Quasigroups and Related Systems **17** (2009), 283 – 290

Congruences on an inverse AG^{**}-groupoid via the natural partial order

Petar V. Protić

In memory of **Nebojša Stevanović (1962–2009)**, my colleague and dear friend.

Abstract. In this paper we first describe natural partial order on an inverse AG^{**}-groupoid. With it we introduce a notion of pseudo normal congruence pair and normal congruence pair and describe congruences.

1. Introduction

A groupoid S on which the following is true

$$(\forall a,b,c \in S) \quad ab \cdot c = cb \cdot a,$$

is called an *Abel-Grassmann's groupoid* (*AG-groupoid*) [8] (or in some papers Left almost semigroups (*LA-semigroups*)) [3]. It is easy to verify that in every AG-groupoid *medial law* $ab \cdot cd = ac \cdot bd$ holds. Thus, AG-groupoids belong to the wider class of medial groupoids.

We denote the set of all idempotents of S by $E(S)$.

Abel-Grassmann's groupoid S satisfying

$$(\forall a,b,c \in S) \quad a \cdot bc = b \cdot ac$$

is an AG^{**}-*groupoid.* It is obvious that in AG^{**}-groupoid for $a,b,c,d \in S$

$$ab \cdot cd = c(ab \cdot d) = c(db \cdot a) = db \cdot ca.$$

2000 Mathematics Subject Classification: 20N02
Keywords: Abel-Grassmann's groupoid, natural partial order, congruence pair
Supported by Grant ON 144013 of Ministry of science through Math. Inst. SANU

If AG-groupoid S has the left identity e, then

$$a \cdot bc = ea \cdot bc = eb \cdot ac = b \cdot ac,$$

so S is an AG^{**}-groupoid.

In [5] an AG-groupoid S is called an *inverse AG-groupoid* if for every $a \in S$ there exists $a' \in S$ such that $a = aa' \cdot a$ and $a' = a'a \cdot a'$. Then a' is an inverse element of a, and by $V(a)$ we shall mean the set of all inverses of a. It is easy to prove that if $a' \in V(a)$, $b' \in V(b)$, then $a'b' \in V(ab)$ and that aa' or $a'a$ are not necessarily idempotents.

Remark 1. In [1] it is proved that in an AG^{**}-groupoid S the set $E(S)$ is a semilattice (Remark 2). Also, in [1] it is proved that in an inverse AG^{**}-groupoid for $a \in S$, by Remark 3, we have $|V(a)| = 1$. If a^{-1} is a unique inverse for a, then by Lemma 1 $aa^{-1}, a^{-1}a \in E(S)$ if and only if $aa^{-1} = a^{-1}a$.

The following proposition is is trivially true.

Proposition 1. *Let S be an inverse AG^{**}-groupoid and ρ congruence relation on S. Then $S/_\rho$ is an inverse AG^{**}-groupoid. Also, if $a, b \in S$ then $a\rho b$ if and only if $a^{-1}\rho b^{-1}$.* $\square$

2. Natural partial order

In this section we define a natural partial relation on inverse AG^{**}-groupoid S and prove some of its properties.

Theorem 1. *If S is an inverse AG^{**}-groupoid, then the relation*

$$a \leqslant b \Longleftrightarrow a = aa^{-1} \cdot b \tag{1}$$

on S is a natural partial order relation and it is compatible.

Proof. The proof that $\leqslant$ is reflexive is obvious. For antisymmetry let us suppose that $a \leqslant b$ and $b \leqslant a$. Then $a = aa^{-1} \cdot b$ and $b = bb^{-1} \cdot a$, and

$$a = aa^{-1} \cdot b = aa^{-1} \cdot (bb^{-1} \cdot a) = bb^{-1} \cdot (aa^{-1} \cdot a) = bb^{-1} \cdot a = b,$$

imply antisymmetry.

