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Prolongations of quasigroups by middle translations

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Abstract. This article is a continuation of the study of prolongations of quasigroups and Latin squares. Now using complete mappings and middle translations we present various characterizations of prolongations of quasigroups described in [4]. Based on these characterizations we find isotopic prolongations.

1. Introduction

Let $Q = \{1, 2, 3, \ldots, n\}$ be a finite set, φ and ψ permutations of Q. The composition of permutations is defined as $\varphi\psi(x) = \varphi(\psi(x))$. Permutations will be written in the form of cycles, and cycles will be separated by points:

$$\varphi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 1 & 2 & 5 & 4 & 6 \end{pmatrix} = (132.45.6.)$$

$Q(\cdot)$ always denotes a quasigroup.

Definition 1.1. A permutation φ_i of the set Q such that

$$x \cdot \varphi_i(x) = i, \quad i \in Q \tag{1}$$

all $x \in Q$ is called the *track* of an element i.

It is clear that for a quasigroup $Q(\cdot)$ of order n the set of permutations

$$\{\varphi_1, \varphi_2, \varphi_3, \ldots, \varphi_n\},$$

satisfying (1) uniquely determines its Latin square (i.e., its multiplication table). Therefore, we can identify $Q(\cdot)$ with the above set of permutations and write

$$Q(\cdot) = \{\varphi_1, \varphi_2, \varphi_3, \ldots, \varphi_n\}.$$

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In [3] the following very useful result is proved

Lemma 1.2. *Let* $Q(\cdot) = \{\varphi_1, \varphi_2, \ldots, \varphi_n\}$ *and* $Q(\circ) = \{\psi_1, \psi_2, \ldots, \psi_n\}$ *be two quasigroups. Then for any bijections* $\alpha, \beta, \gamma : Q \to Q$ *satisfying the identity* $\gamma(x \circ y) = \alpha(x) \cdot \beta(y)$ *we have* $\psi_{\gamma(i)} = \beta\varphi_i\alpha^{-1}$. □

Definition 1.3. Any mapping σ of Q defines on a quasigroup $Q(\cdot)$ a new mapping $\bar{\sigma}$ such that

$$\bar{\sigma}(x) = x \cdot \sigma(x), \qquad x \in Q.$$

The number $rg(\sigma) = |\bar{\sigma}(Q)|$, where $\bar{\sigma}(Q) = \{\bar{\sigma}(x) \,|x \in Q\}$, is called the *range of a mapping* σ on a quasigroup $Q(\cdot)$.

If $\bar{\sigma}(Q) = Q$, i.e., $rg(\sigma) = n = |Q|$, then we say that σ is a *complete mapping.* A quasigroup having at least one complete mapping is called *admissible.* If $rg(\sigma) = n - 1$, the mapping σ is called *quasicomplete.* Every track has range 1.

For a quasicomplete mapping σ we define its *defect* $def(\sigma)$ putting

$$def(\sigma) = Q - \bar{\sigma}(Q).$$

If $def(\sigma) = d$, then $\bar{\sigma}^{-1}(d) = \emptyset$. In this case $a = \bar{\sigma}(a_1) = \bar{\sigma}(a_2)$ for some $a, a_1, a_2 \in Q$.

It is clear that a permutation φ of Q can be extended to $Q' = Q \cup \{q\}$, where $q \notin Q$, by putting $\varphi(q) = q$.

2. Classical prolongation

The *classical* method of prolongation of admissible quasigroups proposed by V. D. Belousov [1] is based on the following construction. Let $Q(\cdot)$ be a fixed admissible quasigroup, σ its complete mapping. The operation $(\circ)$ on $Q' = Q \cup \{q\}$ is defined by the formula:

$$x \circ y = \begin{cases} x \cdot y & \text{if } x \neq q,\ y \neq q,\ y \neq \sigma(x), \\ \bar{\sigma}\sigma^{-1}(y) & \text{if } x = q,\ y \neq q, \\ \bar{\sigma}(x) & \text{if } x \neq q,\ y = q, \\ q & \text{if } y = \sigma(x),\ x \in Q'. \end{cases} \qquad (2)$$

We say that this prolongation is *induced* by a complete mapping σ.

By Δ_σ we denote the set

$$\Delta_\sigma = \{\sigma_1, \sigma_2, \ldots, \sigma_n \mid \sigma_i = \left(q, \bar{\sigma}^{-1}(i)\right)\},$$

where $Q(\cdot)$ is a quasigroup, $q \notin Q$ and σ is a complete mapping of $Q(\cdot)$.

Below we present the new description of this prolongation. But first we make some useful observations.

Note first that

$$x_i \cdot \sigma(x_i) = \bar{\sigma}(x_i) = i. \tag{3}$$

Hence

$$x_i = \bar{\sigma}^{-1}(i). \tag{4}$$

Thus the transposition σ_i in Δ_σ can be written in the form $\sigma_i = (q, x_i)$. Moreover, from (1) it follows $x_i \cdot \varphi_i(x_i) = i$, which together with (3) gives

$$\sigma(x_i) = \varphi_i(x_i). \tag{5}$$

Now we can give the new characterization of the classical prolongation.

