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Products of the symmetric or alternating groups with $\mathbf{L}_3(3)$

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Abstract. The structure of simple groups G with proper subgroups A and B such that $G = AB$, where B is isomorphic to $L_3(3)$ and A is isomorphic to the alternating or symmetric group on $n \geqslant 5$ letters, is described.

1. Introduction

Let A and B be proper subgroups of a group G. If $G = AB$, then G is called a *factorizable group*. In this case G is also called the *product of A and B*. In [1] page 13, the question of finding all the factorizable groups is raised. This is in general a hard question. We should remark that there are groups which are not factorizable. For example by [11] the smallest Janko simple group J_1 of order 175560 is not a factorizable group. Of course an infinite group whose proper subgroups are finite is not a factorizable group as well, one may recall a Tarski group for this purpose. In what follows we will assume G is a finite group.

A factorization $G = AB$ is called *maximal* if both factors A and B are maximal subgroups of G. In [11] all the maximal factorizations of all the finite simple groups and their automorphism groups are found. A factorization $G = AB$ with the condition $A \cap B = 1$ is called an *exact factorization*. In [15] the authors found all the exact factorizations of the alternating and the symmetric groups. In [13] all the factorizations of the alternating and the symmetric groups were found with both factors simple. In [7] an interesting application of exact factorization is given. The authors show that an exact factorization of a finite group leads to the construction of a biperfect Hopf algebra, and then they find such a factorization for the Mathieu group M_{24}, where $A \cong M_{23}$ and $B \cong 2^4 : \mathbb{A}_7$, both perfect groups (a group G is called *perfect* if $G' = G$).

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The involvement of the alternating or the symmetric group in a factorization received attention in the past. In [9] all finite groups $G = AB$, where A and B are isomorphic to the alternating group on 5 letters are classified and in [12] factorizable groups where one factor is a non-abelian simple group and the other factor is isomorphic to the alternating group on 5 letters are classified. In [14] factorizable finite groups are classified in the case where one factor is simple and the other factor is almost simple. In [5] all finite groups $G = AB$, where $A \cong \mathbb{A}_6$ and $B \cong \mathbb{S}_n$, $n \geqslant 5$, are determined. Similarly all finite groups $G = AB$, $A \cong \mathbb{A}_7$ and $B \cong \mathbb{S}_n$, $n \geqslant 5$, were found in [3]. Also in [6] we determined the structure of a finite factorizable group with one factor a simple group and the other factor isomorphic to the symmetric group on 6 letters. In [4] we obtained the structure of groups G which factor as $G = AB$, where A is isomorphic to an alternating group and B is isomorphic to a symmetric group on more than 5 letters. Motivated by the above results, in this paper we find the structure of simple groups G with a factorization $G = AB$, where $A = L_3(3)$ and B isomorphic to an alternating or symmetric group on more than 5 letters. Throughout the paper all groups are assumed to be finite. Notation for the names of the finite simple groups is taken from [2].

2. Preliminary results

In the following we quote two results from [14] which are useful when dealing with factorizable groups.

Lemma 1. *Let A and B be subgroups of a group G. Then the following statements are equivalent:*

(a) $G = AB$.

(b) *A acts transitively on the coset space $\Omega(G : B)$ of right cosets B in G.*

(c) *B acts transitively on the coset space $\Omega(G : A)$ of right cosets of A in G.*

(d) *$(\pi_A, \pi_B) = 1$, where π_A and π_B are the permutation characters of G on $\Omega(G : A)$ and $\Omega(G : B)$, respectively.*

Lemma 2. *Let G be a permutation group on a set Ω of size n. Suppose the action of G on Ω is k-homogeneous, $1 \leqslant k \leqslant n$. If a subgroup H of G acts on Ω k-homogeneously, then $G = G_{(\Delta)}H$, where Δ is a k-subset of Ω and $G_{(\Delta)}$ denotes its global stabilizer.*

