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Vague Lie subalgebras over a vague field

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Abstract. The concept of a vague subfield and some of its fundamental properties are introduced. We then introduce the vague Lie subalgebra over a vague field and present some of its properties. In particular, different methods of constructions of such vague sets are given.

1. Introduction

The concept of fuzzy set was first initiated by Zadeh [14] in 1965 and since then, fuzzy set has become an important tool in studying scientific subjects, in particular, it can be applied in a wide variety of disciplines such as Computer Science, Medical Science, Management Science, Social Science, Engineering and so on. In fact, if we let U be a universe of discourse, then a fuzzy set A is a class of objects of U along with a membership function A. The grade of membership of $x(x \in U)$ in the universe U is 1, but the grade of membership of x in a fuzzy subset A (of U) is a real number in [0, 1] denoted by $\mu_A(x)$ which signifies that x is a member of the fuzzy set A up to certain extent. The degree of membership could be zero or more and at most one. The greater $\mu_A(x)$ means the greater is the truth of the statement that the element x belongs to the set A.

Different authors from time to time have made a number of generalizations of Zadeh fuzzy set theory [14]. Recently, the notion of Vague Set (VS) was introduced by Gau and Buehrer in [10]. This is because in most cases of judgments, the evaluation is done by human beings and so the certainty is a limitation of knowledge or intellectual functionaries. Naturally, every decision-maker hesitates more or less on every evaluation activity. For example, in order to judge whether a patient has cancer or not, a medical doctor (the decision-maker) will hesitate because of the fact that a fraction of evaluation he thinks in favor of the truthness, another fraction in favor

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of the falseness and the rest part remains undecided to him. This is the breaking philosophy in the notion of vague set theory introduced by Gau and Buehrer in [10]. The notions of fuzzy ideals and fuzzy subalgebras of Lie algebras over a field were considered in [13] by Yehia. In this paper, we first introduce the concept of a vague subfield and study some fundamental properties. Then we introduce the notion of a vague Lie subalgebra over a vague field and present some properties. Finally, we give some important properties of a vague Lie subalgebra over a vague field of different types and describe some methods of constructions for such vague sets. The definitions and terminologies that we used in this paper are standard. For other notations, terminologies and applications, the readers are refereed to [1, 3, 4, 6, 7, 10, 11].

2. Preliminaries

Throughout this paper, L is a Lie algebra and X is a field. It is clear that the multiplication of a Lie algebra is not necessary associative, that is, $[[x,y],z]=[x,[y,z]]$ does not hold in general, however it is *anti- commutative*, that is, $[x,y]=-[y,x]$.

Let μ be a *fuzzy set* on L, that is, a map $\mu : L \to [0,1]$.

Definition 2.1. [12] A *fuzzy set* F of X is called a *fuzzy field* if

(1) $(\forall\, m,n \in X)(F(m-n) \geqslant \min\{F(m),F(n)\})$,

(2) $(\forall\, m,n \in X, n \neq 0)(F(mn^{-1}) \geqslant \min\{F(m),F(n)\})$.

Definition 2.2. [10] A *vague set* (in short, VS) A in the universe L is a pair (t_A, f_A), where $t_A : L \to [0,1]$, $f_A : L \to [0,1]$ are true and false memberships, respectively such that $t_A(x)+f_A(x) \leqslant 1$ for all $x \in L$. The interval $[t_A(x), 1-f_A(x)]$ is called the *vague value* of x in A, and is denoted by $V_A(x)$.

Definition 2.3. [10] Let $A=(t_A,f_A)$ and $B=(t_B,f_B)$ be two vague sets. Then we define:

(3) $\overline{A}=(f_A, 1-t_A)$,

(4) $A \subset B \Leftrightarrow V_A(x) \leqslant V_B(x)$, i.e., $t_A(x) \leqslant t_B(x)$ and $1-f_A(x) \leqslant 1-f_B(x)$,

(5) $A=B \Leftrightarrow V_A(x)=V_B(x)$,

(6) $C = A \cap B \Leftrightarrow V_C(x) = \min(V_A(x), V_B(x))$,

(7) $C = A \cup B \Leftrightarrow V_C(x) = \max(V_A(x), V_B(x))$

for all $x \in L$.