Theorem 2.1. *Let $Q(\cdot) = \{\varphi_1, \varphi_2, \varphi_3, \ldots, \varphi_n\}$ be a quasigroup with a complete mapping σ. The quasigroup $Q'(\circ) = \{\psi_1, \psi_2, \psi_3, \ldots, \psi_n, \psi_q\}$ coincides with the prolongation of $Q(\cdot)$ defined by (2) if and only if*

$$\begin{cases} \psi_i = \varphi_i \sigma_i & i = 1, 2, \ldots, n, \\ \psi_q = \sigma, \end{cases} \tag{6}$$

where $\sigma_i = \left(q, \bar{\sigma}^{-1}(i)\right)$.

Proof. The first, the second and the third row of (2) is equivalent to the first row of (6). The last row of (2) is equivalent to the second row of (6).

Indeed, the first row of (2) says that

$$x \circ y = x \cdot y \quad \text{for} \quad x \neq q,\ y \neq q,\ y \neq \sigma(x).$$

This for $y = \varphi_i(x)$ gives

$$x \circ \varphi_i(x) = x \cdot \varphi_i(x) = i.$$

Since $x \circ \psi_i(x) = i$, the last implies

$$\varphi_i = \psi_i \quad \text{for} \quad x \neq q,\ y \neq q,\ y \neq \sigma(x). \tag{7}$$

If $x = q$, then from the second row of (2), we obtain

$$q \circ y = \bar{\sigma}\sigma^{-1}(y) \quad \text{for} \quad y \neq q,\ y \in Q, \tag{8}$$

whence putting $y = \psi_i(q)$ we get $\bar{\sigma}\sigma^{-1}\psi_i(q) = q \circ \psi_i(q) = i$. Thus $\sigma^{-1}\psi_i(q) = \bar{\sigma}^{-1}(i)$. But $x_i = \bar{\sigma}^{-1}(i)$, so $\psi_i(q) = \sigma(x_i)$, and consequently

$$\psi_i(q) = \varphi_i(x_i). \tag{9}$$

In the case $y = q$, according to the third row of (2), we have $x \circ q = \bar{\sigma}(x)$. If $x \circ q = i$, then $\bar{\sigma}(x) = i$, i.e., $x = \bar{\sigma}^{-1}(i)$, which, by (4), implies $x = x_i$. Hence $x_i \circ q = i$. But on the other hand, by the definition of ψ_i, we have $x_i \circ \psi_i(x_i) = i$. So,

$$\psi_i(x_i) = q. \tag{10}$$

Let us consider the first row of (6).

If $x \neq q$ and $x \neq x_i$, then $\sigma_i = (q, x_i)$ can be eliminated from $\psi_i = \varphi_i\sigma_i$. Hence $\psi_i = \varphi_i$. This coincides with (7) and corresponds to the first row of (2).

If $x = q$, then $\psi_i(q) = \varphi_i\sigma_i(q) = \varphi_i(x_i)$. This coincides with (9) and corresponds to the second row of (2).

If $x = x_i$, then $\psi_i(x_i) = \varphi_i\sigma_i(x_i) = \varphi_i(q) = q$. This coincides with (10) and corresponds to the third row of (2).

So, the first, the second and the third rows of (2) correspond to the first row of (6).

Let us consider the fourth row of (2). Then $y = \sigma(x)$ and $x \circ \sigma(x) = q$. But, by the definition, $x \circ \psi_q(x) = q$. Thus $\psi_q(x) = \sigma(x)$. This means that the fourth row of (2) is equivalent to the second row of (6). □

Since quasigroups may have several complete mappings, the natural question is: *when prolongations obtained from the same quasigroup by different complete mappings are isotopic.*

Below we give the partial answer to this question. Our answer is based on the concept of the equivalence of complete mappings.

Remind (see for example [1]) that two permutations ρ and σ of a quasigroup $Q(\cdot)$ are *equivalent* if

$$\rho = \beta\sigma\alpha^{-1}$$

for some autotopism $T = (\alpha, \beta, \gamma)$ of $Q(\cdot)$.

Proposition 2.2. *Any mapping equivalent to a complete mapping of a quasigroup $Q(\cdot)$ also is a complete mapping of $Q(\cdot)$.*

Proof. Let $\rho = \beta\sigma\alpha^{-1}$, where σ is a complete mapping of a quasigroup $Q(\cdot)$ and $T = (\alpha, \beta, \gamma)$ its autotopism.

Let $\{\varphi_1, \varphi_2, \varphi_3, \ldots, \varphi_n\}$ be tracks of $Q(\cdot)$. We can choose elements $x_1, x_2, \ldots, x_n$ of Q for which $\sigma(x_i) = \varphi_i(x_i)$.