Let us now suppose that $a \leqslant b$ and $b \leqslant c$. Then $a = aa^{-1} \cdot b$, $b = bb^{-1} \cdot c$, and

$$\begin{aligned} a &= aa^{-1} \cdot b = aa^{-1}(bb^{-1} \cdot c) = ((aa^{-1} \cdot a)a^{-1})(bb^{-1} \cdot c) \\ &= (a^{-1}a \cdot aa^{-1})(bb^{-1} \cdot c) = (a^{-1}a \cdot bb^{-1})(aa^{-1} \cdot c) \\ &= b(a^{-1}a \cdot b^{-1}) \cdot (aa^{-1} \cdot c) = b(aa^{-1} \cdot b)^{-1} \cdot (aa^{-1} \cdot c) \\ &= ba^{-1} \cdot (aa^{-1} \cdot c) = ca^{-1} \cdot (aa^{-1} \cdot b) = ca^{-1} \cdot a = aa^{-1} \cdot c, \end{aligned}$$

imply that $a \leqslant c$. Hence transitivity holds and $\leq$ is a partial order on S.

Let $a \leqslant b$ and $c \in S$. Then

$$\begin{aligned} ca &= c(aa^{-1} \cdot b) = (cc^{-1} \cdot c)(aa^{-1}) \cdot b = (cc^{-1} \cdot aa^{-1}) \cdot cb \\ &= (ca \cdot c^{-1}a^{-1}) \cdot cb = (ca \cdot (ca)^{-1}) \cdot cb, \end{aligned}$$

and so the relation $\leq$ is left compatible. Also, since

$$\begin{aligned} ac &= (aa^{-1} \cdot b)c = (aa^{-1} \cdot b)(cc^{-1} \cdot c) = (aa^{-1} \cdot cc^{-1}) \cdot bc \\ &= (ac \cdot a^{-1}c^{-1}) \cdot bc = (ac \cdot (ac)^{-1}) \cdot bc, \end{aligned}$$

therefore the relation $\leq$ is right compatible. Hence, $\leq$ is compatible. □

Corollary 1. *Let S be an inverse AG^{**}-groupoid and $a, b \in S$. Then*

$$a \leqslant b \iff aa^{-1} = ba^{-1}.$$

Proof. If $a \leqslant b$ then by (1) we have

$$aa^{-1} = (aa^{-1} \cdot b)a^{-1} = a^{-1}b \cdot aa^{-1} = a^{-1}a \cdot ba^{-1} = b(a^{-1}a \cdot a^{-1}) = ba^{-1}.$$

Conversely, for $a, b \in S$, $aa^{-1} = ba^{-1}$ implies that

$$a = aa^{-1} \cdot a = ba^{-1} \cdot a = aa^{-1} \cdot b.$$

So, by (1), $a \leqslant b$. □

3. Normal congruence pair

In this section by S we mean an inverse AG^{**}-groupoid in which for each $a \in S$ we have $aa^{-1} = a^{-1}a$ or equivalently $aa^{-1}, a^{-1}a \in E(S)$.

First, we prove the following consequence of Theorem 1.

Corollary 2. *Let* $a, b \in S$. *Then*

$$a \leqslant b \iff (\exists e \in E(S))\ a = eb.$$

Proof. Let $a, b \in S$. Then $a \leqslant b$ if and only if $a = (aa^{-1})b$. Since $aa^{-1} \in E(S)$, therefore if $e = aa^{-1}$ implies that $a = eb$.

Conversely, let $a, b \in S$ be such that $e \in E(S)$ and $a = eb$. Because $aa^{-1} = a^{-1}a \in E(S)$ and $E(S)$ is a semilattice, we have

$$\begin{aligned} aa^{-1} \cdot b &= (eb \cdot eb^{-1})b = (bb^{-1} \cdot e)b = (bb^{-1} \cdot e)(bb^{-1} \cdot b) \\ &= (bb^{-1} \cdot bb^{-1}) \cdot eb = bb^{-1} \cdot eb = e(bb^{-1} \cdot b) = eb = a \end{aligned}$$

and so $a \leqslant b$. □

Let ρ be a congruence on S. The restriction $\rho|_{E(S)}$ is the *trace* of ρ and it is denoted by trρ. Also, kernel ρ is $\ker\rho = \{a \in S \mid (\exists e \in E(S))\ a\rho e\}$.

