$\mathcal{N}$-quasigroups

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Abstract. The notion of $\mathcal{N}$-quasigroups is introduced, and several properties are investigated. A characterization of an $\mathcal{N}$-quasigroup is given. The notion of translation of $\mathcal{N}$-quasigroups is introduced, and related properties are discussed. Using a class of subquasigroups of a quasigroup, we establish an $\mathcal{N}$-quasigroup.

1. Preliminaries

A *quasigroup* $(G, \cdot)$ is a set G with a binary operation "$\cdot$" such that for each a and b in G there exist unique elements x and y in G such that $a \cdot x = b$ and $y \cdot a = b$. The unique solutions to these equations are written $x = a\backslash b$ and $y = b/a$. The operations "$\backslash$" and "$/$" denote the defined binary opersations of left and right division, respectively. This axiomatization of quasigroups requires existential quantification and hence first order logic. The second definition of a quasigroup is grounded in universal algebra, which prefers that algebraic structures be varieties, i.e., that structures be axiomatized solely by identities. An identity is an equation in which all variables are tacitly universally quantified, and the only operations are the primitive operations proper to the structure. Quasigroups can be axiomatized in this manner if left and right division are taken as primitive.

A quasigroup $(G, \cdot, \backslash, /)$ is a type $(2, 2, 2)$ algebra satisfying the identities:

$$(x \cdot y)/y = x, \quad x\backslash(x \cdot y) = y, \quad (x/y) \cdot y = x, \quad x \cdot (x\backslash y) = y$$

(cf. [1] or [4]). Hence if $(G, \cdot)$ is a quasigroup according to the first definition, then $(G, \cdot, \backslash, /)$ is an equivalent quasigroup in the universal algebra sense. We say also that $(G, \cdot, \backslash, /)$ is an *equasigroup* (i.e., equationally definable quasigroup) [4] or a *primitive quasigroup* [1]. The equasigroup $(G, \cdot, \backslash, /)$

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corresponds to quasigroup $(G,\cdot)$ where

$$x\backslash y = z \longleftrightarrow x\cdot z = y, \qquad x/y = z \longleftrightarrow z\cdot y = x.$$

Unipotent quasigroups, i.e., quasigroups with the identity $x\cdot x = y\cdot y$, are connected with Latin squares which have one fixed element in the diagonal (cf. [2]). Such quasigroups may be defined as quasigroups G with the special element θ satisfying the identity $x\cdot x = \theta$. Obviously, θ is uniquely determined and it is an idempotent, but, in general, it is not the (left, right) neutral element. A non-empty subset S of a quasigroup $\mathcal{G} = (G,\cdot,\backslash,/)$ is called a *subquasigroup* if it is closed with respect to these three operations, i.e., $x*y\in S$ for all $x,y\in S$ and $*\in\{\cdot,\backslash,/\}$.

Denote by $\mathrm{N}(G,[-1,0])$ the collection of functions from a set G to $[-1,0]$. We say that an element of $\mathrm{N}(G,[-1,0])$ is a *negative-valued function* from G to $[-1,0]$ (briefly, $\mathcal{N}$*-function* on G). By an $\mathcal{N}$*-structure* we mean an ordered pair (G,φ) of G and an $\mathcal{N}$-function φ on G. In what follows, let G denote a quasigroup and φ an $\mathcal{N}$-function on G unless otherwise specified.

For any φ and $t\in[-1,0)$, the set

$$C(\varphi;t) := \{x\in G \mid \varphi(x)\leqslant t\}$$

is called a *closed* (φ,t)*-cut* of φ, and the set

$$O(\varphi;t) := \{x\in G \mid \varphi(x) < t\}$$

is called an *open* (φ,t)*-cut* of φ.

The investigation of such algebraic structures is motivated by bipolar-valued fuzzy sets introduced in [3] as a common generalization of intuitionistic fuzzy sets, vague sets and soft sets. Bipolar-valued fuzzy sets are fuzzy sets whose membership degree range is enlarged from the interval $[0,1]$ to the interval $[-1,1]$. In a bipolar-valued fuzzy sets, the membership degree 0 means that elements are irrelevant to the corresponding property, the membership degree $(0,1]$ indicates that elements somewhat satisfy the property, and the membership degree $[-1,0)$ indicates that elements somewhat satisfy the implicit counter-property. Bipolar-valued fuzzy sets and intuitionistic fuzzy sets look similar, but they are different (see [3]).

