N-fuzzy quasi-ideals in ordered semigroups

Asghar Khan, Young Bae Jun and Muhammad Shabir

Abstract. In this paper, we introduce the concept of N-fuzzy quasi-ideals in ordered semigroups and investigate the basic theorem of quasi-ideals of ordered semigroups in terms of N-fuzzy quasi-ideals. We characterize left (resp. right) regular and completely regular ordered semigroups in terms of N-fuzzy quasi-ideals. We define semiprime N-fuzzy quasi-ideals and characterize completely regular ordered semigroups in terms of semiprime N-fuzzy quasi-ideals. We provide characterizations of some semilattices of left and right simple semigroups in terms of N-fuzzy quasi-ideals.

1. Introduction

A fuzzy subset $f$ of a given set $S$ is described, as an arbitrary function $f : S \rightarrow [0, 1]$, where $[0, 1]$ is the usual closed interval of real numbers. This fundamental concept of a fuzzy set was first introduced by Zadeh in his pioneering paper [26] of 1965, provides a natural frame-work for the generalizations of some basic notions of algebra, e.g. logic, set theory, group theory, ring theory, groupoids, real analysis, topology, and differential equations etc. Rosenfeld (see [21]) was the first who considered the case when $S$ is a groupoid. He gave the definition of a fuzzy subgroupoid and the fuzzy left (right, two-sided) ideals of $S$ and justified these definitions by showing that a subset $A$ of a groupoid $S$ is is a subgroupoid or left (right, or two-sided) ideal of $S$ if and only if the characteristic mapping $f_A : S \rightarrow \{0, 1\}$ of $A$ defined by

$$x \mapsto f_A(x) := \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

is respectively, a fuzzy subgroupoid or a fuzzy left (right or two-sided) ideal of $S$. The concept of a fuzzy ideal in semigroups was first developed by

2000 Mathematics Subject Classification: 06F05, 06D72, 08A72
Keywords: N-fuzzy subset; N-fuzzy quasi-ideal; left (right regular), semilattice of left (right) simple semigroups; completely regular ordered semigroup.
Kuroki (see [13-17]). He studied fuzzy ideals, fuzzy bi-ideals, fuzzy quasi-ideals and semiprime fuzzy ideals of semigroups. Fuzzy ideals and Green’s relations in semigroups were studied by McLean and Kummer in [18]. Ahesan et. al in [1] characterized semigroups in terms of fuzzy quasi-ideals. A systematic exposition of fuzzy semigroups was given by Mordeson, Malik and Kuroki appeared in [20], where one can find theoretical results on fuzzy semigroups and theoretical results on fuzzy semigroups and their use in fuzzy coding, fuzzy finite state machines and fuzzy languages. The monograph given by Mordeson and Malik [19] deals with the applications of fuzzy approach to the concepts of automata and formal languages. Fuzzy sets in ordered semigroups/ordered groupoids were first introduced by Kehayopulu and Tsingelis in [8]. They also introduced the concepts of fuzzy bi-ideals and fuzzy quasi-ideals in ordered semigroups in (see [9]).

The quasi-ideals in rings and semigroups were studied by Stienfeld in [25] and Kehayopulu extended this concept in ordered semigroups and defined a quasi-ideal of an ordered semigroup $S$ as a non-empty subset $Q$ of $S$ such that

1. $(QS) \cap (SQ) \subseteq Q$ and
2. If $a \in Q$ and $S \ni b \leq a$ then $b \in Q$.

The purpose of this paper, is to initiate and study the new sort of fuzzy quasi-ideals called $N$-fuzzy quasi-ideals in ordered semigroups. We characterize regular, left and right simple ordered semigroups and completely regular ordered semigroups in terms of $N$-fuzzy quasi-ideals. In this respect, we prove that: An ordered semigroup $S$ is regular, left and right simple if and only if every $N$-fuzzy quasi-ideal of $S$ is a constant $N$-function. We also prove that $S$ is completely regular if and only if for every $N$-fuzzy quasi-ideal $f$ of $S$ we have $f(a) = f(a^2)$ for every $a \in S$. We define semiprime $N$-fuzzy quasi-ideal in ordered semigroups and prove that an ordered semigroup $S$ is completely regular if and only if every $N$-fuzzy quasi-ideal $f$ of $S$ is semiprime. Next, we characterize semilattices of left and right simple ordered semigroup in terms of $N$-fuzzy quasi-ideals. We prove that an ordered semigroup $S$ is a semilattice of left and right simple if and only if for every $N$-fuzzy quasi-ideal $f$ of $S$, we have $f(a) = f(a^2)$ and $f(ab) = f(ba)$ for every $a, b \in S$. In the last of this paper, we discuss ordered semigroups having the property $a \leq a^2$ for all $a \in S$ and prove that an ordered semigroup $S$ (having the property $a \leq a^2$ for all $a \in S$) is a semilattice of left and right simple ordered semigroup if and only if for every $N$-fuzzy quasi-ideal $f$ of $S$ we have $f(ab) = f(ba)$ for all $a, b \in S$. 


2. Some basic definitions and results

By an ordered semigroup (or po-semigroup) we mean a structure \((S, \cdot, \leq)\) in which

\[(OS1)\] \((S, \cdot)\) is a semigroup,
\[(OS2)\] \((S, \leq)\) is a poset,
\[(OS3)\] \((∀a, b, x ∈ S)(a \leq b \implies ax \leq bx \text{ and } xa \leq xb)\).

Let \((S, \cdot, \leq)\) be an ordered semigroup. For \(A ⊆ S\), we denote
\[\{t ∈ S | t ≤ h \text{ for some } h ∈ A\}\] and \(AB := \{ab | a ∈ A, b ∈ B\}\).

A non-empty subset \(Q\) of \(S\) is called a \textit{quasi-ideal} (see [8]) of \(S\) if:

1. \((QS) \cap (SQ) ⊆ Q\) and
2. If \(a ∈ Q\) and \(S ⊢ b ≤ a\) then \(b ∈ Q\).

Let \(A, B ⊆ S\). Then \(A \subseteq (A], (A][B] ⊆ (AB), ((A]) = (A]\) and \((([A][B])] \subseteq (AB]\) (see [8]).