If ρ is a congruence relation on S, then $\ker\rho$ is a subgroupoid of S and $E(S) \subseteq \ker\rho$ it is, $\ker\rho$ is a *full* subgroupoid of S. Also, trρ is a congruence on semillatice $E(S)$.

Definition 1. Let K be a full subgroupoid of S and τ a congruence on $E(S)$ satisfying the following condition:

(i) For all $a \in S, b \in K$, $b \leqslant a$ and $aa^{-1}\tau bb^{-1}$ imply $a \in K$.

We call (K, τ) a *pseudo normal congruence pair for* S. If, in addition,

(ii) For every $a \in K$, there exists $b \in S$ with $b \leqslant a$, $aa^{-1}\tau\, bb^{-1}$ and $b^{-1} \in K$,

then (K, τ) is called a *normal congruence pair for* S.

For pseudo normal congruence pair (K, τ), we define a relation

$$a\rho_{(K,\tau)}b \iff ab^{-1}, a^{-1}b, ba^{-1}, b^{-1}a \in K,\ aa^{-1} \cdot b^{-1}b\,\tau\, aa^{-1}\tau\, bb^{-1}.$$

Lemma 1. *Let* (K, τ) *be a pseudo normal congruence pair of* S, $a, b \in S$. *If* $a\,\rho_{(K,\tau)}b$ *and* $b \in K$, *then* $a \in K$.

Proof. From $a\,\rho_{(K,\tau)}b$ we have $ab^{-1} \in K$ and $aa^{-1} \cdot bb^{-1}\tau\, aa^{-1}\tau\, bb^{-1}$. Since $b \in K$, it follows that $ab^{-1} \cdot b = bb^{-1} \cdot a \in K$.

We prove that $ab^{-1} \cdot b \leqslant a$. Here

$$\begin{aligned} ((ab^{-1} \cdot b)(ab^{-1} \cdot b)^{-1})a &= ((ab^{-1} \cdot b)(a^{-1}b \cdot b^{-1}))a = ((bb^{-1} \cdot a)(b^{-1}b \cdot a^{-1}))a \\ &= ((bb^{-1} \cdot b^{-1}b)aa^{-1})a = (bb^{-1} \cdot aa^{-1})a \\ &= (aa^{-1} \cdot bb^{-1})a = (aa^{-1} \cdot bb^{-1})(aa^{-1} \cdot a) \\ &= (aa^{-1} \cdot aa^{-1})(bb^{-1} \cdot a) = aa^{-1}(bb^{-1} \cdot a) \\ &= bb^{-1}(aa^{-1} \cdot a) = bb^{-1} \cdot a = ab^{-1} \cdot b. \end{aligned}$$

Hence, by (1), it follows that $ab^{-1} \cdot b \leq a$.

Also

$$\begin{aligned}(ab^{-1} \cdot b)(ab^{-1} \cdot b)^{-1} &= (ab^{-1} \cdot b)(a^{-1}b \cdot b^{-1}) \\ &= (ab^{-1} \cdot a^{-1}b)bb^{-1} = (aa^{-1} \cdot b^{-1}b)bb^{-1} \\ &= (bb^{-1} \cdot b^{-1}b) \cdot aa^{-1}) = bb^{-1} \cdot aa^{-1} \tau aa^{-1},\end{aligned}$$

whence by Definition 1 (i) it follows that $a \in K$. □

Theorem 2. *If (K, τ) is a pseudo normal congruence pair for S, then $\rho_{(K,\tau)}$ is a congruene on S with*

$$\ker \rho_{(K,\tau)} = \{a \in K \mid (\exists b \in S),\ a \geqslant b,\ aa^{-1}\tau\, bb^{-1},\ b^{-1} \in K\} \tag{2}$$

and the trace is equal to τ. Moreover, if (K_1, τ_1) and (K_2, τ_2) are pseudo congruence pairs for S with $K_1 \subseteq K_2$ and $\tau_1 \subseteq \tau_2$, then $\rho_{(K_1,\tau_1)} \subseteq \rho_{(K_2,\tau_2)}$.

Proof. Let (K, τ) be a pseudo normal congruence pair for S and $\rho = \rho_{(K,\tau)}$. Since K is full it follows that ρ is reflexive. Obviously, ρ is symmetric. We verify that ρ is transitive after we prove that ρ is compatible.