2. $\mathcal{N}$-quasigroups

In what follows, let G denote a quasigroup and φ an $\mathcal{N}$-function on G unless otherwise specified.

Definition 2.1. By a *quasigroup* of G based on φ (briefly, $\mathcal{N}$*-quasigroup of* G), we mean an $\mathcal{N}$-structure (G, φ) such that every non-empty closed (φ, t)-cut $C(\varphi; t)$, where $t \in [-1, 0)$, of φ is a subquasigroup of G.

Example 2.2. Let $G = \{1, 2, 3, 4\}$ be a set with the following Cayley table:

$\cdot$	1	2	3	4
1	2	1	3	4
2	1	2	4	3
3	4	3	1	2
4	3	4	2	1

Then $(G, \cdot)$ is a quasigroup. The $\backslash$-operation and the $/$-operation on G are given by the following Cayley tables respectively:

$\backslash$	1	2	3	4
1	2	1	3	4
2	1	2	4	3
3	3	4	2	1
4	4	3	1	2

$/$	1	2	3	4
1	2	1	3	4
2	1	2	4	3
3	4	3	1	2
4	3	4	2	1

Define an $\mathcal{N}$-function φ on G by

G	1	2	3	4
φ	-0.7	-0.7	-0.4	-0.4

It is routine to check that (G, φ) is an $\mathcal{N}$-quasigroup of G.

Example 2.3. Consider a quasigroup $(\mathbb{Z}, -)$ where $\mathbb{Z}$ is the set of all integers. Let φ be an $\mathcal{N}$-function on $\mathbb{Z}$ defined by

$$\varphi(x) = \begin{cases} -0.6 & \text{if } \ x \in 2\mathbb{Z}, \\ -0.3 & \text{otherwise} \end{cases}$$

for all $x \in \mathbb{Z}$. Then $(\mathbb{Z}, \varphi)$ is an $\mathcal{N}$-quasigroup of $\mathbb{Z}$.

We first give a characterization of an $\mathcal{N}$-quasigroup of G.

Theorem 2.4. *Let (G, φ) be an $\mathcal{N}$-structure of G and φ. Then (G, φ) is an $\mathcal{N}$-quasigroup of G if and only if it satisfies:*

$$\varphi(x * y) \leqslant \max\{\varphi(x), \varphi(y)\} \tag{1}$$

for all $x, y \in G$ and $* \in \{\cdot, \backslash, /\}$.

Proof. Assume that (G, φ) is an $\mathcal{N}$-quasigroup of G, that is, $C(\varphi; t)$ is a non-empty subquasigroup of G for all $t \in [-1, 0)$. If the inequality (1) is not valid for some $* \in \{\cdot, \backslash, /\}$, then there exist $a, b \in G$ and $t_0 \in [-1, 0)$ such that $\varphi(a*b) > t_0 \geqslant \max\{\varphi(a), \varphi(b)\}$. It follows that $a, b \in C(\varphi; t_0)$ and $a*b \notin C(\varphi; t_0)$. This is a contradiction since $C(\varphi; t_0)$ is a subquasigroup of G. Therefore the inequality (1) is valid for all $* \in \{\cdot, \backslash, /\}$.

Conversely, suppose that the inequality (1) is true for all $* \in \{\cdot, \backslash, /\}$ and $x, y \in G$. Let $t \in [-1, 0)$ be such that $C(\varphi; t) \neq \emptyset$. Let $x, y \in C(\varphi; t)$. Then $\varphi(x) \leqslant t$ and $\varphi(y) \leqslant t$. It follows from (1) that

$$\varphi(x * y) \leqslant \max\{\varphi(x), \varphi(y)\} \leqslant t$$

so that $x * y \in C(\varphi; t)$ for all $* \in \{\cdot, \backslash, /\}$. Hence $C(\varphi; t)$ is a subquasigroup of G, and so (G, φ) is an $\mathcal{N}$-quasigroup of G. □

Corollary 2.5. *If (G, φ) is an $\mathcal{N}$-quasigroup of G, then every non-empty open (φ, t)-cut of G is a subquasigroup of G for all $t \in [-1, 0)$.*

Proof. Straightforward. □

Let φ and ψ be $\mathcal{N}$-functions on G. The *union* $\varphi \cup \psi$ and the *intersection* $\varphi \cap \psi$ of φ and ψ are defined by

$$(\forall x \in G)((\varphi \cup \psi)(x) = \max\{\varphi(x), \psi(x)\}),$$

$$(\forall x \in G)((\varphi \cap \psi)(x) = \min\{\varphi(x), \psi(x)\}),$$

respectively.