Definition 2.4. [10] A vague set $A = (t_A, f_A)$ of a set L is called

(8) the *zero vague set* if $t_A(x) = 0$ and $f_A(x) = 1$ for all $x \in L$,

(9) the *unit vague set* if $t_A(x) = 1$ and $f_A(x) = 0$ for all $x \in L$,

(10) the *α-vague set* if $t_A(x) = \alpha$ and $f_A(x) = 1 - \alpha$ for all $x \in L$, $\alpha \in (0, 1)$.

We also denote the zero vague and the unit vague value by intervals $\mathbf{0} = [0, 0]$ and $\mathbf{1} = [1, 1]$, respectively.

For α, $\beta \in [0, 1]$, we define the $(\alpha, \beta)-$cut and the α-cut of a vague set.

Definition 2.5. [6] Let $A = (t_A, f_A)$ be vague set of a universe L. Then the $(\alpha, \beta)-$ cut of a vague set A is a crisp set $A_{(\alpha,\beta)}$ of L given by

$$A_{(\alpha,\beta)} = \{x \in L : V_A(x) \geqslant [\alpha, \beta]\}.$$

Obviously, $A_{(0,0)} = L$. The (α, β)-cuts are also the vague-cuts of the vague set A. The α-cut of the vague set $A = (t_A, f_A)$ is a crisp set A_α of L given by $A_\alpha = A_{(\alpha,\alpha)}$. Note that $A_0 = L$. Clearly, $A_\alpha = \{x \in L : t_A(x) \geqslant \alpha\}$.

By an *interval number* D, we mean an interval $[a^-, a^+]$ with $0 \leqslant a^- \leqslant a^+ \leqslant 1$. The set of all interval numbers is denoted by $D[0, 1]$. The interval $[a, a]$ is identified with the fuzzy number $a \in [0, 1]$.

For any two interval numbers $D_1 = [a_1^-, b_1^+]$ and $D_2 = [a_2^-, b_2^+]$, we define

$$\min(D_1, D_2) = \min([a_1^-, b_1^+], [a_2^-, b_2^+]) = [\min\{a_1^-, a_2^-\}, \min\{b_1^+, b_2^+\}],$$

$$\max(D_1, D_2) = \max([a_1^-, b_1^+], [a_2^-, b_2^+]) = [\max\{a_1^-, a_2^-\}, \max\{b_1^+, b_2^+\}],$$

and put

- $D_1 \leqslant D_2 \Longleftrightarrow a_1^- \leqslant a_2^-$ and $b_1^+ \leqslant b_2^+$,
- $D_1 = D_2 \Longleftrightarrow a_1^- = a_2^-$ and $b_1^+ = b_2^+$,
- $D_1 < D_2 \Longleftrightarrow D_1 \leqslant D_2$ and $D_1 \neq D_2$,
- $mD = m[a_1^-, b_1^+] = [ma_1^-, mb_1^+]$, where $0 \leqslant m \leqslant 1$.

It can be easily verified that $(D[0,1], \leqslant, \vee, \wedge)$ forms a complete lattice under the set inclusion with $[0,0]$ as its least element and $[1,1]$ as its greatest element.

3. Vague fields

Definition 3.1. A *vague set* $F = (t_F, f_F)$ of X is said to be a *vague subfield* of the field X if the following conditions are satisfied:

(11) $(\forall\, m, n \in X)(V_F(m-n) \geqslant \min\{V_F(m), V_F(n)\})$,

(12) $(\forall\, m, n \in X, n \neq 0)(V_F(mn^{-1}) \geqslant \min\{V_F(m), V_F(n)\})$,

that is,

$$(13)\quad \begin{cases} t_A(m-n) \geqslant \min\{t_A(m), t_A(n)\}, \\ 1 - f_A(m-n) \geqslant \min\{1 - f_A(m), 1 - f_A(n)\}, \end{cases}$$

$$(14)\quad \begin{cases} t_A(mn^{-1}) \geqslant \min\{t_F(m), t_A(n)\}, \\ 1 - f_A(mn^{-1}) \geqslant \min\{1 - f_F(m), 1 - f_A(n)\}, \end{cases}$$