\(\emptyset = A ⊆ S\) is called a subsemigroup of \(S\) if \(A^2 ⊆ A\), and a \textit{right} (resp. \textit{left}) \textit{ideal} of \(S\) if (1) \(AS ⊆ A\) (resp. \(SA ⊆ A\)) and (2) \(a ∈ A, S ⊢ b ≤ a\) imply \(b ∈ A\). If \(A\) is both a right and a left ideal of \(S\), then it is called an \textit{ideal}. A subsemigroup \(B\) of \(S\) is called a \textit{bi-ideal} of \(S\) if:

1. \(BSB ⊆ B\) and
2. \(a ∈ B, S ⊢ b ≤ a\) imply \(b ∈ B\).

By a \textit{negative fuzzy subset} (briefly, an \(N\)-\textit{fuzzy subset}) of \(S\) we mean a function \(f : S → [-1, 0]\). An \(N\)-fuzzy subset \(f\) of \(S\) is called an \(N\)-\textit{fuzzy left} (resp. \textit{right}) \textit{ideal} of \(S\) if:

1. \(x ≤ y \implies f(x) ≤ f(y)\) and
2. \(f(xy) ≤ f(x)\) (resp. \(f(xy) ≤ f(x)\))

for all \(x, y ∈ S\).

If \(f\) is both an \(N\)-fuzzy left and an \(N\)-fuzzy right ideal of \(S\), then it is called an \(N\)-\textit{fuzzy ideal} of \(S\).

For a non-empty \textit{family} of \(N\)-fuzzy subsets \(\{f_i\}_{i ∈ I}\) of an ordered semigroup \(S\), the \(N\)-fuzzy subsets \(\bigwedge_{i ∈ I} f_i\) and \(\bigvee_{i ∈ I} f_i\) of \(S\) are defined as follows:

\[
\left(\bigwedge_{i ∈ I} f_i\right)(x) := \inf_{i ∈ I}\{f_i(x)\}, \quad \left(\bigvee_{i ∈ I} f_i\right)(x) := \sup_{i ∈ I}\{f_i(x)\}.
\]

The \textit{characteristic} \(N\)-\textit{function} \(\kappa_A\) of \(\emptyset ≠ A ⊆ S\) is given by:

\[
\kappa_A(x) := \begin{cases} 
-1 & \text{if } x ∈ A, \\
0 & \text{if } x \notin A.
\end{cases}
\]

For \(N\)-fuzzy subsets \(f\) and \(g\) of \(S\) we define the \(N\)-\textit{composition} \(fN g\) by

\[
fN g(x) := \begin{cases} 
\bigwedge_{(y, z) ∈ A_x} \max\{f(y), g(z)\} & \text{if } A_x \neq \emptyset, \\
0 & \text{if } A_x = \emptyset.
\end{cases}
\]
where
\[ A_x := \{(y, z) \in S \times S \mid x \leq yz\}. \]

The set \( NF(S) \) of all \( \mathcal{N} \)-fuzzy subsets of \( S \) with such defined \( \mathcal{N} \)-composition and the relation
\[ f \preceq g \text{ if and only if } f(x) \leq g(x) \text{ for all } x \in S \]
is an ordered semigroup denoted by \( (NF(S), \preceq) \). The fuzzy subsets \( \beta(x) = 0 \) and \( \alpha(x) = -1 \) (for all \( x \in S \)) the greatest and least element of \( (NF(S), \preceq) \). The fuzzy subset \( \beta \) is the zero element of \( (NF(S), \preceq) \)(that is, \( fN\beta = \beta \) for every \( f \in NF(S) \)). Obviously, \( f_S = \alpha \) and \( f_\emptyset = \beta \).

**Definition 2.1.** (cf. [11]). Let \( S \) be an ordered semigroup. An \( \mathcal{N} \)-fuzzy subset \( f \) of \( S \) is called an \( \mathcal{N} \)-fuzzy subsemigroup of \( S \) if
\[ f(xy) \leq \max\{f(x), f(y)\} \text{ for all } x, y \in S. \]

**Definition 2.2.** (cf. [11]). Let \( S \) be an ordered semigroup. An \( \mathcal{N} \)-fuzzy subsemigroup \( f \) of \( S \) is called an \( \mathcal{N} \)-fuzzy bi-ideal of \( S \) if:
1. \( x \preceq y \) implies \( f(x) \preceq f(y) \).
2. \( f(xay) \leq \max\{f(x), f(y)\} \) for all \( x, a, y \in S \).

**Proposition 2.3.** (cf. [11]). Let \( S \) be an ordered semigroup and \( A, B \subseteq S \). Then
\[ (a) \ B \subseteq A \text{ if and only if } \kappa_B \leq \kappa_A. \]
\[ (b) \ \kappa_A \vee \kappa_B = \kappa_{A \cap B}. \]
\[ (c) \ \kappa_A \wedge \kappa_B = \kappa_{[AB]} . \]

**Lemma 2.4.** (cf. [22]). Let \( S \) be an ordered semigroup. Then every quasi-ideal of \( S \) is subsemigroup of \( S \).

**Lemma 2.5.** (cf. [24]). An ordered semigroup \( (S, \cdot, \preceq) \) is a semilattice of left and right simple semigroups if and only if for all quasi-ideals \( A, B \) of \( S \), we have
\[ (A^2) = A \text{ and } (AB) = (BA). \]

An ordered semigroup \( S \) is called regular (see [5]) if for every \( a \in S \) there exists \( x \in S \) such that \( a \preceq axa \) or equivalently, (1) \((\forall a \in S)(a \in (aSa)]\) and (2) \((\forall A \subseteq S)(A \subseteq (ASA)]\).

An ordered semigroup \( S \) is called left (resp. right) simple (see [9]) if for every left (resp. right) ideal \( A \) of \( S \), we have \( A = S \).
**Lemma 2.6.** (cf. [9, Lemma 3]). An ordered semigroup $S$ is left (resp. right) simple if and only if $(Sa) = S$ (resp. $(aS) = S$) for every $a \in S$.

An ordered semigroup $S$ is called left (resp. right) regular (see [4]) if for every $a \in S$ there exists $x \in S$ such that $a \leq xa$ (resp. $a \leq a^2x$) or equivalently, $(1) \ (\forall a \in S)(a \in (Sa^2))$ and $(2) \ (\forall A \subseteq S)(A \subseteq (SA^2))$. An ordered semigroup $S$ is called completely regular if it is regular, left regular and right regular [6].

If $S$ is an ordered semigroup and $\emptyset \neq A \subseteq S$, then the set $(A \cup (AS \cap SA)) \setminus \emptyset$ is the quasi-ideal of $S$ generated by $A$. If $A = \{x\} \ (x \in S)$, we write $(x \cup (xS \cap Sx)) \setminus \emptyset$ instead of $(\{x\} \cup (\{x\}S \cap S\{x\})) \setminus \emptyset$.