Since $\varphi_{\gamma(i)} = \beta\varphi_i\alpha^{-1}$ (see Lemma 1.2), for $y_1, y_2, \ldots, y_n$ such that $y_i = \alpha(x_i)$, $i = 1, 2, \ldots, n$, we have

$$\rho(y_i) = \beta\sigma\alpha^{-1}(y_i) = \beta\sigma(x_i) = \beta\varphi_i(x_i) = \varphi_{\gamma(i)}\alpha(x_i) = \varphi_{\gamma(i)}(y_i).$$

Thus $\rho(y_i) = \varphi_{\gamma(i)}(y_i)$.

Multiplying this equation by y_i we obtain

$$y_i \cdot \rho(y_i) = y_i \cdot \varphi_{\gamma(i)}(y_i) = \gamma(i),$$

which proves that ρ is a complete mapping. □

Corollary 2.3. *If α is an automorphism of a quasigroup $Q(\cdot)$ and σ is its complete mapping, then $\rho = \alpha\sigma\alpha^{-1}$ also is a complete mapping of $Q(\cdot)$.* □

Corollary 2.4. *If $T = (\alpha, \beta, \gamma)$ is an autotopism of $Q(\cdot)$ and σ_0 is its complete mapping, then*

$$\sigma_k = \beta^k\sigma_0\alpha^{-k}$$

is a complete mapping of $Q(\cdot)$. □

Corollary 2.5. *Equivalent permutations have the same range.* □

Theorem 2.6. *Classical prolongations induced by equivalent complete mappings are isotopic.*

Proof. Let $Q'_\sigma(\circ) = \{\psi_1, \psi_2, \ldots, \psi_n, \psi_q\}$ and $Q'_\rho(*) = \{\omega_1, \omega_2, \ldots, \omega_n, \omega_q\}$ be prolongations of a quasigroup $Q(\cdot) = \{\varphi_1, \varphi_2, \ldots, \varphi_n\}$ induced by complete mappings σ and $\rho = \beta\sigma\alpha^{-1}$, respectively. Then, according to Theorem 2.1, we have

$$\begin{cases} \psi_i = \varphi_i\sigma_i & i = 1, 2, \ldots, n, \\ \psi_q = \sigma, \end{cases} \qquad \text{and} \qquad \begin{cases} \omega_i = \varphi_i\rho_i & i = 1, 2, \ldots, n, \\ \omega_q = \rho, \end{cases}$$

where $\sigma_i = (q, x_i)$, $\rho_i = (q, y_i)$.

Let us consider two sequences $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$ of elements of Q such that

$$\varphi_i(x_i) = \sigma(x_i) \quad \text{and} \quad \varphi_i(y_i) = \rho(y_i) \tag{11}$$

for $i = 1, 2, \ldots, n$.

Since $T = (\alpha, \beta, \gamma)$ is an autotopism of $Q(\cdot)$ such that $\rho = \beta\sigma\alpha^{-1}$, by Lemma 1.2, we obtain $\varphi_{\gamma(i)} = \beta\varphi_i\alpha^{-1}$. Thus

$$\varphi_{\gamma(i)}\alpha(x_i) = \beta\varphi_i(x_i) = \beta\sigma(x_i) = \rho\alpha(x_i),$$

i.e., $\varphi_{\gamma(i)}\alpha(x_i) = \rho\alpha(x_i)$, which for $\alpha(x_i) = z$ gives $\varphi_{\gamma(i)}(z) = \rho(z)$. But, by the assumption $\varphi_{\gamma(i)}\left(y_{\gamma(i)}\right) = \rho\left(y_{\gamma(i)}\right)$ (see (11)), hence $z = y_{\gamma(i)}$. Therefore

$$\alpha(x_i) = y_{\gamma(i)} \tag{12}$$

for every $i = 1, 2, \ldots, n$.

Now we will prove that

$$\alpha\sigma_i\alpha^{-1} = \rho_{\gamma(i)}. \tag{13}$$

Indeed, according to the definition $\sigma_i = (q, x_i)$ and $\rho_{\gamma(i)} = \left(q, y_{\gamma(i)}\right)$. Thus

$$\alpha\sigma_i\alpha^{-1}\left(y_{\gamma(i)}\right) = \alpha\sigma_i(x_i) = \alpha(q) = q,$$
$$\alpha\sigma_i\alpha^{-1}(q) = \alpha\sigma_i(q) = \alpha(x_i) = y_{\gamma(i)}.$$

This means that $\alpha\sigma_i\alpha^{-1} = \left(q, y_{\gamma(i)}\right) = \rho_{\gamma(i)}$. So, (13) is valid.