Theorem 2.6. *If (G, φ) and (G, ψ) are $\mathcal{N}$-quasigroups of G, then $(G, \varphi \cup \psi)$ is also an $\mathcal{N}$-quasigroup of G.*

Proof. Let $x, y \in G$ and $* \in \{\cdot, \backslash, /\}$. Then

$$\begin{aligned}(\varphi \cup \psi)(x * y) &= \max\{\varphi(x * y), \psi(x * y)\}\\ &\leqslant \max\{\max\{\varphi(x), \varphi(y)\}, \max\{\psi(x), \psi(x)\}\}\\ &= \max\{\max\{\varphi(x), \psi(x)\}, \max\{\varphi(y), \psi(y)\}\}\\ &= \max\{(\varphi \cup \psi)(x), (\varphi \cup \psi)(y)\}.\end{aligned}$$

Therefore $(G, \varphi \cup \psi)$ is an $\mathcal{N}$-quasigroup of G. □

The following example shows that $(G, \varphi \cap \psi)$ is not an $\mathcal{N}$-quasigroup of G although (G, φ) and (G, ψ) are $\mathcal{N}$-quasigroups of G.

Example 2.7. Let $G = \{1, 2, 3, 4, 5, 6\}$ be a set with the following Cayley table:

$\cdot$	1	2	3	4	5	6
1	1	2	3	6	5	4
2	4	6	1	5	2	3
3	6	5	4	1	3	2
4	5	4	2	3	6	1
5	3	1	6	2	4	5
6	2	3	5	4	1	6

Then $(G, \cdot)$ is a quasigroup. Define two $\mathcal{N}$-functions φ and ψ on G by

G	1	2	3	4	5	6
φ	-0.7	-0.4	-0.4	-0.4	-0.4	-0.4
ψ	-0.3	-0.3	-0.3	-0.3	-0.3	-0.8

Then (G, φ) and (G, ψ) are $\mathcal{N}$-quasigroups of G. Note that if $t \in [-0.7, -0.4)$, then $C(\varphi \cap \psi; t) = \{1, 6\}$ is not a subquasigroup of G. Hence $(G, \varphi \cap \psi)$ is not an $\mathcal{N}$-quasigroup of G.

Proposition 2.8. *Let G be a unipotent quasigroup. If (G, φ) is an $\mathcal{N}$-quasigroup of G, then $\varphi(\theta) \leqslant \varphi(x)$ for all $x \in G$.*

Proof. Since $x \cdot x = \theta$ for all $x \in G$, we have

$$\varphi(\theta) = \varphi(x \cdot x) \leqslant \max\{\varphi(x), \varphi(x)\} = \varphi(x)$$

for all $x \in G$ by (1). □

Proposition 2.9. *Let (G, φ) be an $\mathcal{N}$-quasigroup of G. For any $* \in \{\cdot, \backslash, /\}$ and $x, y \in G$, we have*

$$\max\{\varphi(x * y), \varphi(x)\} = \max\{\varphi(x * y), \varphi(y)\} = \max\{\varphi(x), \varphi(y)\}. \tag{2}$$

Proof. We first consider the case when $*$ is the quasigroup multiplication. Since $(x \cdot y)/y = x$ for all $x, y \in G$, we get

$$\begin{aligned}\max\{\varphi(x\cdot y),\varphi(y)\} &\leqslant \max\{\max\{\varphi(x),\varphi(y)\},\varphi(y)\}\\ &= \max\{\varphi(x),\varphi(y)\}\\ &= \max\{\varphi((x\cdot y)/y),\varphi(y)\}\\ &\leqslant \max\{\max\{\varphi(x\cdot y),\varphi(y)\},\varphi(y)\}\\ &= \max\{\varphi(x\cdot y),\varphi(y)\}\end{aligned}$$