Assume now that $a\rho b$ and let $c \in S$. Then

$$ac \cdot (bc)^{-1} = ac \cdot b^{-1}c^{-1} = ab^{-1} \cdot cc^{-1} \subseteq K \cdot E(S) \subseteq K.$$

Similarly,

$$(ac)^{-1} \cdot bc, bc \cdot (ac)^{-1}, (bc)^{-1} \cdot ac \in K.$$

Next we have

$$\begin{aligned}(ac \cdot (ac)^{-1})((bc)^{-1} \cdot bc)) &= (ac \cdot (bc)^{-1})((ac)^{-1} \cdot bc) \\ &= (ac \cdot b^{-1}c^{-1})(a^{-1}c^{-1} \cdot bc) \\ &= (ab^{-1} \cdot cc^{-1})(a^{-1}b \cdot c^{-1}c) \\ &= (ab^{-1} \cdot a^{-1}b)(cc^{-1} \cdot cc^{-1}) \\ &= (aa^{-1} \cdot b^{-1}b)cc^{-1}\tau\, aa^{-1} \cdot cc^{-1} \\ &= ac \cdot a^{-1}c^{-1} = ac \cdot (ac)^{-1}.\end{aligned}$$

By symmetry, it follows that

$$(ac \cdot (ac)^{-1})((bc)^{-1} \cdot bc)\, \tau\, bc \cdot (bc)^{-1},$$

whence $ac\, \rho bc$. Thus ρ is right compatible. Analogously, ρ is left compatible. Hence, ρ is compatible.

Now, suppose that $a\,\rho\,b$ and $b\,\rho\,c$. Then by right compatibility $ac^{-1}\,\rho\,bc^{-1}$ and $bc^{-1}\,\rho\,cc^{-1}$. Since $cc^{-1}\in E(S)\subseteq K$ and $bc^{-1}\rho cc^{-1}$, we have $bc^{-1}\in K$ by Lemma 1, and subsequently $ac^{-1}\in K$. Similarly, $aa^{-1}\rho ba^{-1}, ba^{-1}\rho ca^{-1}$ yield $ca^{-1}\in K$ by Lemma 1.

Similarly, by left compatibility, from $a\rho b$ and $b\rho c$ we have $a^{-1}a\rho a^{-1}b$, $a^{-1}b\rho a^{-1}c$, $c^{-1}a\rho c^{-1}b$ and $c^{-1}b\rho c^{-1}c$. So by Lemma 1 it follows that $a^{-1}c, c^{-1}a\in K$.

Also $a\rho b$, $b\rho c$ yields

$$a^{-1}a\cdot bb^{-1}\tau aa^{-1}\tau bb^{-1},\quad b^{-1}b\cdot cc^{-1}\tau bb^{-1}\tau cc^{-1}$$

and by transitivity it follows that $aa^{-1}\tau cc^{-1}$. Moreover,

$$(bb^{-1}\cdot cc^{-1})(aa^{-1}\cdot cc^{-1}) = (bb^{-1}\cdot aa^{-1})cc^{-1}\tau\, aa^{-1}\cdot cc^{-1},$$
$$(bb^{-1}\cdot cc^{-1})(aa^{-1}\cdot cc^{-1}) = (bb^{-1}\cdot aa^{-1})cc^{-1}\tau bb^{-1}\cdot cc^{-1}\tau\, cc^{-1},$$

whence $aa^{-1}\cdot cc^{-1}\tau cc^{-1}$.

Now, $ac^{-1}, a^{-1}c, ca^{-1}, c^{-1}a\in K$, $aa^{-1}\cdot cc^{-1}\tau\, aa^{-1}\tau\, cc^{-1}$ is equivalent to $a\rho c$. Hence, ρ is a transitive relation and so is a congruence.

It is apparent that for $e,f\in E(S)$, $e\rho f$ if and only if $e\tau f$ whence $\mathrm{tr}\rho=\tau$.

We let

$$H=\{a\in K\mid (\exists b\in S)\ a\geqslant b, b^{-1}\in K, aa^{-1}\tau bb^{-1}\}$$

and we show that $\ker\rho = H$.