Moreover,

$$\begin{aligned}\beta\psi_i\alpha^{-1} &= \beta(\varphi_i\sigma_i)\alpha^{-1} = \beta\varphi_i\left(\alpha^{-1}\alpha\right)\sigma_i\alpha^{-1} \\ &= (\beta\varphi_i\alpha^{-1})(\alpha\sigma_i\alpha^{-1}) = \varphi_{\gamma(i)}\rho_{\gamma(i)} = \omega_{\gamma(i)},\end{aligned}$$

and

$$\beta\psi_q\alpha^{-1} = \beta\sigma\alpha^{-1} = \rho = \omega_q.$$

From the above it follows that $T = (\alpha, \beta, \gamma)$ is an isotopism between $Q'(\circ)$ and $Q'(*)$. This completes the proof. $\square$

Example 2.7. Consider the quasigroup $Q(\cdot)$ with the multiplication table:

$\cdot$	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	1	4	3	6	5
3	3	4	5	6	1	2
4	4	3	6	5	2	1
5	5	6	2	1	4	3
6	6	5	1	2	3	4

This quasigroup can be written as $Q(\cdot) = \{\varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5, \varphi_6\}$, where

$$\varphi_1 = (1.2.3546.),\quad \varphi_2 = (12.3645.),\quad \varphi_3 = (13.24.56.),$$
$$\varphi_4 = (14.23.5.6.),\quad \varphi_5 = (15.26.3.4.),\quad \varphi_6 = (16.25.34.).$$

Choose the following two complete mappings of this quasigroup:

$$\rho_0 = (1.3.6.245.) \quad \text{and} \quad \theta = (1.23456.)\,.$$

Then $\bar{\rho}_0 = (1.23564.)$ and $\bar{\theta} = (1.24.365.)$.

Consider now the autotopism $T = (\alpha, \beta, \gamma)$ of $Q(\cdot)$, where

$$\alpha = (163254.)\,,\quad \beta = (164.253.)\,,\quad \gamma = (146235.)\,.$$

We can construct this autotopism used, for example, the method proposed in [3].

Let $\rho_1 = \beta\rho_0\alpha^{-1} = T(\rho_0)$ and $T^k(\rho_0) = \rho_k$ for $k \geqslant 1$. Then

$$T(\rho_0) = (1345.2.6.) = \rho_1,\qquad T(\rho_1) = (13462.5.) = \rho_2,$$
$$T(\rho_2) = (143562.) = \rho_3,\qquad T(\rho_3) = (12356.4.) = \rho_4,$$
$$T(\rho_4) = (1.4.2365.) = \rho_5,\qquad T(\rho_5) = (1.3.6.245.) = \rho_0.$$

By Corollary 2.4 all mappings $\rho_1, \rho_2, \ldots, \rho_5$ are completeon $Q(\cdot)$. These mappings are equivalent. Hence, by Theorem 2.6, all prolongations $Q'_{\rho_i}(\circ)$ of $Q(\cdot)$ are isotopic.

We select two prolongations $Q'_{\rho_0}(\circ)$ and $Q'_{\rho_3}(*)$. For simplicity ρ_0 will be denoted by σ, ρ_3 – by τ. Since $\beta^3\sigma\alpha^{-3} = \tau$, ρ and τ are equivalent, and $\bar{\sigma} = (1.23564.)$, $\bar{\tau} = (14653.2.)$.

Let $Q'_\sigma(\circ) = \{\psi_1, \psi_2, \ldots, \psi_6, \psi_7\}$, $Q'_\tau(*) = \{\omega_1, \omega_2, \ldots, \omega_6, \omega_7\}$. Then $q = 7$ and

$$\Delta_\sigma = \{\sigma_1 = (7,1),\ \sigma_2 = (7,4),\ \sigma_3 = (7,2),\ \sigma_4 = (7,6),\ \sigma_5 = (7,3),\ \sigma_6 = (7,5)\},$$
$$\Delta_\tau = \{\tau_1 = (7,3)\,,\ \tau_2 = (7,2)\,,\ \tau_3 = (7,5)\,,\ \tau_4 = (7,1)\,,\ \tau_5 = (7,6)\,,\ \tau_6 = (7,4)\}.$$

According to (2), the multiplication table of $Q'_\sigma(\circ)$ has the form:

∘	1	2	3	4	5	6	7
1	7	2	3	4	5	6	1
2	2	1	4	7	6	5	3
3	3	4	5	6	7	2	1
4	4	3	7	5	1	1	6
5	5	6	2	1	4	7	3
6	6	7	1	2	3	4	5
7	2	5	6	4	1	3	7

By Theorem 2.1, $Q'_\sigma(\circ)$ has the following tracks:

$$\begin{aligned}
\psi_1 &= \varphi_1\sigma_1 = (1.2.3546.7.)\,(7,1) \ = (17.2.3546.)\,,\\
\psi_2 &= \varphi_2\sigma_2 = (12.3645.7.)\,(7,4) \ = (12.36475.)\,,\\
\psi_3 &= \varphi_3\sigma_3 = (13.24.56.7.)\,(7,2) \ = (13.274.56.)\,,\\
\psi_4 &= \varphi_4\sigma_4 = (14.23.5.6.7.)\,(7,6) = (14.23.5.67.)\,,\\
\psi_5 &= \varphi_5\sigma_5 = (15.26.3.4.7.)\,(7,3) = (15.26.37.4.)\,,\\
\psi_6 &= \varphi_6\sigma_6 = (16.25.43.7.)\,(7,5) \ = (16.257.43.)\,,\\
\psi_7 &= \ \ \sigma \ \ = (1.3.6.245.7.)\,.
\end{aligned}$$