Since $L_3(3)$ has a 2-transitive action on 13 points, using Lemma 2 we obtain the factorization $\mathbb{A}_{13} = L_3(3)\mathbb{A}_{11}$ involving $L_3(3)$. Transitive actions of $L_3(3)$ corresponds to the indices of its subgroups. According to [2] maximal subgroups of $L_3(3)$ have the following shapes: $3^2 : GL_2(3)$, $\mathbb{S}_4$ and $13 : 3$. Using these we can verify that $L_3(3)$ has proper subgroups with the following orders only: 1,2,3,4,6,8,9,12,13,16,18,24,27,36,39,48,54,72,144,216,432. Therefore the indices of proper subgroups of $L_3(3)$ are as follows: 13, 26, 39, 78, 104, 117, 144, 156, 208, 234, 312, 351, 432, 468, 624, 702, 936, 1404, 1872, 2808, 5616.

Now using the above information we prove the following Lemma.

Lemma 3. *Let $\mathbb{A}_m$ denote the alternating group of degree m. If $\mathbb{A}_m = AB$ is a factorization of $\mathbb{A}_m$ with A a non-abelian simple group and $B \cong L_3(3)$, then one of the following occurs:*

(a) $\mathbb{A}_m = \mathbb{A}_{m-1}L_3(3)$ *where* $m =$ 13, 26, 39, 78, 104, 117, 144, 156, 208, 234, 312, 351, 432, 468, 624, 702, 936, 1404, 1872, 2808, 5616.

(b) $\mathbb{A}_{13} = \mathbb{A}_{11}L_3(3)$.

Proof. Let $\mathbb{A}_m = AB$, where A is a simple group and $B \cong L_3(3)$. Obviously $m \geqslant 13$. By [11], Theorem D, we have the following two cases.

(i) $\mathbb{A}_{m-k} \trianglelefteq A \trianglelefteq \mathbb{S}_{m-k} \times \mathbb{S}_k$ for some k with $1 \leqslant k \leqslant 5$, and B k-homogenous on m letters.

Since $B \cong L_3(3)$ it is clear that $k = 1$ or 2. If $k = 2$, then $m = 13$, and from $\mathbb{A}_{11} \trianglelefteq A \trianglelefteq \mathbb{S}_{11} \times \mathbb{S}_2$ and the simplicity of A we obtain $A = \mathbb{A}_{11}$ and (b) occurs. If $k = 1$, then $A = \mathbb{A}_{m-1}$ and the factorization $\mathbb{A}_m = \mathbb{A}_{m-1}B$ corresponds to transitive actions of B on m letters. Since we have already found indices of subgroups of $B \cong L_3(3)$, hence m is one of the numbers in case (a) and all possibilities in case (a) occur.

(ii) $\mathbb{A}_{m-k} \trianglelefteq B \trianglelefteq \mathbb{S}_{m-k} \times \mathbb{S}_k$ for some k with $1 \leqslant k \leqslant 5$, and A is k-homogenous on m letters.

Since B is a simple group we obtain $\mathbb{A}_{m-k} = 1$, the trivial group. Therefore $m - k = 1$, and from $1 \leqslant k \leqslant 5$ we get $2 \leqslant m \leqslant 6$, contradicting $m \geqslant 13$. □

Lemma 4. *Let $\mathbb{A}_m = AB$, where A is isomorphic to a symmetric group $\mathbb{S}_n$ and $B \cong L_3(3)$. Then $m = 13$ and $n = 11$ and we have the factorization $\mathbb{A}_{13} = \mathbb{S}_{11}L_3(3)$.*

Proof. The proof is the same as the proof of Lemma 3. □

3. The main result

According to the classification theorem for the finite simple groups every finite simple non-abelian group G is isomorphic to one of the following: alternating group $\mathbb{A}_m$, $m \geqslant 5$; a sporadic group or a group of Lie type. Therefore to see if G has an appropriate factorization we have to go through all the members of the above list. In the Lemmas 3 and 4 we dealt with the case of the alternating group. Here the other cases will be examined.

