exists x_i for some $1 \leqslant i \leqslant n$ such that $a \in U_{F_0}[x_i]$. Hence $F_0 \ll x_i^* \oplus a$ and $F_0 \ll a^* \oplus x_i$.

Conversely, let $M \subseteq \bigcup\limits_{\alpha\in\Gamma} O_\alpha$, where O_α is an open set in for each $\alpha \in \Gamma$ and $a \in M$ be arbitrary. By assumption there exists $x_i \in X$ which $F_0 \ll x_i^* \oplus a$ and $F_0 \ll a^* \oplus x_i$. Thus $a \in U_{F_0}[x_i]$. Hence $M \subseteq \bigcup\limits_{i=1}^{n} U_{F_0}[x_i]$. On the other hand $M \subseteq \bigcup\limits_{\alpha\in\Gamma} O_\alpha$, then for each $x_i \in X$ there exists O_{α_i} such that $x_i \in O_{\alpha_i}$. Therefore $U_{F_0}[x] \subseteq O_{\alpha_i}$. Hence $M \subseteq \bigcup\limits_{i=1}^{n} O_{\alpha_i}$ and the topological space(M,τ) is compact. □

Theorem 3.32. *If F_0^c is finite, then the topological space (M,τ) is compact.*

Proof. Let $M \subseteq \bigcup\limits_{\alpha\in\Gamma} O_\alpha$, where O_α is an open set in M for each $\alpha \in \Gamma$ and $F_0^c = \{x_1, x_2, \ldots, x_n\}$. Since $1 \in F_0$, there exists $\alpha_0 \in \Gamma$ such that $1 \in O_{\alpha_0}$. Hence $F_0 = U_{F_0}[1] \subseteq O_{\alpha_0}$. Also there exist $\alpha_1,\ \alpha_2, \ldots, \alpha_n \in \Gamma$ such that $x_1 \in O_{\alpha_1},\ x_2 \in O_{\alpha_2},\ \ldots,\ x_n \in O_{\alpha_n}$. Hence $M \subseteq \bigcup\limits_{i=0}^{n} O_{\alpha_i}$ and the topological space (M,τ) is compact. □

Theorem 3.33. *(M,τ) is a Hausdorff topological space if and only if $F_0 = \{1\}$.*

Proof. Let the topological space (M,τ) be Hausdorff. Then there exist open set O_1 and O_2 such that $x \in O_1$, $1 \in O_2$, $O_1 \cap O_2 = \emptyset$, $U_{F_0}[x] \subseteq O_1$ and $U_{F_0}[1] \subseteq O_2$. Hence we have $U_{F_0}[x] \cap U_{F_0}[1] = \emptyset$ for each $1 \neq x \in M$. So $x \notin U_{F_0}[1]$ and then $F_0 \not\ll x^* \oplus 1$ or $F_0 \not\ll 1^* \oplus x$. Since $F_0 \ll x^* \oplus 1$, we have $F_0 \not\ll 1^* \oplus x$. Thus $y \notin F_0$ for each $y \in 0 \oplus x$. Since $x \in 0 \oplus x$, then $x \notin F_0$. Hence for each $1 \neq x \in M$ we have $x \notin F_0$ and then $F_0 = \{1\}$.

Conversely, let $F_0 = \{1\}$. Then $\{1\}$ is a S-reflexive hyper MV-filter of M. Thus τ is a discrete topology by Theorem 3.17. Hence the topological space (M,τ) is Hausdorff. □

Theorem 3.34. *The following are equivalent:*

(1) *M is T_0,*

(2) *M is* T_1,
(3) *M is Hausdorff.*

Proof. (1) → (2) Suppose that $x \neq y$. If there exists an open set O such that $x \in O$ and $y \notin O$, then we have $U_{F_0}[x] \subseteq O$ and $y \notin U_{F_0}[x]$. Hence $x \notin U_{F_0}[y]$. Similarly, we can show that if there exists an open set O such that $y \in O$ and $x \notin O$, there exist open neighborhoods V of x and W of y such that $y \notin V$ and $x \notin W$.
(2) → (3) Suppose that $x \neq y$. Then there exist open set O_1 and O_2 such that $x \in O_1$, $y \notin O_1$, $y \in O_2$ and $x \notin O_2$. Hence $U_{F_0}[x] \subseteq O_1$, $y \notin U_{F_0}[x]$, $U_{F_0}[y] \subseteq O_2$ and $x \notin U_{F_0}[y]$. We claim $U_{F_0}[x] \cap U_{F_0}[y] = \emptyset$. Let $z \in U_{F_0}[x] \cap U_{F_0}[y]$. Then $x \sim_{F_0} z$ and $z \sim_{F_0} y$. Hence $x \sim_{F_0} y$. So $y \in U_{F_0}[x]$ which is contradiction. Hence M is Hausdorff.
(2) → (3) It is clear. □

4. Spectral topology on hyper MV-algebras

Definition 4.1. A proper hyper MV-filter P of M is called a *strong prime hyper MV-filter* of M, if $P \subseteq x^* \oplus y$ or $P \subseteq y^* \oplus x$ for all $x, y \in M$.