Similarly, $Q'_\tau(*)$ has the multiplication table:

$*$	1	2	3	4	5	6	7
1	1	2	3	7	5	6	4
2	7	1	4	3	6	5	2
3	3	4	5	6	7	2	1
4	4	3	7	5	1	1	6
5	5	6	2	1	4	7	3
6	6	7	1	2	3	4	5
7	2	5	6	4	1	3	7

and tracks:

$$\begin{aligned}
\omega_1 &= \varphi_1\tau_1 = (1.2.3546.7.)\,(7,3) \ = (1.2.37546.)\,,\\
\omega_2 &= \varphi_2\tau_2 = (12.3645.7.)\,(7,2) \ = (127.3645.)\,,\\
\omega_3 &= \varphi_3\tau_3 = (13.24.56.7.)\,(7,5) \ = (13.24.576.)\,,\\
\omega_4 &= \varphi_4\tau_4 = (14.23.5.6.7.)\,(7,1) = (174.23.5.6.)\,,\\
\omega_5 &= \varphi_5\tau_5 = (15.26.3.4.7.)\,(7,6) = (15.267.3.4.)\,,\\
\omega_6 &= \varphi_6\tau_6 = (16.25.43.7.)\,(7,4) \ = (16.25.473.)\,,\\
\omega_7 &= \ \ \tau \ \ = (143562.7.)\,.
\end{aligned}$$

$T^3 = (\alpha^3, \beta^3, \gamma^3)$ is an isotopism between $Q'_\sigma(\circ)$ and $Q'_\tau(*)$. Hence, by (3) from [3], we have $\beta^3\psi_i\alpha^{-3} = \omega_{\gamma^3(i)}$. This fact can be deduced from the above calculations because $\beta^3 = \varepsilon$, $\alpha^{-3} = (12.34.56.7.)$, $\gamma^3 = (12.24.56.7.)$ and, as it is not difficult to see, $\psi_i\alpha^{-3} = \omega_{\gamma^3(i)}$ for every $i \in Q'$.

Consider now the prolongation $Q'_\theta(\times)$ of $Q(\cdot)$ induced by the complete mapping $\theta = (1.23456.)$. Then $\bar\theta = (1.24.365.)$, $\bar\theta^{-1} = (1.24.356.)$ and

$$\Delta_\theta = \{\theta_1 = (7,1)\,,\ \theta_2 = (7,4)\,,\ \theta_3 = (7,5)\,,\ \theta_4 = (7,2)\,,\ \theta_5 = (7,6)\,,\ \theta_6 = (7,3)\}.$$

Quasigroups $Q'_\theta(\times)$ and $Q'_\tau(*)$ are not isotopic because complete mappings θ and τ are not equivalent. $Q'_\theta(\times)$ has the multiplication table:

×	1	2	3	4	5	6	7
1	7	2	3	4	5	6	1
2	2	1	7	3	6	5	4
3	3	4	5	7	1	2	6
4	4	3	6	5	7	1	2
5	5	6	2	1	4	7	3
6	6	7	1	2	3	4	5
7	1	5	4	6	2	3	7

and tracks:

$$\begin{aligned}
\lambda_1 &= \varphi_1\theta_1 = (1.2.3546.7.)\,(7,1) \;= (17.2.3546.)\,,\\
\lambda_2 &= \varphi_2\theta_2 = (12.3645.7.)\,(7,4) \;= (12.36475.)\,,\\
\lambda_3 &= \varphi_3\theta_3 = (13.24.56.7.)\,(7,5) \;= (13.24.576.)\,,\\
\lambda_4 &= \varphi_4\theta_4 = (14.23.5.6.7.)\,(7,2) = (14.273.5.6.)\,,\\
\lambda_5 &= \varphi_5\theta_5 = (15.26.3.4.7.)\,(7,6) = (15.267.3.4.)\,,\\
\lambda_6 &= \varphi_6\theta_6 = (16.25.43.7.)\,(7,3) \;= (16.25.437.)\,,\\
\lambda_7 &= \quad\theta \quad = (1.23456.7.)\,.
\end{aligned}$$

The fact that these two quasigroups are non-isotopic can be also deduced from Theorem 2.5 in [3]. □

3. Prolongations by fixed element

Consider now the prolongation proposed by G. B. Belyavskaya (see [2] or [4]). This prolongation is a modyfication of a prolongation proposed by V. D. Belousov.