**Lemma 2.7.** (cf. [6]). An ordered semigroup $S$ is completely regular if and only if $A \subseteq (A^2SA^2)$ for every $A \subseteq S$. Equivalently, if $a \in (a^2Sa^2)$ for every $a \in S$.

**Lemma 2.8.** (cf. [24]). An ordered semigroup $(S, \cdot, \leq)$ is a semilattice of left and right simple semigroups if and only if for every right ideal $A$ and every left ideal $B$ of $S$, we have $A \cap B = (AB)$.

**Lemma 2.9.** Let $(S, \cdot, \leq)$ be an ordered semigroup and let $A, B$ be quasi-ideals of $S$. Then $(AB)$ is a bi-ideal of $S$.

### 3. $N$-fuzzy quasi-ideals

In this section we prove the basic theorem which characterizes ordered semigroup in terms of $N$-fuzzy quasi-ideals.

**Definition 3.1.** Let $(S, \cdot, \leq)$ be an ordered semigroup. An $N$-fuzzy subset $f$ of $S$ is called an $N$-fuzzy quasi-ideal of $S$ if

1. $(fN \alpha) \cup (\alpha N f) \geq f$.
2. $x \leq y$, then $f(x) \leq f(y)$ for all $x, y \in S$.

The set

$$L(f; t) := \{x \in S | f(x) \leq t\}$$

is called a level subset of $f$.

**Theorem 3.2.** (cf. [11]). Let $S$ be an ordered semigroup and $f$ an $N$-fuzzy subset of $S$. Then $\forall t \in [-1, 0)$, $L(f; t)(\neq \emptyset)$ is a bi-ideal if and only if $f$ is an $N$-fuzzy bi-ideal.

**Theorem 3.3.** Let $(S, \cdot, \leq)$ be an ordered semigroups and $\emptyset \neq A \subseteq S$. Then $A$ is a quasi-ideal of $S$ if and only if the characteristic $N$-function $\kappa_A$ of $A$ is an $N$-fuzzy quasi-ideal of $S$. 
Proof. Suppose that $A$ is a quasi-ideal of $S$. Then $(\kappa_A N \alpha) \cup (\alpha N \kappa_A) \succeq \kappa_A$. Indeed:

$$(\kappa_A N \alpha) \cup (\alpha N \kappa_A) = (\kappa_A N \kappa_S) \cup (\kappa_S N \kappa_A) = \kappa_{(AS)} \cup \kappa_{(SA)} = \kappa_{(AS) \cap (SA)}$$

by Proposition 2.3.

Since $(AS) \cap (SA) \subseteq A$, then by Proposition 2.3, we have $\kappa_{(AS) \cap (SA)} \succeq \kappa_A$. Thus $(\kappa_A N \alpha) \cup (\alpha N \kappa_A) \succeq \kappa_A$. Let $x, y \in S$ be such that $x \leq y$. If $\kappa_A(y) = 0$. Then $\kappa_A(x) \leq \kappa_A(y)$, because $\kappa_A(x) \leq 0 \forall x \in S$. If $\kappa_A(y) = -1$, then $y \in A$. Since $S \ni x \leq y \in A$, we have $x \in A$, then $\kappa_A(x) = -1$ and hence $\kappa_A(x) \leq \kappa_A(y)$.

Conversely, assume that $\kappa_A$ is an $\mathcal{N}$-fuzzy quasi-ideal of $S$. Then $A$ is a quasi-ideal of $S$ since $(\kappa_A N \alpha) \cup (\alpha N \kappa_A) \succeq \kappa_A$ implies $(\kappa_A N \kappa_S) \cup (\kappa_S N \kappa_A) \succeq \kappa_A$. By Proposition 2.3, $\kappa_S N \kappa_A = \kappa_{(SA)}$ and $\kappa_A N \kappa_S = \kappa_{(AS)}$, then $\kappa_{(AS)} \cup \kappa_{(SA)} = \kappa_{(AS) \cap (SA)}$ and we have $\kappa_{(AS) \cap (SA)} \succeq \kappa_A$. Thus $(AS) \cap (SA) \subseteq A$.

If $x \in A$ and $S \ni y \leq x$. Since $y \leq x$ and $\kappa_A$ is an $\mathcal{N}$-fuzzy quasi-ideal of $S$, we have $\kappa_A(y) \leq \kappa_A(x)$. Since $x \in A$ then $\kappa_A(x) = -1$, and hence $\kappa_A(y) = -1$, i.e., $y \in A$. □

Theorem 3.4. Let $S$ be an ordered semigroup and $f$ an $\mathcal{N}$-fuzzy subset of $S$. Then $\forall t \in [-1, 0)$, $L(f; t)(\not= \emptyset)$ is a quasi-ideal if and only if $f$ is an $\mathcal{N}$-fuzzy quasi-ideal.

Proof. Assume that $f$ is an $\mathcal{N}$-fuzzy quasi-ideal of $S$. Let $x, y \in S$ be such that $x \leq y$. If $y \in L(f; t)$, then $f(y) \leq t$. Since $x \leq y$ and $f(x) \leq f(y) \leq t$ then $x \in L(f; t)$. Let $a \in S$ be such that $a \in (L(f; t)S \cap (SL(f; t))$ then $a \in (L(f; t)S)$ and $a \in (SL(f; t))$. Then $a \leq xy$ and $a \leq x'y'$ for some $x, y' \in L(f; t)$ and $x', y \in S$ and so $(x, y) \in A_a$ and $(x', y') \in A_a$. Since $A_a \not= \emptyset$, we have

$$f(a) \leq ((fN \alpha) \cup (\alpha N f))(a) = \max \{(fN \alpha)(a), (\alpha N f)(a)\}$$

$$= \max \left[ \bigwedge_{(p,q) \in A_a} \max \{f(p), \alpha(q)\}, \bigwedge_{(p', q') \in A_a} \max \{\alpha(p'), f(q')\} \right]$$

$$\leq \max \left[ \max \{f(a), \alpha(b)\}, \max \{\alpha(a'), f(b')\} \right]$$

$$= \max \left[ \max \{f(x), -1\}, \max \{-1, f(y')\} \right] = \max \left[ f(x), f(y') \right].$$

Since $x, y' \in L(f; t)$ we have $f(x) \leq t$ and $f(y') \leq t$. Then

$$f(a) \leq \max \left[ f(x), f(y') \right] \leq t,$$
of such that $f$ contradict. Thus $f(x) \leq f(y)$ for all $x \leq y$.