Let $a\in H$, then there exists $b\in K$ such that $b\leqslant a$, $b^{-1}\in H$ and $aa^{-1}\,\tau\, bb^{-1}$. By (1) $b\leqslant a$ it implies that $b=bb^{-1}\cdot a$. We next prove that $a\rho bb^{-1}$ that is

$$bb^{-1}\cdot a^{-1},\ a^{-1}\cdot bb^{-1},\ bb^{-1}\cdot a,\ a\cdot bb^{-1}\in K,\ bb^{-1}\cdot aa^{-1}\tau aa^{-1}\tau\, bb^{-1}.$$

Now $b=bb^{-1}\cdot a\in K$ and $b^{-1}=bb^{-1}\cdot a^{-1}\in K$. Also we have $a\cdot bb^{-1}\in K\cdot E(S)\subseteq K$ and

$$a^{-1}\cdot bb^{-1}=(a^{-1}a\cdot a^{-1})bb^{-1}=(bb^{-1}\cdot a^{-1})a^{-1}a\in K\cdot E(S)\subseteq K.$$

Conversely, let $a\in\ker\rho$. Then $a\rho e$ for some $e\in E(S)$. If $b=ea$, then $b\leqslant a$ by Corollary 2 and $b=ea\in E(S)\cdot K\subseteq K$. From $a\rho e$ it follows that $aa^{-1}=ea^{-1}=b^{-1}$ and since $aa^{-1}\in K$ we have by Lemma 1 that $b^{-1}\in K$. Because $b,b^{-1}\in K$ we have $bb^{-1}=b^{-1}b\in K$ and so $b\rho b^{-1}$. Now

$$\begin{aligned} bb^{-1}\rho b^{-1}b^{-1} &= ea^{-1}\cdot ea^{-1}\rho aa^{-1}\cdot ea^{-1}\\ &= e(a^{-1}a\cdot a^{-1}) = ea^{-1}\rho aa^{-1}\end{aligned}$$

Thus $a\in H$ implies that $\ker\rho\subseteq H$, that is $H=\ker\rho$. □

Theorem 3. *If* (K,τ) *is a normal congruence pair for* S*, then* $\rho_{(K,\tau)}$ *is a congruence on* S *with kernel* K *and trace* τ*. Conversely, if* ρ *is a congruence on* S*, then* $(\ker\rho, \mathrm{tr}\rho)$ *is a normal congruence pair for* S *and* $\rho = \rho_{(\ker\rho,\mathrm{tr}\rho)}$.

Proof. Let (K,τ) be a normal congruence pair and let $\rho = \rho_{(K,\tau)}$. Then by Theorem 2, ρ is a congruence with trace equal to τ and $\ker\rho$ as in (2). Thus $\ker\rho \subseteq K$. Now let $a \in K$. Then by Definition 1 (ii) there exist $b \in S$, $b \leqslant a$, $b^{-1} \in K$ and $bb^{-1}\tau aa^{-1}$ such that $a \in \ker\rho$ due to Theorem 2. Thus $K = \ker\rho$.

Conversely, let ρ be a congruence on S and let $K = \ker\rho$, $\tau = \mathrm{tr}\rho$. Then K is a full subgroupoid of S and τ is a congruence on $E(S)$.

Let $a \in S$, $b \in K$ and $a \geqslant b$. Suppose that $aa^{-1}\rho bb^{-1}$. Then $b = bb^{-1}\cdot a$ (by (1)). From $aa^{-1}\rho\, bb^{-1}$ it follows that $a\rho\,(bb^{-1})a$ and by above argument we have $a\rho b$. Hence $a \in b\rho \subseteq \ker\rho = K$. Thus (i) from the Definition 1 holds for (K,τ) and that it is a pseudo congruence pair for S.

Let $a \in K$. Then there exists $e \in E(S)$ with $a\rho\, e$. If $b = ea$, then $b \leqslant a$ by Corollary 2. From $a\rho\, e$ it follows that $ea\rho\, e$ whence $b\rho\, e$ and so $a\rho\, b$. Now $a^{-1}\rho\, b^{-1}$ by Proposition 1 and so $aa^{-1}\rho\, bb^{-1}$. Moreover, from $a\rho\, e$ follows that $aa^{-1}\rho\, ea^{-1} = (ea)^{-1} = b^{-1}$, that is $b^{-1} \in K$. Hence, (K,τ) is a congruence pair for S.