Let $Q(\cdot)$ be a quasigroup with a complete mapping σ. Then for an arbitrary fixed element $a \in Q$ there exists an uniquely determined element $x_a \in Q$ such that

$$x_a \cdot \sigma(x_a) = a. \tag{14}$$

Using this fixed element $a \in Q$ and a complete mapping σ we can define the prolongation $Q'_{\sigma,a}(\circ)$ of $Q(\cdot)$ by putting

$$x \circ y = \begin{cases} x \cdot y & y \neq \sigma(x), \ x, y \in Q, \\ q & y = \sigma(x), \ x \neq x_a, \\ a & x = x_a, \ y = \sigma(x_a), \\ \bar{\sigma}(x) & x \neq x_a, \ y = q, \\ \bar{\sigma}\sigma^{-1}(y) & x = q, \ y \neq \sigma(x_a), \\ q & x = q, \ y = \sigma(x_a), \\ q & x = x_a, \ y = q, \\ a & x = y = q. \end{cases} \tag{15}$$

It is clear that for one complete mapping σ of $Q(\cdot)$ one can find $n = |Q|$ different elements satisfying (14). So, one complete mapping induces n different prolongations of the form $Q'_{\sigma,a}(\circ)$. Obviously, some of these prolongations can be isotopic.

First observe that the prolongation $Q'_{\sigma,a}(\circ)$ can be characterized by their tracks. Namely, we have

Theorem 3.1. *Let $Q(\cdot) = \{\varphi_1, \varphi_2, \ldots, \varphi_n\}$ be a quasigroup with a complete mapping σ. The quasigroup $Q'(\circ) = \{\psi_1, \psi_2, \ldots, \psi_n, \psi_q\}$ coincides with the prolongation $Q'_{\sigma,a}(\circ)$ of $Q(\cdot)$ defined by* (15) *if and only if*

$$\begin{cases} \psi_i = \varphi_i \sigma_i, & i \neq a, q, \\ \psi_a = \varphi_a, \\ \psi_q = \sigma\sigma_a, \end{cases} \tag{16}$$

where $\sigma_a = (q, x_a)$ and $\sigma_i = (q, \bar{\sigma}^{-1}(i))$.

Proof. The proof of this theorem is similar to the proof of Theorem 2.1. □

Using this theorem we can prove

Theorem 3.2. *Let $T = (\alpha, \beta, \gamma)$ be an autotopism of a quasigroup $Q(\cdot)$ with a complete mapping σ. If $\beta\sigma\alpha^{-1} = \rho$, then prolongations $Q'_{\sigma,a}(\circ)$ and $Q'_{\rho,b}(*)$, where $b = \gamma(a)$, are isotopic.*

Proof. Let $Q'_{\sigma,a}(\circ) = \{\psi_1, \psi_2, \ldots, \psi_n, \psi_q\}$ and $Q'_{\rho,b}(*) = \{\omega_1, \omega_2, \ldots, \omega_n, \omega_q\}$, where $b = \gamma(a)$. In this case

$$\begin{cases} \psi_i = \varphi_i \sigma_i, & i \neq a, q, \\ \psi_a = \varphi_a, \\ \psi_q = \sigma\sigma_a, \end{cases} \quad \text{and} \quad \begin{cases} \omega_i = \varphi_i \rho_i, & i \neq b, q, \\ \omega_b = \varphi_b, \\ \omega_q = \sigma\rho_b, \end{cases}$$

where $\sigma_a = (q, x_a)$, $\rho_b = (q, y_b)$ and $\sigma_i = (q, x_i)$, $\rho_i = (q, y_i)$.

Since $\varphi_{\gamma(i)} = \beta\varphi_i\alpha^{-1}$ for any autotopism $T = (\alpha, \beta, \gamma)$ of $Q(\cdot)$ (Lemma 1.2), for $i \neq a, q$ we have

$$\beta\psi_i\alpha^{-1} = \beta\varphi_i\sigma_i\alpha^{-1} = \beta\varphi_i\alpha^{-1}\alpha\sigma_i\alpha^{-1} = \varphi_{\gamma(i)}\rho_{\gamma(i)} = \omega_{\gamma(i)},$$

where $\alpha\sigma_i\alpha^{-1} = \rho_{\gamma(i)}$ (as in the proof of Theorem 2.6).

For $i = a$ we obtain $\beta\psi_a\alpha^{-1} = \varphi_{\gamma(a)} = \varphi_b = \omega_b$. Similarly, if $i = q$, then

$$\beta\psi_q\alpha^{-1} = \beta\sigma\sigma_a\alpha^{-1} = \beta\sigma\alpha^{-1}\alpha\sigma_a\alpha^{-1} = \sigma\alpha\sigma_a\alpha^{-1} = \sigma\rho_{\gamma(a)} = \sigma\rho_b = \omega_q.$$

From the above it follows that $T = (\alpha, \beta, \gamma)$ is an isotopism between $Q'_{\sigma,a}(\circ)$ and $Q'_{\rho,b}(*)$. □

Corollary 3.3. *Is σ is a complete mapping on $Q(\cdot)$ such that $\alpha\sigma\alpha^{-1} = \rho$ for some automorphism α of $Q(\cdot)$, then prolongations $Q'_{\sigma,a}(\circ)$ and $Q'_{\rho,b}(*)$, where $b = \alpha(a)$, are isomorphic.* □

Example 3.4. Consider the group $Q(\cdot)$ isomorphic to the group $\mathbb{Z}_5$.