Let $x \leq y \in S$ be such that $f(x) > f(y)$, then $\exists t \in [-1,0]$ such that $f(x) > t \geq f(y)$ then $y \in L(f;t)$ but $x \notin L(f;t)$. This is a contradiction. Thus $f(x) \leq f(y)$ for all $x \leq y$.

Let $x \in S$ be such that $f(x) > ((fNa) \vee (aNf))(x)$, then $\exists t \in [-1,0]$ such that $f(x) > t \geq ((fNa) \vee (aNf))(x) = \max((fNa)(x), (aNf)(x))$.

Then $(fNa)(x) \leq t$ and $(aNf)(x) \leq t$ and hence $x \in (L(f;t)S]$ and $x \in (SL(f;t)]$, so $x \in (L(f;t)S] \cap (SL(f;t)]$. But $(L(f;t)S] \cap (SL(f;t)] \subseteq L(f;t)$, hence $x \in L(f;t)$, i.e., $f(x) \leq t$. This is a contradiction. Thus $f(x) \leq ((fNa) \vee (aNf))(x)$.

\begin{proof}
\end{proof}

**Example 3.5.** Let $S = \{a, b, c, d, f\}$ be an ordered semigroup with the following multiplication

\[
\begin{array}{cccccc}
  & a & b & c & d & f \\
  a & a & a & a & a & a \\
b & a & b & a & d & a \\
c & a & f & c & c & f \\
d & a & c & d & d & b \\
f & a & f & a & c & a \\
\end{array}
\]

and let $\leq := \{(a, a), (a, b), (a, c), (a, d), (a, f), (b, b), (c, c), (d, d), (f, f)\}$.

The quasi-ideals of $S$ are: $\{a\}$, $\{a, b\}$, $\{a, c\}$, $\{a, d\}$, $\{a, f\}$, $\{a, b, f\}$, $\{a, c, f\}$ and $S$ (see [3]). Define $f : S \to [-1,0]$ by $f(a) = -0.8$, $f(b) = -0.6$, $f(d) = -0.5$, $f(c) = f(f) = -0.3$. Then

$$L(f;t) := \begin{cases}
  S & \text{if } t \in [-0.3,0], \\
  \{a,b,d\} & \text{if } t \in [-0.5,-0.3), \\
  \{a,b\} & \text{if } t \in [-0.6,-0.5), \\
  \{a\} & \text{if } t \in [-0.8,-0.6), \\
  \emptyset & \text{if } t \in [-1,-0.8).
\end{cases}$$

is a quasi-ideal and by Theorem 3.4, $f$ is an $\mathcal{N}$-fuzzy quasi-ideal of $S$.

**Lemma 3.6.** Every $\mathcal{N}$-fuzzy quasi-ideal is an $\mathcal{N}$-fuzzy bi-ideal.

\begin{proof}
\end{proof}
which completes the proof.

Let Proposition 3.8.

and by Theorem 3.4, $f$ is a $\mathcal{N}$-fuzzy quasi-ideal of $S$. Thus $f$ is an $\mathcal{N}$-fuzzy bi-ideal of $S$.

The converse of above Lemma is not true, in general.

Example 3.7. Consider the semigroup $S = \{a, b, c, d\}$

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with the order $\leq := \{(a, a), (b, b), (c, c), (d, d), (a, b)\}$.

Then $\{a, d\}$ is a bi-ideal but not a quasi-ideal of $S$. For an $\mathcal{N}$-fuzzy set $f$ defined by $f(a) = f(d) = -0.3$, $f(b) = f(c) = -0.7$ we have

$$L(f; t) := \begin{cases} 
S & \text{if } t \in [-0.3, 0], \\
\{a, d\} & \text{if } t \in [-0.7, -0.3], \\
\emptyset & \text{if } t \in [-0.7, -1]. 
\end{cases}$$

Then $L(f; t)$ is a bi-ideal of $S$ and by Theorem 3.4, $f$ is an $\mathcal{N}$-fuzzy bi-ideal of $S$. Moreover, $L(f; t)$ is a bi-ideal of $S$ but not a quasi-ideal for all $t \in [-0.7, -0.3]$ and by Theorem 3.4, $f$ is not an $\mathcal{N}$-fuzzy quasi-ideal of $S$.

Proposition 3.8. If $(S, \leq)$ is an ordered semigroup and $f_1, f_2, g_1, g_2$ are $\mathcal{N}$-fuzzy subsets of $S$ such that $g_1 \preceq f_1$ and $g_2 \preceq f_2$, then $g_1 N g_2 \preceq f_1 N f_2$.

Proof. Let $a \in S$. If $A_a = \emptyset$ then $f_1 N f_2(a) = 0 \succeq g_1 N g_2(a)$. If $A_a \neq \emptyset$ then

$$f_1 N f_2(a) = \bigwedge_{(y, z) \in A_a} \max \{f_1(y), f_2(z)\} \succeq \bigwedge_{(y, z) \in A_a} \max \{g_1(y), g_2(z)\} = g_1 N g_2(a),$$

which completes the proof.

From the above Proposition we see that the set of all $\mathcal{N}$-fuzzy subsets of an ordered semigroup is a complete lattice.
4. Characterizations of regular ordered semigroups

In this section, we prove that an ordered semigroup $S$ is regular, left and right simple if and only if every $\mathcal{N}$-fuzzy quasi-ideal $f$ of $S$ is a constant $\mathcal{N}$-function. We define semiprime $\mathcal{N}$-fuzzy quasi-ideals of ordered semigroups and prove that an ordered semigroup $(S,\cdot,\leq)$ is completely regular if and only if every $\mathcal{N}$-fuzzy quasi-ideal $f$ of $S$ is a semiprime $\mathcal{N}$-fuzzy quasi-ideal of $S$.

**Theorem 4.1.** For an ordered semigroup $S$ the following are equivalent:

(i) $S$ is regular, left and right simple.

(ii) Every $\mathcal{N}$-fuzzy quasi-ideal of $S$ is a constant $\mathcal{N}$-function.

**Proof.** (i) $\Rightarrow$ (ii). Let $S$ be regular, left and right simple ordered semigroup. Let $f$ be an $\mathcal{N}$-fuzzy quasi-ideal of $S$. We consider the set $E_{\Omega} := \{e \in S | e^2 \geq e\}$. Then $E_{\Omega}$ is non-empty. In fact, if $a \in S$, since $S$ is regular, then there exists $x \in S$ such that $a \leq ax$. We consider the element $ax$ of $S$. Then $(ax)^2 = (ax)x \geq ax$, and so $ax \in E_{\Omega}$.