Remark 4.2. We note that if P is a strong prime hyper MV-filter of M and $x, y \in M$ such that $x \neq y$, then P can not be a subset of both $x^* \oplus y$ and $y^* \oplus x$.

Example 4.3. Consider Example 3.3 and hyper MV-filters $P_1 = \{1\}$ and $P_2 = \{1, b\}$. We can show that P_1 and P_2 are strong prime hyper MV-filters of M.

Proposition 4.4. *The intersection of any family of strong prime hyper MV-filters of a hyper MV-algebra M is a strong prime hyper MV-filter of M.*

Proof. $\bigcap_{\alpha \in \Gamma} P_\alpha$ is a hyper MV-filter of M by Proposition 2.8. Let $x, y \in M$. If $x = y$, then it is clear that $\bigcap_{\alpha \in \Gamma} P_\alpha \subseteq x^* \oplus y$ and $\bigcap_{\alpha \in \Gamma} P_\alpha \subseteq y^* \oplus x$. Now let $x \neq y$ and let there exist $\alpha, \beta \in \Gamma$ such that $P_\alpha \subseteq x^* \oplus y$ and $P_\beta \subseteq y^* \oplus x$. Since $1 \in P_\alpha$ and $1 \in P_\beta$, then $x \ll y$ and $y \ll x$. By (hMV5) $x = y$ which is a contradiction. Hence $\bigcap_{\alpha \in \Gamma} P_\alpha \subseteq x^* \oplus y$ or $\bigcap_{\alpha \in \Gamma} P_\alpha \subseteq y^* \oplus x$. □

Definition 4.5. A proper hyper MV-filter P of M is called a *prime hyper MV-filter* of M if $P \ll x^* \oplus y$ or $P \ll y^* \oplus x$ for all $x, y \in M$.

Proposition 4.6. *A strong prime hyper MV-filter is prime.*

Proof. Use Proposition 2.2. □

Remark 4.7. The converse of Proposition 4.6 may not be true. Also the intersection of prime hyper MV-filters may not be a prime hyper MV-filter of a hyper MV-algebra M. Consider the following example.

Example 4.8. Let $M = \{0, a, b, 1\}$. Consider the following tables:

$\oplus$	0	a	b	1
0	$\{0\}$	$\{0, a\}$	$\{0, b\}$	$\{0, a, b, 1\}$
a	$\{0, a\}$	$\{0, a\}$	$\{0, a, b, 1\}$	$\{0, a, b, 1\}$
b	$\{0, b\}$	$\{0, a, b, 1\}$	$\{0, b\}$	$\{0, a, b, 1\}$
1	$\{0, a, b, 1\}$	$\{0, a, b, 1\}$	$\{0, a, b, 1\}$	$\{0, a, b, 1\}$

$*$	0	a	b	1
	1	b	a	0

Then $< M, \oplus, *, 0 >$ is a hyper MV-algebra, $P_1 = \{a, 1\}$ and $P_2 = \{1, b\}$ are prime hyper MV-filters of M but are not strong prime hyper MV-filters M because $P_1, P_2 \nsubseteq a^* \oplus b$ and $P_1, P_2 \nsubseteq b^* \oplus a$. Also $P_1 \cap P_2 = \{1\}$ is neither a prime hyper MV-filter nor a strong prime hyper MV-filter of M. Thus M has no strong prime hyper MV-filter.

Theorem 4.9. *Let $f : M_1 \to M_2$ be an epimorphism of hyper MV-algebras. Then*

(1) *if P is a (strong) prime hyper MV-filter of M_1 which contains* $\ker f$, *then $f(P)$ is a (strong) prime hyper MV-filter of M_2,*
(2) *if P is a (strong) prime hyper MV-filter of M_2, then $f^{-1}(P)$ is a (strong) prime hyper MV-filter of M_1,*
(3) *the map $P \longmapsto f(P)$ is one to one corresponding between (strong) prime hyper MV-filters of M_1 which contain* $\ker f$ *and (strong) prime hyper MV-filter of M_2.*

Proof. (1) Let $f(P) = M_2$. Since $P \neq M_1$, there exists some $x \in M_1$ such that $x \notin P$. So $f(x) \in M_2 = f(P)$. Thus there exists some $a \in P$ such that $f(x) = f(a)$ and then $1 \in f(x) \oplus f(a)^* = f(x \oplus a^*)$. So there exists some $t \in a^* \oplus x$ such that $f(t) = 1$, i.e., $t \in \ker f \subseteq P$. Then $P \ll a^* \oplus x$. Thus $x \in P$ which is a contradiction. Hence $f(P) \neq M_2$. By Theorem 2.10 $f(P)$ is a hyper MV-filter of M_2.