$\cdot$	1	2	3	4	5
1	1	2	3	4	5
2	2	3	4	5	1
3	3	4	5	1	2
4	4	5	1	2	3
5	5	1	2	3	4

This group has the following tracks: $\varphi_1 = (1.25.34.)$, $\varphi_2 = (12.35.4.)$, $\varphi_3 = (13.45.2.)$, $\varphi_4 = (14.23.5.)$, $\varphi_5 = (15.24.3.)$.

The permutation $\sigma = (1.2453.)$ is a complete mapping of this quasigroup, $\alpha = (1.2354.)$ is its automorphism such that $\alpha\sigma\alpha^{-1} = \sigma$. Since $\bar{\sigma} = (1.25.34.) = \bar{\sigma}^{-1}$, for $a = 1$ the formula (15) gives the prolongation $Q'_{\sigma,1}(\circ)$, where:

$\circ$	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	3	4	6	1	5
3	3	6	5	1	2	4
4	4	5	1	2	6	3
5	5	1	6	3	4	2
6	6	4	2	5	3	1

$\psi_1 = \varphi_1 = (1.25.34.6.)$,
$\psi_2 = \varphi_2\sigma_2 = (12.356.4.)$,
$\psi_3 = \varphi_3\sigma_3 = (13.465.2.)$,
$\psi_4 = \varphi_4\sigma_4 = (14.236.5.)$,
$\psi_5 = \varphi_5\sigma_5 = (15.264.3.)$,
$\psi_6 = \sigma\sigma_1 = (16.2453.)$.

For $a = 2$ and the same σ we obtain the second prolongation $Q'_{\sigma,2}(*)$:

$*$	1	2	3	4	5	6
1	6	2	3	4	5	1
2	2	3	4	6	1	5
3	3	6	5	1	2	4
4	4	5	1	2	6	3
5	5	1	2	3	4	6
6	1	4	6	5	3	2

$$\begin{aligned}
\omega_1 &= \varphi_1\sigma_1 = (16.25.34.),\\
\omega_2 &= \ \sigma_2 \ = (12.35.4.6.),\\
\omega_3 &= \varphi_3\sigma_3 = (13.465.2.),\\
\omega_4 &= \varphi_4\sigma_4 = (14.236.5.),\\
\omega_5 &= \varphi_5\sigma_5 = (15.264.3.),\\
\omega_6 &= \sigma\sigma_2 = (1.24563.).
\end{aligned}$$

From Theorem 2.5 in [3] it follows that these two prolongations are not isotopic, as $\alpha(1) \neq 2$.

Observe by the way, that for the automorphism α we have $\alpha(2) = 3$, $\alpha(3) = 5$, $\alpha(5) = 4$, $\alpha(4) = 2$, which, by Corollary 3.3, means that for this quasigroup $Q'_{\sigma,2} \cong Q'_{\sigma,3} \cong Q'_{\sigma,5} \cong Q'_{\sigma,4}$. The isomorphism $Q'_{\sigma,k} \to Q'_{\sigma,\alpha(k)}$ for $k = 2, 3, 4$, coincides with α. □

4. Prolongations by quasicomplete mappings

The method of prolongations of quasigroups having at least one quasicomplete mapping was proposed in [4].

Let $Q(\cdot)$ be an arbitrary finite quasigroup with a quasicomplete mapping σ. Then $def(\sigma) = d$ and for some $a \in Q$ there are two different elements a_1, a_2 of Q such as $\bar\sigma(a_1) = \bar\sigma(a_2) = a$. Choosing one of these elements (for example a_1) we obtain the prolongation $Q'_{a_1}(\circ)$ of $Q(\cdot)$ defined by:

$$x \circ y = \begin{cases} x \cdot y & x, y \in Q,\ y \neq \sigma(x), \\ q & x \in Q - \{a_1\},\ y = \sigma(x), \\ a & x = a_1,\ y = \sigma(x), \\ \bar\sigma(x) & x \in Q - \{a_1, a_2\},\ y = q, \\ \bar\sigma\sigma^{-1}(y) & y = q,\ y \neq \sigma(a_1), y \neq \sigma(a_2), \\ q & x = a_1,\ y = q \ \text{ or } \ x = q,\ y = \sigma(a_1), \\ a & x = a_2,\ y = q \ \text{ or } \ x = q,\ y = \sigma(a_2), \\ d & x = y = q. \end{cases} \tag{17}$$

Replacing a_1 by a_2 we obtain the second prolongation of $Q(\cdot)$ which may not be isotopic to the first.