(A) We first prove that $f$ is a constant $\mathcal{N}$-function on $E_{\Omega}$. That is, $f(e) = f(t)$ for every $t \in E_{\Omega}$. In fact: Since $S$ is left and right simple, we have $(Sl) = S$ and $(tS) = S$. Since $e \in S$ then $e \in (Sl)$ and $e \in (tS)$. Then $e \leq xt$ and $e \leq ty$ for some $x, y \in S$. If $e \leq xt$ then $e^2 = ee \leq (xt)(xt) = (xt)x$ and $(xt, t) \in A_2$. If $e \leq ty$ then $e^2 = ee \leq (ty)(ty) = t(yt)$ and $(t, yt) \in A_2$. Since $A_2 \neq \emptyset$, and $t$ is an $\mathcal{N}$-fuzzy quasi-ideal of $S$, we have

$$f(e^2) \leq [(fN\alpha) \vee (\alpha N\alpha)](e^2) = \max(\{fN\alpha \cdot e^2, (\alpha N\alpha)(e^2)\})$$

$$= \max \left[ \max_{(y_1, z_1) \in A_2} \{f(y_1), \alpha(z_1)\}, \max_{(y_2, z_2) \in A_2} \{\alpha(z_2), f(y_2)\} \right]$$

$$\leq \max \{\max \{f(t), \alpha(yt)\}, \max \{f(t), \alpha(xt)\}\}$$

$$= \max \{\max \{f(t), -1\}, \max \{f(t), -1\}\} = \max \{f(t), f(t)\} = f(t).$$

Since $e \in E_{\Omega}$, we have $e^2 \geq e$ and $f(e^2) \geq f(e)$. Thus $f(e) \leq f(t)$. On the other hand since $S$ is left and right simple and $e \in S$, we have $S = (Se)$ and $S = (es)$. Since $t \in S$ we have $t \in (Se)$ and $t \in (es)$. Then $t \leq ze$ and $t \leq es$ for some $z, s \in S$. If $t \leq ze$ then $t^2 = tt \leq (ze)(ze) = (ze)e$ and $(ze, e) \in A_2$. If $t \leq es$ then $t^2 = tt \leq (es)(es) = e(se)$ and $(e, ses) \in A_2$. Since $A_2 \neq \emptyset$, we have

$$f(t^2) \leq [(fN\alpha) \vee (\alpha Nf)](t^2) = \max(\{fN\alpha)(t^2), (\alpha Nf)(t^2)\})$$

$$= \max \left[ \max_{(p_1, q_1) \in A_2} \{f(p_1), \alpha(q_1)\}, \max_{(p_2, q_2) \in A_2} \{\alpha(p_2), f(q_2)\} \right]$$

$$\leq \max \{\max \{f(e), \alpha(xe)\}, \max \{\alpha(ye), f(e)\}\}$$

$$= \max \{\max \{f(e), -1\}, \max \{-1, f(e)\}\} = \max \{f(e), f(e)\} = f(e).$$

Since $t \in E_{\Omega}$ then $t^2 \geq t$ and $f(t^2) \geq f(t)$. Thus $f(t) \leq f(e)$. Consequently, $f(t) = f(e)$. 
(B). Now, we prove that \( f \) is a constant \( N \)-function on \( S \). That is, \( f(t) = f(a) \) for every \( a \in S \). Since \( S \) is regular and \( a \in S \), there exists \( x \in S \) such that \( a \leq axa \).

We consider the elements \( ax \) and \( xa \) of \( S \). Then by (OS3), we have
\[
(ax)^2 = (axa) \geq ax \quad \text{and} \quad (xa) = x(axa) \geq xa,
\]
then \( ax, xa \in E_\Omega \) and by (A) we have \( f(ax) = f(t) \) and \( f(xa) = f(t) \). Since
\[
(ax)(xa) = axa \geq a,
\]
then \( (ax, xa) \in A_\pi \) and \( (axa)(xa) \geq axa \), then \( (ax, xa) \in A_\pi \) and hence \( A_\pi \neq \emptyset \). Since \( f \) is an \( N \)-fuzzy quasi-ideal of \( S \), we have
\[
f(a) = f(ax) = f(xa).
\]

Since \( S \) is left and right simple we have \( (Sa) = S \), and \( (aS) = S \). Since \( t \in S \), we have \( t \in (Sa) \) and \( t \in (aS) \). Then \( t \leq pa \) and \( t \leq qa \) for some \( p, q \in S \). Then \( (p, a) \in A_\tau \) and \( (a, q) \in A_\tau \). Since \( A_\tau \neq \emptyset \) and \( f \) is an \( N \)-fuzzy quasi-ideal of \( S \), we have
\[
f(t) = f(ax) = f(xa).
\]
Thus \( f(t) \leq f(a) \) and \( f(t) = f(a) \).

(ii) \( \Rightarrow \) (i). Let \( a \in S \). Then the set \( (aS) \) is a quasi-ideal of \( S \). Indeed, (a) \( (aS) \cap (Sa) \subseteq (aS) \) and (b) If \( x \in (aS) \) and \( S \ni y \leq x \in (aS) \), then \( y \in ((aS) = (aS) \). Since \( (aS) \) is quasi-ideal of \( S \), by Theorem 3.3, the characteristic \( N \)-function \( \kappa_{(aS)} \) of \( (aS) \) is an \( N \)-fuzzy quasi-ideal of \( S \). By hypothesis, \( \kappa_{(aS)} \) is a constant \( N \)-function, so \( \kappa_{(aS)}(x) = -1 \) or \( \kappa_{(aS)}(x) = -1 \) for every \( x \in S \).

Let \( (aS) \subseteq S \) and \( a \) be an element of \( S \) such that \( a \not\in (aS) \), then \( \kappa_{(aS)}(x) = 0 \). On the other hand, since \( a^2 \in (aS) \) then \( \kappa_{(aS)}(a^2) = 1 \). A contradiction to the fact that \( \kappa_{(aS)} \) is a constant \( N \)-function. Thus \( (aS) = S \). By symmetry we can prove that \( (Sa) = S \).

Since \( a \in S \) and \( S = (aS) = (Sa) \), we have \( a \in (aS) \subseteq (aSa) \), hence \( S \) is regular. \( \square \)
Theorem 4.2. An ordered semigroup \((S, \cdot, \leq)\) is completely regular if and only if for every \(N\)-fuzzy quasi-ideal \(f\) of \(S\) we have \(f(a) = f(a^2)\) for every \(a \in S\).