Example 3.2. Consider a field $X = \{0, 1, w, w^2\}$, where $w = \frac{-1+\sqrt{-3}}{2}$, with the following Cayley tables:

$+$	0	1	w	w^2
0	0	1	w	w^2
1	1	0	w^2	w
w	w	w^2	0	1
w^2	w^2	w	1	0

.	0	1	w	w^2
0	0	0	0	0
1	0	1	w	w^2
w	0	w	w^2	1
w^2	0	w^2	1	w

It can be easily seen that the vague set

$$\{(0, [0.3,\ 0.2]), (1, [0.4,\ 0.5]), (w, [0.3,\ 0.6]), (w^2, [0.5,\ 0.4])\}$$

forms a vague subfield of the field X. □

The following Lemmas can be easily proved and hence we omit their proofs.

Lemma 3.3. *If $F = (t_F, f_F)$ is a vague subfield of X, then*

$$V_F(0) \geqslant V_F(1) \geqslant V_F(m) = V_F(-m) \quad \textit{for } m \in X, \quad \textit{and}$$

$$V_F(-m) = V_F(m^{-1}) \quad \textit{for } m \in X - \{0\}.$$

Lemma 3.4. *A vague set $A = (t_A, f_A)$ of X is a vague subfield of X if and only if t_A and $1 - f_A$ are fuzzy subfields.*

Proposition 3.5. *If A and B are vague subfields of X, then $A \cap B$ is a vague subfield of X.*

Proof. Let m, $n \in X$. Then we have

$$\begin{aligned} t_{A\cap B}(m-n) &= \min\{t_A(m-n), t_B(m-n)\} \\ &\geqslant \min\{\min\{t_A(m), t_A(n)\}, \min\{t_B(m), t_B(n)\}\} \\ &= \min\{\min\{t_A(m), t_B(m)\}, \min\{t_A(n), t_B(n)\}\} \\ &= \min\{t_{A\cap B}(m), t_{A\cap B}(n)\}, \end{aligned}$$

and hence, we derive that

$$\begin{aligned} t_{A\cap B}(mn^{-1}) &= \min\{t_A(mn^{-1}), t_B(mn^{-1})\} \\ &\geqslant \min\{\min\{t_A(m), t_A(n)\}, \min\{t_B(m), t_B(n)\}\} \\ &= \min\{\min\{t_A(m), t_B(m)\}, \min\{t_A(n), t_B(n)\}\} \\ &= \min\{t_{A\cap B}(m), t_{A\cap B}(n)\}, \end{aligned}$$

$$\begin{aligned} 1-f_{A\cap B}(m-n) &= \min\{1-f_A(m-n), 1-f_B(m-n)\} \\ &\geqslant \min\{\min\{1-f_A(m), 1-f_A(n)\}, \min\{1-f_B(m), 1-f_B(n)\}\} \\ &= \min\{\min\{1-f_A(m), 1-f_B(m)\}, \min\{1-f_A(n), 1-f_B(n)\}\} \\ &= \min\{1-f_{A\cap B}(m), 1-f_{A\cap B}(n)\}, \end{aligned}$$

$$\begin{aligned} 1-f_{A\cap B}(mn^{-1}) &= \min\{1-f_A(mn^{-1}), 1-f_B(mn^{-1})\} \\ &\geqslant \min\{\min\{1-f_A(m), 1-f_A(n)\}, \min\{1-f_B(m), 1-f_B(n)\}\} \\ &= \min\{\min\{1-f_A(m), 1-f_B(m)\}, \min\{1-f_A(n), 1-f_B(n)\}\} \\ &= \min\{1-f_{A\cap B}(m), 1-f_{A\cap B}(n)\}. \end{aligned}$$

Therefore, we have proved that $A \cap B$ is indeed a vague subfield of X. □

Proposition 3.6. *The zero vague set, unit vague set and α- vague set are all vague subfields of X.*