Lemma 5. *Let G be a sporadic finite simple group Then it is impossible to write $G = AB$ where $B \cong L_3(3)$ and A isomorphic to an alternating or symmetric group on more than 5 letters.*

Proof. Let G be a sporadic simple group. First we assume $G = AB$ where A is isomorphic to a simple alternating group and $B \cong L_3(3)$. Since in this case both factors A and B are simple we can use [8] to see there is no possibilities for A and B.

Secondly we assume $G = AB$ where A is isomorphic to the symmetric group $\mathbb{S}_n$, $n \geqslant 5$, and $B \cong L_3(3)$. By [11] factorizable sporadic simple groups G whose orders are divisible by 13 are Ru, Suz, Co_1. By [2] the structure of maximal subgroups of these groups are known. Therefore using [2] we see that if $\mathbb{S}_n \leqslant Ru$ or Suz, then $n \leqslant 6$, and if $\mathbb{S}_n \leqslant Co_1$, then $n \leqslant 8$. Now taking into account each of the above sporadic groups G and considering the order of AB, $A \cong \mathbb{S}_n$, $B \cong L_3(3)$, a contradiction is reached and the Lemma is proved. □

Simple groups of Lie type are divided into two large families called the *classical groups* and the *exceptional groups of Lie type.* According to [11] factorizations of exceptional groups of Lie type are given in Theorem B from which it follows that none of these groups have the desired factorization. Therefore we are left with the projective special linear, symplectic, unitary and orthogonal groups.

Lemma 6. *The decomposition $L_m(q) = AB$ where $A \cong \mathbb{A}_n$ or $\mathbb{S}_n$, $n \geqslant 5$, $B \cong L_3(3)$ is impossible.*

Proof. Let $L_m(q) = AB$ where $A \cong \mathbb{A}_n$ or $\mathbb{S}_n$, $n \geqslant 5$, and $B \cong L_3(3)$. By [10] the minimum degree of a projective modular representation of $\mathbb{A}_n$ or $\mathbb{S}_n$ is $n-2$ and therefore $m \geqslant n-2$ which implies $n \leqslant m+2$. First we consider the case $q = 2$. In this case the 2-part of $|L_m(2)|$ is equal to $2^{m(m-1)/2}$. If the 2-part of $\mathbb{S}_n$ is 2^a, then it is well-known that $a = [\frac{n}{2}] + [\frac{n}{4}] + \cdots \leqslant \frac{n}{2} + \frac{n}{4} + \cdots = n$, hence the 2-part of AB is at most $2^{(n+3)}$. Therefore we must have $m(m-1)/2 \leqslant n+3$ from which it follows that $n \geqslant (m-3)(m+2)/2$. If $m > 5$, then from the last inequality we obtain $n > m+2$ which contradicts the condition $n \leqslant m+2$. If $m \leqslant 5$ then since 13 does not divide the order of $L_m(2)$, hence $L_3(3)$ cannot be involved in $L_m(2)$. Therefore the condition $q = 2$ is ruled out. Hence in the following we will assume $q > 2$ and distinguish two cases:

(i) $n \geqslant 9$. Since $n \geqslant 9$ we will obtain $m \geqslant 7$. For any natural number $k \geqslant 2$ we have $\frac{q^k-1}{q-1} = q^{k-1} + \cdots + q + 1 \geqslant 3^{k-1} + \cdots + 3 + 1 = \frac{3^k-1}{2} \geqslant k+2$. Hence $q^k - 1 \geqslant (k+2)(q-1) \geqslant (k+2)q^{1/2}$. But

$$|L_m(q)| = \frac{1}{d} q^{m(m-1)/2}(q^m - 1)\cdots(q^2 - 1),$$

where $d = (m, q-1)$. Therefore using the above inequality and the fact that $d \leqslant q - 1 < q$ we obtain:

$$|L_m(q)| > \frac{1}{6}(m+2)! q^{(m^2-3)/2} \tag{1}$$

From $L_m(q) = AB$ and $n \leqslant m+2$ we obtain: $|L_m(q)| < |A| \times |B| \leqslant |\mathbb{S}_n| \times |L_3(3)| \leqslant 2^4.3^3.13(m+2)!$. Therefore

$$|L_m(q)| \leqslant 2^4.3^3.13(m+2)! \tag{2}$$

Combining inequalities (1) and (2) results: $q^{(m^2-3)/2} < 2^5.3^4.13$. Since $m \geqslant 7$ we can write $q^{(7^2-3)/2} \leqslant q^{(m^2-3)/2} < 33696$, which implies $q^{23} < 33696$, a contradiction, and case (i) is proved.