Proof. Let \(S\) be a completely regular ordered semigroup and \(f\) an \(N\)-fuzzy quasi-ideal of \(S\). Since \(S\) is left and right regular we have \(a \in (Sa^2)\) and \(a^2 \in (a^2S)\) for every \(a \in S\). Then there exists \(x, y \in S\) such that \(a \leq xa^2\) and \(a \leq a^2y\). Then \((x, a^2), (a^2, y) \in A_a\). Since \(A_a \neq \emptyset\), we have

\[
f(a) \leq ((fNa) \vee (aNf))(a) = \max[(fNa)(a), (aNf)(a)]
= \max[\max\{f(y), \alpha(z)\}, \max\{\max\{\alpha(y), f(z)\}, \max\{\alpha(y), f(z)\}\}]
\leq \max[\max\{f(a^2), \alpha(y)\}, \max\{\alpha(x), f(a^2)\}]
= \max\{\max\{f(a^2), -1\}, \max\{\alpha(x), f(a^2)\}\}
= \max\{f(a^2), f(a^2)\} = f(a^2) = f(aa) \leq \max\{f(a), f(a)\} = f(a).
\]

Thus \(f(a) = f(a^2)\).

Conversely, let \(a \in S\). We consider the quasi-ideal \(Q(a^2)\) generated by \(a^2\) \((a \in S)\). That is, the set \(Q(a^2) = (a^2 \cup (a^2S \cap Sa^2))\). By Theorem 3.3, the characteristic \(N\)-function \(\kappa_{Q(a^2)}\) is an \(N\)-fuzzy quasi-ideal of \(S\). By hypothesis

\[
\kappa_{Q(a^2)}(a) = \kappa_{Q(a^2)}(a^2).
\]

Since \(a^2 \in Q(a^2)\), we have \(\kappa_{Q(a^2)}(a^2) = -1\) then \(\kappa_{Q(a^2)}(a) = -1\) and \(a \in Q(a^2) = (a^2 \cup (a^2S \cap Sa^2))\). Then \(a \leq a^2\) or \(a \leq a^2x\) and \(a \leq ya^2\) for some \(x, y \in S\). If \(a \leq a^2\) then \(a \leq a^2 = aa \leq a^2a^2 = aana^2 \leq a^2a^2yy = a^2Sa^2\) and \(a \in (a^2Sa^2)\). If \(a \leq a^2x\) and \(a \leq ya^2\) then \(a \leq (a^2x)(ya^2) = a^2(xy)a^2 \leq a^2Sa^2\) and \(a \in (a^2Sa^2)\). \(\square\)

A subset \(T\) of an ordered semigroup \(S\) is called semi-prime if for every \(a \in S\) from \(a^2 \in T\) it follows \(a \in T\).

Definition 4.3. An \(N\)-fuzzy subset \(f\) of an ordered semigroup \((S, \cdot, \leq)\) is called semi-prime if \(f(a) \leq f(a^2)\) for all \(a \in S\).

Theorem 4.4. An ordered semigroup \((S, \cdot, \leq)\) is completely regular if and only if every \(N\)-fuzzy quasi-ideal \(f\) of \(S\) is semi-prime.

Proof. Let \(S\) be a completely regular ordered semigroup and \(f\) an \(N\)-fuzzy quasi-ideal of \(S\). Let \(a \in S\). Then \(f(a) \leq f(a^2)\). Indeed, since \(S\) is left and right regular, there exist \(x, y \in S\) such that \(a \leq xa^2\) and \(a \leq a^2y\) then \((x, a^2) \in A_a\) and \((a^2, y) \in A_a\). Since \(A_a \neq \emptyset\), then we have

\[
f(a) \leq ((fNa) \vee (aNf))(a) = \max[(fNa)(a), (aNf)(a)]
= \max[\max\{f(y), \alpha(z)\}, \max\{\max\{\alpha(y), f(z)\}, \max\{\alpha(y), f(z)\}\}]
\]

Therefore, \(f(a) = f(a^2)\).
\[ \leq \max \left[ \max \{ f(a^2), a(y) \}, \max \{ a(x), f(a^2) \} \right] \]
\[ = \max \left[ \max \{ f(a^2), -1 \}, \max \{ -1, f(a^2) \} \right] = \max \{ f(a^2), f(a^2) \} = f(a^2). \]

To prove the converse, let \( f \) be an \( N \)-fuzzy quasi-ideal of \( S \) such that \( f(a) \leq f(a^2) \) for all \( a \in S \). We consider the quasi-ideal \( Q(a^2) \) generated by \( a^2 (a \in S) \). That is, the set \( Q(a^2) = \{ a^2 \cup (a^2 S \cap Sa^2) \} \). Then by Theorem 3.3, \( \kappa_{Q(a^2)} \) is an \( N \)-fuzzy quasi-ideal of \( S \). By hypothesis \( \kappa_{Q(a^2)}(a) \leq \kappa_{Q(a^2)}(a^2) \). Since \( a^2 \in Q(a^2) \), we have \( \kappa_{Q(a^2)}(a^2) = -1 \) and \( \kappa_{Q(a^2)}(a) = -1 \) we get \( a \in Q(a^2) \). Then \( a \leq a^2 \) or \( a \leq a^2 p \) and \( a \leq qa^2 \) for some \( p, q \in S \). If \( a \leq a^2 \) then
\[ a \leq a^2 = aa \leq a^2 a^2 = aaa^2 \leq a^2 aa^2 \in a^2 Sa^2 \text{ and } a \in (a^2 Sa^2). \]

If \( a \leq a^2 p \) and \( a \leq qa^2 \). Then \( a \leq (a^2 p)(qa^2) = a^2(pq)a^2 \in a^2 Sa^2 \) and \( a \in (a^2 Sa^2) \). \( \square \)

5. Some semilattices of simple ordered semigroups

A subsemigroup \( F \) of an ordered semigroup \( S \) is called a filter of \( S \) if:

(i) \( a, b \in S \) and \( ab \in F \) implies \( a \in F \) and \( b \in F \).

(ii) If \( a \in F \) and \( c \in S \) such that \( c \geq a \) then \( c \in F \). (see \([7]\)).