This construction is a generalization of the previous constructions. Prolongations of $Q(\cdot)$ obtained by these three constructions are not isotopic in general (for details see [4]).

Theorem 4.1. *Let $Q(\cdot) = \{\varphi_1, \varphi_2, \dots, \varphi_n\}$ be a quasigroup with a quasicomplete mapping σ. The quasigroup $Q'(\circ) = \{\psi_1, \psi_2, \dots, \psi_n, \psi_q\}$ coincides with the prolongation of $Q(\cdot)$ obtained by the formula* (17) *if and only if*

$$\begin{cases} \psi_i = \varphi_i \sigma_i & \text{for} \quad i \neq a, q, d, \\ \psi_d = \varphi_d \varepsilon, & \\ \psi_a = \varphi_a \sigma_{a_1} & \text{or} \quad \varphi_a \sigma_{a_2}, \\ \psi_q = \sigma \sigma_{a_2} & \text{or} \quad \sigma \sigma_{a_1}, \end{cases} \tag{18}$$

where $\sigma_i = \left(q, \bar{\sigma}^{-1}(i)\right)$ for $i \notin \{d, a_1, a_2\}$, $\sigma_d = \varepsilon$, $\sigma_{a_1} = (q, a_1)$ and $\sigma_{a_2} = (q, a_2)$.

Proof. The proof of this theorem is similar to the proof of Theorem 2.1. □

Theorem 4.2. *Let $T = (\alpha, \beta, \gamma)$ be an autotopism of a quasigroup $Q(\cdot)$ with a quasicomplete mapping σ. If $\beta\sigma\alpha^{-1} = \sigma$, $\bar{\sigma}(a_1) = \bar{\sigma}(a_2) = a$ and $\gamma(a) = a$, then prolongations $Q'_{a_1}(\circ)$ and $Q'_{a_2}(*)$ induced by σ are isotopic.*

Proof. The proof of this theorem is similar to the proof of Theorem 3.2. □

Example 4.3. Consider the quasigroup $Q(\cdot)$, where

$\cdot$	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	3	1	5	6	4
3	3	1	2	6	4	5
4	4	6	5	1	3	2
5	5	4	6	2	1	3
6	6	5	4	3	2	1

$$\varphi_1 = (1.23.4.5.6.),$$
$$\varphi_2 = (12.3.465.),$$
$$\varphi_3 = (13.2.456.),$$
$$\varphi_4 = (14.2635.),$$
$$\varphi_5 = (15.2436.),$$
$$\varphi_6 = (16.2534.).$$

The permutation $\sigma = (1425.3.6)$ is a quasicomplete mapping of this quasigroup,

$$\bar{\sigma} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 6 & 2 & 6 & 5 & 1 \end{pmatrix}, \qquad \bar{\sigma}^{-1} = \begin{pmatrix} 1 & 2 & 4 & 5 & 6 & 6 \\ 6 & 3 & 1 & 5 & 2 & 4 \end{pmatrix},$$

$def(\sigma) = 3$, $a = 6$, $a_1 = 2$, $a_2 = 4$, and $\sigma_1 = (7, 6)$, $\sigma_2 = (7, 3)$, $\sigma_3 = \varepsilon$, $\sigma_4 = (7, 1)$, $\sigma_5 = (7, 5)$, $\sigma_{a_1} = (7, 2)$, $\sigma_{a_2} = (7, 4)$.

Then $Q'_{a_1}(\circ) = \{\psi_1, \psi_2, \dots, \psi_7\}$ and $Q'_{a_2}(*) = \{\omega_1, \omega_2, \dots, \omega_7\}$, where

$$\begin{cases} \psi_i = \varphi_i \sigma_i \ \text{ for } i = 1, 2, 4, 5, \\ \psi_3 = \varphi_3 \varepsilon, \\ \psi_6 = \varphi_6 \sigma_{a_1}, \\ \psi_7 = \sigma \sigma_{a_2}, \end{cases} \qquad \begin{cases} \omega_i = \varphi_i \sigma_i \ \text{ for } i = 1, 2, 4, 5, \\ \psi_3 = \varphi_3 \varepsilon, \\ \omega_6 = \varphi_6 \sigma_{a_2}, \\ \omega_7 = \sigma \sigma_{a_1}. \end{cases}$$

It is not difficult to verity that $T = (\alpha, \beta, \gamma)$, where $\alpha = (15.24.36.)$, $\beta = (14.25.36.)$ and $\gamma = (12.45.3.6.)$ is an autotopism of a quasigroup $Q(\cdot)$. For this autotopism we have $\beta\sigma\alpha^{-1} = \sigma$, $\gamma(6) = 6$, which means that prolongations $Q'_{a_1}(\circ)$ and $Q'_{a_2}(*)$ are isotopic. This isotopism has the form $T = (\alpha, \beta, \gamma)$, where $\alpha = (15.24.36.7.)$, $\beta = (14.25.36.7.)$ and $\gamma = (12.45.3.6.7.)$. □

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