and so

$$\max\{\varphi(x\cdot y),\varphi(y)\} = \max\{\varphi(x),\varphi(y)\} \tag{3}$$

for all $x, y \in G$. Note that $x\backslash (x\cdot y) = y$ for all $x, y \in G$. Using (1), we have

$$\begin{aligned}\max\{\varphi(x\cdot y),\varphi(x)\} &\leqslant \max\{\max\{\varphi(x),\varphi(y)\},\varphi(x)\}\\ &= \max\{\varphi(x),\varphi(y)\}\\ &= \max\{\varphi(x),\varphi(x\backslash (x\cdot y))\}\\ &\leqslant \max\{\varphi(x),\max\{\varphi(x),\varphi(x\cdot y)\}\}\\ &= \max\{\varphi(x\cdot y),\varphi(x)\}\end{aligned}$$

which implies that

$$\max\{\varphi(x\cdot y),\varphi(x)\} = \max\{\varphi(x),\varphi(y)\} \tag{4}$$

for all $x, y \in G$.

We now discuss the case when $*$ is the left division. Then for any $x, y \in X$, we obtain

$$\max\{\varphi(x\backslash y),\varphi(x)\} \leqslant \max\{\max\{\varphi(x),\varphi(y)\},\varphi(x) = \max\{\varphi(x),\varphi(y)\}$$

by using (1). Since $x\cdot(x\backslash y) = y$ for all $x, y \in G$, we have

$$\begin{aligned}\max\{\varphi(x),\varphi(y)\} &= \max\{\varphi(x),\varphi(x\cdot(x\backslash y))\}\\ &\leqslant \max\{\varphi(x),\max\{\varphi(x),\varphi(x\backslash y)\}\}\\ &= \max\{\varphi(x),\varphi(x\backslash y)\}.\end{aligned}$$

Hence $\max\{\varphi(x\backslash y),\varphi(x) = \varphi(\varphi(x),\varphi(y)\}$ for all $x, y \in G$. Since

$$x\backslash y = z \longleftrightarrow x\cdot z = y$$

for all $x, y, z \in G$, we know, by using (3), that

$$\begin{aligned}\max\{\varphi(x\backslash y), \varphi(y)\} &= \max\{\varphi(z), \varphi(x \cdot z)\}\\ &= \max\{\varphi(z), \varphi(x)\}\\ &= \max\{\varphi(x\backslash y), \varphi(x)\}\\ &= \max\{\varphi(x), \varphi(y)\}.\end{aligned}$$

We finally consider the case when $*$ is the right division. Then

$$\max\{\varphi(x/y), \varphi(y)\} \leqslant \max\{\max\{\varphi(x), \varphi(y)\}, \varphi(y)\} = \max\{\varphi(x), \varphi(y)\}.$$

Using (1) and the identity $x = (x/y) \cdot y$, we obtain

$$\begin{aligned}\max\{\varphi(x), \varphi(y)\} &= \max\{\varphi((x/y) \cdot y), \varphi(y)\}\\ &\leqslant \max\{\max\{\varphi(x/y), \varphi(y)\}, \varphi(y)\}\\ &= \max\{\varphi(x/y), \varphi(y)\}.\end{aligned}$$

Therefore

$$\max\{\varphi(x/y), \varphi(y)\} = \max\{\varphi(x), \varphi(y)\} \tag{5}$$

for all $x, y \in G$. Note that $x/y = u$ implies $u \cdot y = x$ for all $u, x, y \in G$. Then

$$\begin{aligned}\max\{\varphi(x/y), \varphi(x)\} &= \max\{\varphi(u), \varphi(u \cdot y)\}\\ &= \max\{\varphi(u), \varphi(y)\}\\ &= \max\{\varphi(x/y), \varphi(y)\}\\ &= \max\{\varphi(x), \varphi(y)\}\end{aligned}$$

by (4) and (5). This completes the proof. □

Corollary 2.10. *Let (G, φ) be an $\mathcal{N}$-quasigroup of G. For any $x, y \in G$, if $\varphi(x) < \varphi(y)$ then $\varphi(x * y) = \varphi(x) = \varphi(y * x)$ for all $* \in \{\cdot, \backslash, /\}$.*

Proof. Straightforward. □

For any element w of G, we consider the set

$$G_w := \{x \in G \mid \varphi(x) \leqslant \varphi(w)\}.$$

Obviously, $w \in G_w$, and so G_w is non-empty.