For \( x \in S \), we denote by \( N(x) \) the least filter of \( S \) generated \( x (x \in S) \) and by \( N \) the equivalence relation
\[ N := \{ (x, y) \in S \times S \mid N(x) = N(y) \}. \]

Let \( S \) be an ordered semigroup. An equivalence relation \( \sigma \) on \( S \) is called congruence if \( (a, b) \in \sigma \) implies \( (ac, bc) \in \sigma \) and \( (ca, cb) \in \sigma \) for every \( c \in S \). A congruence \( \sigma \) on \( S \) is called semilattice congruence if \( (a^2, a) \in \sigma \) and \( (ab, ba) \in \sigma \) for each \( a, b \in S \) (see \([7]\)). If \( \sigma \) is a semilattice congruence on \( S \) then the \( \sigma \)-class \( \{x\}_\sigma \) of \( S \) containing \( x \) is a subsemigroup of \( S \) for every \( x \in S \) (see \([7]\)). An ordered semigroup \( S \) is called a semilattice of left and right simple semigroups if there exists a semilattice congruence \( \sigma \) on \( S \) such that the \( \sigma \)-class \( \{x\}_\sigma \) of \( S \) containing \( x \) is a left and right simple subsemigroup of \( S \) for every \( x \in S \).

Equivalent definition:

There exists a semilattice \( Y \) and a family \( \{ S_\alpha \}_{\alpha \in \mathbb{Y}} \) of left and right simple subsemigroups of \( S \) such that:

(i) \( S_\alpha \cap S_\beta = \emptyset \forall \alpha, \beta \in Y, \alpha \neq \beta \),

(ii) \( S = \bigcup_{\alpha \in Y} S_\alpha \),

(iii) \( S_\alpha S_\beta \subseteq S_{\alpha \beta} \forall \alpha, \beta \in Y \).

**Theorem 5.1.** An ordered semigroup \( (S, \cdot, \leq) \) is a semilattice of left and right simple semigroups if and only if for every \( N \)-fuzzy quasi-ideal \( f \) of \( S \), we have \( f(a) = f(a^2) \) and \( f(ab) = f(ba) \) for all \( a, b \in S \).
Lemma 2.7.

Then by hypothesis, there exists a semilattice \( Y \) and a family \( \{S_\alpha\}_{\alpha \in Y} \) of left and right simple subsemigroups of \( S \) satisfying (i), (ii) and (iii).

(A) Let \( f \) be an \( N \)-fuzzy quasi-ideal of \( S \) and \( a \in S \). Since \( a \in S = \bigcup_{\alpha \in Y} S_\alpha \), then there exists \( \alpha \in Y \) such that \( a \in S_\alpha \). Since \( S_\alpha \) is left simple, we have \( S_\alpha = \{S_\alpha a\} \). Since \( a \in S_\alpha \), then \( a \in \{S_\alpha a\} \) and so \( a \leq xa \) for some \( x \in S_\alpha \). Since \( x \in S_\alpha \), we have \( x \in (S_\alpha a) \), then \( x \leq ya \) for some \( y \in S_\alpha \). Thus \( a \leq xa \leq (ya)a = ya^2 \) and we have \( (y, a^2) \in A_\alpha \). Also \( S \) is right simple, we have \( S_\alpha = (aS_\alpha) \), since \( a \in S_\alpha \) then \( a \in (aS_\alpha) \) and we have \( a \leq az \) for some \( z \in S_\alpha \). Since \( z \in S_\alpha \) we have \( z \in (aS_\alpha) \) then \( z \leq at \) for some \( t \in S \). Thus \( a \leq az \leq a(at) = a^2t \), and we have \( a(t, a) \in A_\alpha \). Since \( f \) is an \( N \)-fuzzy quasi-ideal of \( S \) and \( A_\alpha \neq \emptyset \), we have

\[
\begin{align*}
f(a) \leq ((fNa) \vee (aNf))(a) &= \max\{fN(a), (aNf)(a)\} \\
&= \max\left[ \bigwedge_{(y_1, z_1) \in A_\alpha} \max\{f(y_1), (\alpha(z_1)\}, \bigwedge_{(y_2, z_2) \in A_\alpha} \max\{\alpha(y_2), f(z_2)\}\right] \\
&\leq \max\left[ \max\{f(a^2), \alpha(t)\}, \max\{\alpha(y), f(a^2)\}\right] \\
&= \max\left[ \max\{f(a^2), -1\}, \max\{-1, f(a^2)\}\right] = \max [f(a^2), f(a^2)] = f(a^2).
\end{align*}
\]

On the other hand, since every \( N \)-fuzzy quasi-ideal is an \( N \)-fuzzy subsemigroup of \( S \), we have \( f(a^2) = f(aa) \leq \max\{f(a), f(a)\} = f(a) \). Thus \( f(a) = f(a^2) \).

(B) Let \( a, b \in S \). By (A), we have \( f(ab) = f((ab)^2) = f((ab)^4) \). Since \( (AB) = (BA) = A \cap B \), by Lemmas 2.5 an 2.8, we have

\[
\begin{align*}
(ab)^4 &= (aba)(babab) \in Q(aba)Q(babab) \subseteq (Q(aba)Q(babab)) \\
&= (Q(babab)Q(aba)) = ((babab \cup (bababS \cap Sbabab)(aba \cup (abaS \cap Saba)) \\
&\subseteq ((babab \cup (bababS)(aba \cup (Saba)) \subseteq (baS \cup (baS[Sba \cup (Sba)] \\
&= ((baS[Sba]) = ((baSba] = (baSba) = baS \cap Sba
\end{align*}
\]

Then \( (ab)^4 \leq (ba)x \) and \( (ab)^4 \leq y\) for some \( x, y \in S \). Then \( (ba, x) \in A_{(ab)^4} \) and \( (y, ba) \in A_{(ab)^4} \). Since \( f \) is an \( N \)-fuzzy quasi-ideal and \( A_{(ab)^4} \neq \emptyset \), we have

\[
\begin{align*}
f((ab)^4) \leq ((fNa) \vee (aNf))((ab)^4) &= \max\{((fNa)(ab)^4), (aNf)(ab)^4\} \\
&= \max\left[ \bigwedge_{(y_1, z_1) \in A_{(ab)^4}} \max\{f(y_1), (\alpha(z_1))\}, \bigwedge_{(y_2, z_2) \in A_{(ab)^4}} \max\{\alpha(y_2), f(z_2)\}\right] \\
&\leq \max\left[ \max\{f(ba), \alpha(x)\}, \max\{\alpha(y), f(ba)\}\right] \\
&= \max\left[ \max\{f(ba), -1\}, \max\{-1, f(ba)\}\right] = \max [f(ba), f(ba)] = f(ba).
\end{align*}
\]

By symmetry we can prove that \( f(ba) \leq f((ab)^4) = f(ab) \).