(ii) $n \leqslant 8$. In this case using (1) and the inequality $|L_m(q)| \leqslant |A| \times |B| \leqslant 2^4.3^3.13.n!$ we obtain $(m+2)! q^{(m^2-3)/2} < 2^5.3^4.13.n!$ and since $n \leqslant 8$ we obtain

$$(m+2)! q^{(m^2-3)/2} < 2^{12}.3^6.5.7.13 \tag{3}$$

If $m > 5$, then it is easy to see that (3) leads to a contradiction. Therefore $m \leqslant 5$.

If $m = 5$, then from (3) we obtain $q^{11} < 2^9.3^4.13$ which implies $q = 2$, which is not the case. If $m = 4$, then again using inequality (3) we obtain q = 2,3,4,5,7,8 or 9. Now considering orders of the groups $L_m(q)$ in each case a contradiction is obtained. For the case $m = 3$ similar computation is applicable. □

Lemma 7. *The decomposition $S_{2m}(q) = AB$, where $A \cong \mathbb{A}_n$ or $\mathbb{S}_n$, $n \geqslant 5$ and $B \cong L_3(3)$ is impossible.*

Proof. We assume $S_{2m}(q) = AB$, where $A \cong \mathbb{A}_n$ or $\mathbb{S}_n$, $n \geqslant 5$ and $B \cong L_3(3)$. Of course $S_{2m}(q)$ denotes the symplectic group in dimension $2m$ over a field with q elements. Similar to Lemma 6 we distinguish two cases.

(i) $n \geqslant 9$. Again by [10] we have $2m \geqslant n-2$ and hence $n \leqslant 2m+2$ which implies $m \geqslant 4$. By the order of the symplectic group and using the same argument and inequality as in the proof of Lemma 6 we obtain

$$|S_{2m}(q)| = \frac{1}{d}q^{m^2}(q^{2m}-1)\cdots(q^2-1) \geqslant \frac{1}{6d}(2m+2)!q^{m(2m+1)/2},$$

where $d = 1$ or 2. Therefore we obtain the following inequality:

$$|S_{2m}(q)| \geqslant \frac{1}{12}(2m+2)!q^{m(2m+1)/2} \tag{4}$$

But then from $S_{2m}(q) = AB$ and $n \leqslant 2m+2$ we obtain:

$$|S_{2m}(q)| \leqslant |A| \times |B| \leqslant n!.2^4.3^3.13 \leqslant (2m+2)!.2^4.3^3.13 \tag{5}$$

Combining (4) with (5) will result the following inequality:

$$q^{m(2m+1)/2} \leqslant 2^6.3^4.13 \tag{6}$$

Now from $m \geqslant 4$ and using (6) we obtain $q^{18} \leqslant 2^6.3^4.13$ which is a contradiction because $q \geqslant 2$. This proves the Lemma in case (i).

(ii) $n \leqslant 8$. In this case using (5) and the inequality $|S_{2m}(q)| \leqslant 2^4.3^3.13.n!$ we obtain $(2m+2)!q^{m(2m+1)/2} \leqslant 2^6.3^4.13.n! \leqslant 2^6.3^4.13.8!$. Therefore

$$(2m+2)!q^{m(2m+1)/2} \leqslant 2^6.3^4.13.8! \tag{7}$$

from which it follows that if $m \geqslant 4$, then $q^{18} \leqslant \frac{2^5.3^2.13}{5}$ which is a contradiction because q is at least 2. Hence $m = 1$,2 or 3. If $m = 3$, then from (7) we get $q^{21/2} \leqslant 2^6.3^4.13$ which forces $q = 2$. But in this case $\mathbb{S}_6(2)$ does not contain $L_3(3)$. If $m = 2$ then from (7) we obtain $q^5 \leqslant 2^9.3^4.7.13$ which implies $q < 20$, hence $q = 2,3,4,5,7,8,9,11,13,16,17$ and 19. Now a case by case examination of these values result desire contradiction. In the case of $m = 1$ we have $\mathbb{S}_2(q) = L_2(q)$ which is treated in Lemma 6. □