Theorem 2.11. *Let w be an element of G. If (G,φ) is an $\mathcal{N}$-quasigroup of G, then G_w is a subquasigroup of G.*

Proof. Let $x, y \in G_w$. Then $\varphi(x) \leqslant \varphi(w)$ and $\varphi(y) \leqslant \varphi(w)$. It follows from (1) that

$$\varphi(x * y) \leqslant \max\{\varphi(x), \varphi(y)\} \leqslant \varphi(w)$$

so that $x * y \in G_w$ for all $* \in \{\cdot, \backslash, /\}$. Hence G_w is a subquasigroup of G. □

Theorem 2.12. *Let φ be an $\mathcal{N}$-function on G with*

$$\mathrm{Im}(\varphi) = \{t_0, t_1, t_2, \ldots, t_n\},$$

where $t_0 < t_1 < t_2 < \ldots < t_n$. Let $\{Q_k \mid k = 0, 1, 2, \ldots, n\}$ be a class of subquasigroups of G such that

(*i*) $Q_0 \subset Q_1 \subset Q_2 \subset \ldots \subset Q_n = G$,

(*ii*) $\varphi(Q_k^+) = t_k$ *where* $Q_k^+ = Q_k \setminus Q_{k-1}$ *and* $Q_{-1} = \emptyset$ *for* $k = 0, 1, 2, \ldots, n$.

Then (G,φ) is an $\mathcal{N}$-quasigroup of G.

Proof. Let $x, y \in G$. Then $x \in Q_k^+$ and $y \in Q_r^+$ for some $k, r \in \{0, 1, 2, \ldots, n\}$. We may assume that $k \geqslant r$ without loss of generality. Then $x, y \in Q_k$ since $Q_r^+ \subset Q_r \subseteq Q_k$ and $Q_k^+ \subset Q_k$. Since Q_k is a subquasigroup of G, we have $x * y \in Q_k$ for all $* \in \{\cdot, \backslash, /\}$. Hence

$$\varphi(x * y) \leqslant t_k = \max\{t_k, t_r\} = \max\{\varphi(x), \varphi(y)\}$$

for all $* \in \{\cdot, \backslash, /\}$. Therefore (G,φ) is an $\mathcal{N}$-quasigroup of G. □

For any $\mathcal{N}$-function φ on G, we denote

$$\phi := -1 - \inf\{\varphi(x) \mid x \in X\}.$$

For any $\alpha \in [\phi, 0]$, we define $\varphi_\alpha^T(x) = \varphi(x) + \alpha$ for all $x \in G$. Obviously, φ_α^T is a mapping from G to $[-1, 0]$, that is, φ_α^T is an $\mathcal{N}$-function on G. We say that (G, φ_α^T) is an *α-translation* of (G,φ).

Theorem 2.13. *For any $\alpha \in [\phi, 0]$, the α-translation (G, φ_α^T) of an $\mathcal{N}$-quasigroup (G,φ) is an $\mathcal{N}$-quasigroup of G.*

Proof. For any $x, y \in G$ and $* \in \{\cdot, \backslash, /\}$, we have

$$\begin{aligned}\varphi_\alpha^T(x*y) &= \varphi(x*y)+\alpha \\ &\leqslant \max\{\varphi(x),\varphi(y)\}+\alpha \\ &= \max\{\varphi(x)+\alpha,\varphi(y)+\alpha\} \\ &= \max\{\varphi_\alpha^T(x),\varphi_\alpha^T(y)\}.\end{aligned}$$

Therefore (G, φ_α^T) is an $\mathcal{N}$-quasigroup of G. □

Theorem 2.14. *If there exists $\alpha \in [\phi, 0]$ such that α-translation (G, φ_α^T) of (G, φ) is an $\mathcal{N}$-quasigroup of G, then (G, φ) is an $\mathcal{N}$-quasigroup of G.*