Let $x, y \in M_2$. Then there exist $a, b \in M_1$ such tat $f(a) = x$ and $f(b) = y$. Since P is a strong prime hyper MV-filter of M_1, then we have $P \subseteq a^* \oplus b$ or

$P \subseteq b^* \oplus a$. Let $P \subseteq a^* \oplus b$ then $f(P) \subseteq f(a^* \oplus b) = f(a)^* \oplus f(b) = x^* \oplus y$. Hence $f(P)$ is a strong prime hyper MV-filter of M_2.

Similarly, we can show that if P is a prime hyper MV-filter of M_1 which contains $\ker f$, then $f(P)$ is a prime hyper MV-filter of M_2.

(2) Let $f^{-1}(P) = M_1$. Since $P \neq M_2$, there exists $y \in M_2$ such that $y \notin P$. So $f^{-1}(y) \in M_1 = f^{-1}(P)$. Thus there exists some $x \in M_1$ such that $y = f(x) \in P$ which is a contradiction. Hence $f^{-1}(P) \neq M_1$. By Theorem 2.10, $f^{-1}(P)$ is a hyper MV-filter of M_1. Let $a, b \in f^{-1}(P)$. Then $f(a), f(b) \in P$. Hence $P \subseteq f(a)^* \oplus f(b) = f(a^* \oplus b)$ or $P \subseteq f(b)^* \oplus f(a) = f(b^* \oplus a)$. Let $P \subseteq f(a^* \oplus b)$. Then we can show that $f^{-1}(P) \subseteq a^* \oplus b$. Similarly, we can show that if P is a prime hyper MV-filter of M_2, then $f^{-1}(P)$ is a prime hyper MV-filter of M_1.

(3) The proof is straightforward by (1), (2). □

Definition 4.10. The set of all strong prime hyper MV-filters of M is called the *hyper MV-spectrum* and is denoted by $HSpec(M)$.

Theorem 4.11. *Let $C(F) = \{P \in HSpec(M) : F \subseteq P\}$ for each hyper MV-filter F of M. Then $\mathcal{C} = \{C(F) : F \text{ is a hyper } MV-filter \text{ of } M\}$ defines a closed sets family for a topology over $HSpec(M)$. This topology is called spectral topology or Zarisky topology on $HSpec(M)$.*

Proof. (1) Since $C(M) = \emptyset$ and $C(\{1\}) = HSpec(M)$, then $\emptyset, HSpec(M) \in \mathcal{C}$.

(2) Let $\{F_\alpha\}_{\alpha\in\Gamma}$ be a family of hyper MV-filters of M. We claim that $\bigcap_{\alpha\in\Gamma} C(F_\alpha) = C(\langle \bigcup_{\alpha\in\Gamma} F_\alpha\rangle)$. Let P be any strong prime hyper MV-filter of M. Then

$$P \in \bigcap_{\alpha\in\Gamma} C(F_\alpha) \longleftrightarrow \forall\alpha \in \Gamma\ P \in C(F_\alpha) \longleftrightarrow \forall\alpha \in \Gamma\ F_\alpha \subseteq P$$

$$\longleftrightarrow \langle \bigcup_{\alpha\in\Gamma} F_\alpha\rangle \subseteq P \longleftrightarrow P \in C(\langle \bigcup_{\alpha\in\Gamma} F_\alpha\rangle).$$

Hence $\bigcap_{\alpha\in\Gamma} C(F_\alpha) \in \mathcal{C}$.

(3) Let F_1 and F_2 be two hyper MV-filters of M. Then we show that $C(F_1) \cup C(F_2) = C(F_1 \cap F_2)$. If P is a strong prime hyper MV-filter of M, then

$$P \in C(F_1) \cup C(F_2) \to F_1 \subseteq P \text{ or } F_2 \subseteq P \to F_1 \cap F_2 \subseteq P \to P \in C(F_1 \cap F_2).$$

Conversely, let $P \in C(F_1 \cap F_2)$. Then $F_1 \cap F_2 \subseteq P$. Let $F_1 \not\subseteq P$, then there exists $x \in F_1 \backslash P$. Suppose that $y \in F_2$ be arbitrary element. Then

$P \subseteq x^* \oplus y$ or $P \subseteq y^* \oplus x$. If $P \subseteq y^* \oplus x$, then $F_2 \ll y^* \oplus x$. Hence $x \in F_2$ which is a contradiction. Thus $P \subseteq x^* \oplus y$. So $F_1 \ll x^* \oplus y$. It conclude that $y \in F_1 \cap F_2 \subseteq P$. Hence $F_2 \subseteq P$, i.e., $P \in C(F_1) \cup C(F_2)$. □

Remark 4.12. Let $V(F) = HSpec(M) \backslash C(F)$ then

$$\mathcal{V} = \{V(F) : F \text{ is a hyper MV} - \text{filter of } M\}$$

is open sets family in the spectral topological space on $HSpec(M)$.