Lemma 8. *Decomposition of the unitary group or the orthogonal groups as the product of $\mathbb{A}_n$ or $\mathbb{S}_n$, $n \geqslant 5$, with the group $L_3(3)$ is impossible.*

Proof. Since the proof is similar to that of Lemma 6 and 7, we describe the inequalities which are used in the unitary and orthogonal groups only. We have $|U_m(q)| = \frac{1}{d}q^{m(m-1)/2}(q^m-(-1)^m)\cdots(q^2-1)$ where $d=(m,q+1)$. Using the inequality $q^k-(-1)^k \geqslant (k+2)q^{1/2}$ which holds for every positive integer k, we obtain $|U_m(q)| \geqslant \frac{1}{6d}(m+2)!q^{(m^2-1)/2}$ and since $d=(m,q+1) \leqslant q+1 < q^2$, the following inequality holds: $|U_m(q)| > \frac{1}{6}(m+2)!q^{(m^2-5)/2}$. Now using the above inequality and applying the method of proof in Lemma 6 a contradiction is obtained.

Next we consider the orthogonal groups $\Omega_{2m+1}(q)$, $m \geqslant 3$, in odd dimension. We have $|\Omega_{2m+1}(q)| = \frac{1}{d}q^{m^2}(q^{2m}-1)\cdots(q^2-1)$ where $d=(2,q-1)$. Since in this case orders of $\Omega_{2m+1}(q)$ and the symplectic groups $\mathbb{S}_{2m}(q)$ are equal, we can apply the same inequality obtained in Lemma 7 to derive a contradiction.

Finally we will consider the orthogonal groups in even dimensions. These groups are denoted by $O^{\varepsilon}_{2m}(q)$ where $m \geqslant 4$ and $\varepsilon = \pm$. We have $|O^{\varepsilon}_{2m}(q)| = \frac{1}{d}q^{m(m-1)}(q^m+\varepsilon 1)(q^{2m-2}-1)\cdots(q^2-1)$ where $d=(4,q^m+\varepsilon 1)$. Since $m \geqslant 4$, we have $q^m+\varepsilon 1 \geqslant (2m+2)q^{1/2}$. Hence considering the order of $O^{\varepsilon}_{2m}(q)$ and the inequalities $q^k-1 \geqslant (k+2)q^{1/2}$, we obtain $|O^{\varepsilon}_{2m}(q)| \geqslant \frac{1}{6d}q^{(2m^2-m)/2}(2m+2)!$. But $d=(4,q^m+\varepsilon 1) \leqslant 4$, and applying it to the above inequality we obtain $|O^{\varepsilon}_{2m}(q)| \geqslant \frac{1}{24}q^{(2m^2-m)/2}(2m+2)!$. The above inequality is used to obtain a contradiction in assuming a factorization of the kind in the Lemma. The proof of Lemma 8 is complete now. □

In this way we have proved the following theorem which is the main result of this paper.

Theorem 9. *Let G be a finite non-abelian simple group such that $G=AB$, where $A \cong \mathbb{A}_n$ or $\mathbb{S}_n$, $n \geqslant 5$, and $B \cong L_3(3)$. Then the following possibilities occur:*

- (a) $\mathbb{A}_m = \mathbb{A}_{m-1}L_3(3)$, *where* $m = 13$, 26, 39, 78, 104, 117, 144, 156, 208, 234, 312, 351, 432, 468, 624, 702, 936, 1404, 1872, 2808, 5616.
- (b) $\mathbb{A}_{13} = \mathbb{A}_{11}L_3(3)$.
- (c) $\mathbb{A}_{13} = \mathbb{S}_{11}L_3(3)$.

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