Conversely, assume that conditions (1) and (2) are true. Then by (1) and Lemma 2.7, \( S \) is completely regular. Let \( A \) be a quasi-ideal of \( S \) and let \( a \in A \).
Since $S$ is completely and $a \in S$, there exists $x \in S$ such that $a \leq a^2 xa^2$. Then
\[
a \leq a^2 xa^2 \leq (a^2 Sa^2) = a(a(Sa)a) \subseteq a(aSa) \subseteq a(aSa)
\]
\[
= a(aS \cap a) \subseteq A(AS \cap SA) \subseteq A(AS) \subseteq AA,
\]
and so $A \subseteq AA \subseteq (A^2)$. On the other hand, by Lemma 2.4, $A$ is a subsemigroup of $S$, we have $A^2 \subseteq A \Rightarrow (A^2) \subseteq (A) = A$.

Let $A$ and $B$ be any quasi-ideals of $S$ and let $x \in (AB)$, then $x \leq ab$ for some $a \in A$ and $b \in B$. We consider the quasi-ideal $Q(ab) = (ab \cup (abS \cap Sab))$ generated by $ab$. Then by Theorem 3.3, the characteristic $N$-function $\kappa_{Q(ab)}$ of $Q(ab)$ is an $N$-fuzzy quasi-ideal of $S$. By hypothesis $\kappa_{Q(ab)}(ba) = \kappa_{Q(ab)}(ab)$. Since $ab \in Q(ab)$, we have $\kappa_{Q(ab)}(ab) = -1$ and $\kappa_{Q(ab)}(ba) = -1$. Therefore $ba \in Q(ab) = (ab \cup (abS \cap Sab))$ and, by Lemma 2.9,
\[
ba \in (ab \cup (abS \cap Sab)) = (ab \cup (abSba)) \subseteq (AB \cup (ABSAB])
\]
\[
\subseteq (AB \cup ((AB]S(AB)] \subseteq (AB \cup (AB]) = ([AB] = (AB].
\]
Hence $(BA) \subseteq (AB)$. By symmetry we can prove that $(AB) \subseteq (BA)$.

**Lemma 5.2.** Let $(S, \cdot, \leq)$ be an ordered semigroup such that $a \leq a^2$ for all $a \in S$. Then for every $N$-fuzzy quasi-ideal $f$ of $S$ we have $f(a) = f(a^2)$ for every $a \in S$.

**Proof.** Let $a \in S$ such that $a \leq a^2$. Let $f$ be an $N$-fuzzy quasi-ideal of $S$. Since $f$ is an $N$-fuzzy subsemigroup of $S$. Then $f(a) \leq f(a^2) \leq \max\{f(a), f(a)\} = f(a)$.

**Theorem 5.3.** Let $S$ be an ordered semigroup and $a \in S$ such that $a \leq a^2$ for all $a \in S$. Then the following are equivalent:

(i) $ab \in (baS \cap (Sba)$ for each $a, b \in S$.

(ii) For every $N$-fuzzy quasi-ideal $f$ of $S$, we have $f(ab) = f(ba)$ for every $a, b \in S$.

**Proof.** (i) $\Rightarrow$ (ii). Let $f$ be an $N$-fuzzy quasi-ideal of $S$. Since $ab \in (baS \cap (Sba)$, then $ab \in (baS)$ and we have $ab \leq (ba)x$ for some $x \in S$. By (i), we have $(ba)x \in (xbaS \cap (Sxba)$. Then $(ba)x \in (Sxba)$ and we have $(ba)x \leq (yx)(ba)$ and so, $ab \leq (yx)(ba) \Rightarrow (yx)(ba) \in A_{ab}$. Again, since $ab \in (Sba)$, then $ab \leq z(ba)$ for some $z \in S$ and by (i) we have $z(ba) \in (baS \cap (Sba)$, then $z(ba) \leq (ba)(zt)$ for some $t \in S$. So we have $ab \leq (ba)(zt) \Rightarrow (ba, zt) \in A_{ab}$. Since $f$ is an $N$-fuzzy quasi-ideal of $S$ and $A_{ab} \neq \emptyset$, then
\[
f(ab) \leq ((fNa) \vee (aNf))(ab) = \max\{(fNa)(ab), (aNf)(ab)\}
\]
\[
= \max \left[ \bigwedge_{(y_1, z_1) \in A_{ab}} \max\{f(y_1), \alpha(z_1)\}, \bigwedge_{(y_2, z_2) \in A_{ab}} \max\{\alpha(y_2), f(z_2)\} \right]
\]
\[
\leq \max\{\max\{f(ba), \alpha(zt)\}, \max\{\alpha(yx), f(ba)\}\}
\]
\[
= \max\{\max\{f(ba), -1\}, \max\{-1, f(ba)\}\} = \max\{f(ba), f(ba)\} = f(ba).
\]
By symmetry we can prove that $f(ba) \leq f(ab)$.

(ii) $\Rightarrow$ (i). Let $f$ be an $N$-fuzzy quasi-ideal of $S$. Since $a \leq a^2$ for all $a \in S$, by Lemma 5.2, we have $f(a) = f(a^2)$. By (ii), we obtain $f(ba) = f(ab)$ for each $a, b \in S$. By Theorem 5.1, it follows that $S$ is a semilattice of left and right simple semigroups. Thus by hypothesis, there exists a semilattice $Y$ and a family $\{S_\alpha\}_{\alpha \in Y}$ of left and right simple subsemigroups satisfying (i), (ii) and (iii) from the equivalent definition of a semilattice of simple semigroups.

Let $a, b \in S$, we have to show that $a \in (baS) \cap (Sba)$. Let $\alpha, \beta \in Y$ be such that $a \in S_\alpha$ and $b \in S_\beta$. Then $ab \in S_\alpha S_\beta \subseteq S_\alpha \beta$ and $ba \in S_\beta S_\alpha \subseteq S_\beta \alpha = S_\alpha \beta$. Since $S_\alpha \beta$ is left and right simple we have $S_\alpha \beta = (S_\alpha \beta c) c S_\beta$ and $S_\alpha \beta = (c S_\alpha \beta) c S_\beta$ for each $c \in S_\alpha \beta$. Since $ab, ba \in S_\alpha \beta$, we have $ab \in (baS_\alpha \beta) \cap (S_\alpha \beta ba) \subseteq (baS) \cap (Sba)$. This complete the proof.

References


Received December 27, 2008