Notation: For any $x \in M$, we define

$$B(x) = V(\langle x \rangle) = \{P \in HSpec(M) : \langle x \rangle \nsubseteq P\} = \{P \in HSpec(M) : x \notin P\}.$$

Proposition 4.13. *The family $\mathcal{B} = \{B(x) : x \in M\}$ is a basis for the spectral topology on $HSpec(M)$.*

Proof. Let $V(F) \in \mathcal{V}$ for some hyper MV-filter F of M and $P \in V(F)$ be arbitrary. Then

$$P \in V(F) \rightarrow F \nsubseteq P \rightarrow \exists x \in F \ (x \notin P) \rightarrow P \in V(\langle x \rangle) = B(x).$$

Hence $P \in B(x)$. Moreover

$$K \in B(x) = V(\langle x \rangle) \rightarrow \langle x \rangle \nsubseteq K \rightarrow F \nsubseteq K \rightarrow K \in V(F).$$

Therefore there exists $B(x) \in \mathcal{B}$ such that $P \in B(x)$ and $B(x) \subseteq V(F)$. Hence $\mathcal{B} = \{B(x) : x \in M\}$ is a basis for spectral topology on $HSpec(M)$. □

Proposition 4.14. *$(HSpec((M), \mathcal{V})$ is T_1 if and only if there are no two strongly prime hyper MV-filters $P_1, P_2 \subseteq M$ such that $P_1 \subseteq P_2$.*

Proof. Let $(HSpec\,(M), \mathcal{V})$ be T_1 and there exist two strongly prime hyper MV-filters $P_1, P_2 \subseteq M$ such that $P_1 \subseteq P_2$. Then there exists an open set $V(F)$ such that $P_1 \in V(F)$ and $P_2 \notin V(F)$. Since $F \nsubseteq P_1$, then $F \nsubseteq P_2$. Thus $P_2 \in V(F)$ which is a contradiction.

Conversely, let P_1, P_2 be two different element in M. Then by assumption we have $P_1 \nsubseteq P_2$ and $P_2 \nsubseteq P_1$. Hence there exists $x \in P_1$ such that $x \notin P_2$. Thus $P_1 \notin V(\langle x \rangle)$ and $P_2 \in V(\langle x \rangle)$. Also since $P_2 \nsubseteq P_1$, then there exists $y \in P_2$ such that $y \notin P_1$. Thus $P_2 \notin V(\langle y \rangle)$ and $P_1 \in V(\langle y \rangle)$. Hence $(HSpec\,(M), \mathcal{V})$ is T_1. □

Proposition 4.15. *$(HSpec(M), \mathcal{V})$ is a T_0 topological space.*

Proof. The proof is similar to Proposition 4.14. □

Example 4.16. Consider Example 3.3. Then $HSpec(M) = \{P_1, P_2\}$. We can show that $\mathcal{C} = \{\emptyset, \{P_1\}, HSpec(M)\}$, $C(P_1) = \{P_1\}$ and $C(P_2) = \{P_1, P_2\}$. This topology is not T_1.

5. Conclusion

We introduced the notion of a S-reflexive hyper MV-filter to define an equivalence relation on a hyper MV-algebra M. We used this equivalence relation to define a uniform structure and a uniform topology τ on a hyper MV-algebra. Then we investigated some topological properties of uniform topological space. We proved that this topological space is completely regular and locally compact. We showed (1) this topology is discrete if and only if $\{1\}$ is a S-reflexive hyper MV-filter, (2) (M, τ) is connected if and only if M is the only S-reflexive hyper MV-filter of M, (3) (M, τ) is Hausdorff if and only $\{1\}$ is the the intersection of all S-reflexive hyper MV-filters of M and (4) T_0, T_1 and Hausdorff properties are equivalent on the topological space (M, τ). Furthermore, we investigated conditions under which (M, τ) is compact and we proved each S-reflexive hyper MV-filter of M is clopen. We introduced the notions of (strong) prime hyper MV-filter and hyper MV- spectrum on hyper MV-algebras and proved some related results. Then we defined spectral topology on MV- spectrum and obtained a basis for this topology. We proved this topological space is T_0 and is T_1 if and only if there are no two strongly prime filters $P_1, P_2 \subseteq M$ such that $P_1 \subseteq P_2$. In our future research we will consider the notions such as compactness, connectedness and some other properties.

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Received November 29, 2008
Revised January 20, 2009

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