It remains to prove that $\rho = \rho_{(K,\tau)}$. Let $a\rho\, b$. Then

$$ab^{-1}\rho\, bb^{-1},\ b^{-1}a\rho\, b^{-1}b,\ aa^{-1}\rho\, ba^{-1},\ a^{-1}a\rho\, a^{-1}b$$

and so $ab^1, b^{-1}a, ba^{-1}, a^{-1}b \in \ker\rho = K$. Also

$$\begin{aligned} aa^{-1}\cdot bb^{-1}\rho\, a^{-1}b\cdot bb^{-1} &= (bb^{-1}\cdot b)a^{-1} = ba^{-1}\rho\, aa^{-1},\\ aa^{-1}\cdot bb^{-1}\rho\, aa^{-1}\cdot ba^{-1} &= b(aa^{-1}\cdot a) = ba^{-1}\rho\, b^{-1}b = bb^{-1}, \end{aligned}$$

whence it follows that $a\rho_{(K,\tau)}b$ and so $\rho \subseteq a\rho_{(K,\tau)}$.

Let $a\rho_{(K,\tau)}b$. Then $ab^{-1}, a^{-1}b, ba^{-1}, b^{-1}a \in K$, $aa^{-1}\cdot bb^{-1}\tau\, aa^{-1}\tau\, bb^{-1}$, imply that $ab^{-1}\rho e$, $ba^{-1}\rho f$ for some $e, f \in E(S)$. From $aa^{-1}\rho bb^{-1}$, it follows that

$$a\,\rho\, bb^{-1}\cdot a = ab^{-1}\cdot b\,\rho\, eb \qquad \text{and} \qquad b\rho\, aa^{-1}\cdot b = ba^{-1}\cdot a\rho\, fa.$$

Also

$$\begin{aligned} &a\,\rho\, eb\,\rho\, e\cdot fa\,\rho\, e(f\cdot eb) = e(e\cdot fb) = ee(e\cdot fb)\\ &= (fb\cdot e)ee = (fb\cdot e)e = ee\cdot fb = e\cdot fb = f\cdot eb\,\rho\, fa\,\rho\, b \end{aligned}$$

imply that $a\rho b$, that is $\rho_{(K,\tau)} \subseteq \rho$. Then $\rho_{(K,\tau)} = \rho$. □

References

[1] **M. Božinović, P. V. Protić and N. Stevanović**, *Kernel normal system of inverse AG^{**}-groupoids*, Quasigroups and Related Systems **17** (2008), 1 − 8.

[2] **J. Deneš and A. D. Keedwell**, *Latin squares and their applications*, Akadémia Kiadó, Budapest, 1974.

[3] **M. A. Kazim and M. Naseeruddin**, *On almost semigroups*, The Aligarh Bull. Math. **2** (1972), 1 − 7.

[4] **Q. Mushtaq and Q. Iqbal**, *Decomposition of a locally associative LA-semigroup*, Semigroup Forum **41** (1990), 155 − 164.

[5] **Q. Mushtaq and Q. Iqbal**, *Partial oredering and congruences on LA-semigroups*, Indian J. Pure Appl. **22** (1991), 331 − 336.

[6] **M. Petrich**, *Inverse semigroups*, Pure and Applied Mathematics, John Wiley and Sons, New York, 1984.

[7] **M. Petrich and S. Rankin**, *The kernel – trace approach to right congruences on an inverse semigroup*, Trans. Amer. Math. Soc. **330** (1992), 917 − 932.

[8] **P. V. Protić and N. Stevanović**, *On Abel-Grassmann's groupoids (review)*, Proc. Math. Conference, Priština, 1994, 31 − 38.

Received September 4, 2009

Faculty of Civil Engineering
University of Niš
Aleksandra Medvedeva 14
18000 Niš
Serbia
e-mail: pvprotic@yahoo.mail.com