Proof. Assume that (G, φ_α^T) is an $\mathcal{N}$-quasigroup of G for some $\alpha \in [\phi, 0]$. Let $x, y \in G$ and $* \in \{\cdot, \backslash, /\}$. Then

$$\begin{aligned}\varphi(x*y)+\alpha &= \varphi_\alpha^T(x*y) \\ &\leqslant \max\{\varphi_\alpha^T(x),\varphi_\alpha^T(y)\} \\ &= \max\{\varphi(x)+\alpha,\varphi(y)+\alpha\} \\ &= \max\{\varphi(x),\varphi(y)\}+\alpha,\end{aligned}$$

which implies that $\varphi(x*y) \leqslant \max\{\varphi(x), \varphi(y)\}$. Thus (G, φ) is an $\mathcal{N}$-quasigroup of G. □

For any $\mathcal{N}$-function φ on G, $\alpha \in [\phi, 0]$ and $t \in [-1, \alpha]$, let

$$L_\alpha(\varphi; t) := \{x \in G \mid \varphi(x) \leqslant t - \alpha\}.$$

Proposition 2.15. *Let (G, φ) be an $\mathcal{N}$-structure of G and φ, and let $\alpha \in [\phi, 0]$. If (G, φ) is an $\mathcal{N}$-quasigroup of G, then each non-empty $L_\alpha(\varphi; t)$, where $t \in [-1, \alpha]$, is a subquasigroup of G.*

Proof. Assume that (G, φ) is an $\mathcal{N}$-quasigroup of G and let $t \in [-1, \alpha]$ such that $L_\alpha(\varphi; t) \neq \emptyset$. Let $x, y \in L_\alpha(\varphi; t)$. Then $\varphi(x) \leqslant t - \alpha$ and $\varphi(y) \leqslant t - \alpha$. It follows from (1) that

$$\varphi(x*y) \leqslant \max\{\varphi(x), \varphi(y)\} \leqslant t - \alpha$$

so that $x * y \in L_\alpha(\varphi; t)$ for all $* \in \{\cdot, \backslash, /\}$. Hence $L_\alpha(\varphi; t)$ is a subquasigroup of G. □

Theorem 2.16. *Let (G, φ) be an $\mathcal{N}$-structure and $\alpha \in [\phi, 0]$. Then the α-translation (G, φ_α^T) of (G, φ) is an $\mathcal{N}$-quasigroup of G if and only if for all $t \in [-1, \alpha]$ each non-empty $L_\alpha(\varphi; t)$ is a subquasigroup of G.*

Proof. Assume that (G, φ_α^T) is an $\mathcal{N}$-quasigroup of G and let $t \in [-1, \alpha]$ such that $L_\alpha(\varphi; t) \neq \emptyset$. Let $x, y \in L_\alpha(\varphi; t)$. Then $\varphi(x) \leqslant t - \alpha$ and $\varphi(y) \leqslant t - \alpha$. Hence

$$\begin{aligned} \varphi(x * y) + \alpha = \varphi_\alpha^T(x * y) &\leqslant \max\{\varphi_\alpha^T(x), \varphi_\alpha^T(y)\} \\ &= \max\{\varphi(x) + \alpha, \varphi(y) + \alpha\} \\ &= \max\{\varphi(x), \varphi(y)\} + \alpha \leqslant t \end{aligned}$$

for all $* \in \{\cdot, \backslash, /\}$. It follow that $\varphi(x * y) \leqslant t - \alpha$ so that $x * y \in L_\alpha(\varphi; t)$ for all $* \in \{\cdot, \backslash, /\}$. Therefore $L_\alpha(\varphi; t)$ is a subquasigroup of G.

Conversely, let $* \in \{\cdot, \backslash, /\}$. We claim that

$$\varphi_\alpha^T(x * y) \leqslant \max\{\varphi_\alpha^T(x), \varphi_\alpha^T(y)\} \tag{6}$$

for all $x, y \in G$. If (6) is false, then $\varphi_\alpha^T(a * b) > s \geqslant \max\{\varphi_\alpha^T(a), \varphi_\alpha^T(b)\}$ for some $a, b \in G$ and $s \in [-1, \alpha]$. Hence $\varphi(a) \leqslant s - \alpha$ and $\varphi(b) \leqslant s - \alpha$, but $\varphi(a * b) > s - \alpha$. Thus $a, b \in L_\alpha(\varphi; s)$ and $a * b \notin L_\alpha(\varphi; s)$. This is a contradiction, and so (G, φ_α^T) is an $\mathcal{N}$-quasigroup of G. □

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