Proof. Let $A = (t_A,\ f_A)$ be a vague subfield of X. For $m, n \in X$, we have

$$t_A(m-n) \geqslant \min\{t_A(m), t_A(n)\} = \min\{\alpha, \alpha\} = \alpha,$$

$$1-f_A(m-n) \geqslant \min\{1-f_A(m), 1-f_A(n)\} = \min\{\alpha, \alpha\} = \alpha,$$

$$t_A(mn^{-1}) \geqslant \min\{t_A(m), t_A(n)\} = \min\{\alpha, \alpha\} = \alpha,$$
$$1 - f_A(mn^{-1}) \geqslant \min\{1 - t_A(m), 1 - t_A(n)\} = \min\{\alpha, \alpha\} = \alpha.$$

This shows that α-vague set of X is a vague subfield of X. The proofs for the other cases are similar. □

Proposition 3.7. *Let A be a vague subfield of X. Then for $\alpha \in [0,1]$, the vague-cut A_α is a crisp subfield of X.*

Proof. Suppose that $A = (t_A, f_A)$ is a vague subfield of X. For $m, n \in A_\alpha$ we can deduce that

$$t_A(m) \geqslant \alpha, \quad 1 - f_A(m) \geqslant \alpha, \quad t_A(n) \geqslant \alpha, \quad 1 - f_A(n) \geqslant \alpha,$$

so that

$$t_A(m - n) \geqslant \min\{t_A(m), t_A(n)\} \geqslant \min\{\alpha, \alpha\} = \alpha,$$
$$1 - f_A(m - n) \geqslant \min\{1 - f_A(m), 1 - f_A(n)\} \geqslant \min\{\alpha, \alpha\} = \alpha,$$
$$t_A(mn^{-1}) \geqslant \min\{t_A(m), t_A(n)\} \geqslant \min\{\alpha, \alpha\} = \alpha,$$
$$1 - f_A(mn^{-1}) \geqslant \min\{1 - t_A(m), 1 - t_A(n)\} \geqslant \min\{\alpha, \alpha\} = \alpha.$$

This implies that $m-n$, $mn^{-1} \in A_\alpha$. Hence A_α is a crisp subfield of X. □

Proposition 3.8. *Any subfield K of X is a vague-cut subfield of some vague subfield of X.*

Proof. Consider the vague set A of X given by

$$V_A(m) = \begin{cases} [t,\, t] & \text{if } m \in K, \\ [0, 0] & \text{if } m \notin K, \end{cases}$$

where $t \in (0,\, 1)$. Clearly, $A_{(\alpha,\alpha)} = K$. Let $m, n, p \in X$. We now consider the following cases:

Case (i): If $m, n, p \in K$, $p \neq 0$, then $m - n, mp^{-1} \in K$ and

$$V_F(m - n) \geqslant \min\{V_F(m), V_F(n)\} = [t,\, t],$$
$$V_F(mp^{-1}) \geqslant \min\{V_F(m), V_F(p)\} = [t,\, t].$$

Case (ii): If $m, n, p \notin K$, $p \neq 0$, then $V_A(m) = [0, 0] = V_A(n) = V(p)$, and

$$V_F(m - n) \geqslant \min\{V_F(m), V_F(n)\} = [0,\, 0],$$
$$V_F(mp^{-1}) \geqslant \min\{V_F(m), V_F(p)\} = [0,\, 0].$$

Case (iii): If $m \in K$ and $n, p \notin K$, $p \neq 0$, then $V_F(m) = [t, t]$, $V_F(n) = [0, 0] = V_F(p)$, so

$$V_F(m-n) \geqslant \min\{V_F(m), V_F(n)\} = [0, 0],$$
$$V_F(mp^{-1}) \geqslant \min\{V_F(m), V_F(p)\} = [0, 0].$$

Case (iv): If $m \notin K$ and $n, p \in K$, $p \neq 0$, then by using the same argument as in Case 3, we conclude the results. Hence, we have proved that K is a vague field of X. □

Proposition 3.9. *Let K be a vague set of X which is defined by*

$$V_K(m) = \begin{cases} [s, s] & \text{if } m \in K \\ [t, t] & \text{otherwise} \end{cases}$$

for all s, $t \in [0, 1]$ with $s \geqslant t$. Then K is a vague subfield of X if and only if K is a (crisp) subfield of X.

Proof. Let K be a vague subfield of X. If $m, n, p \in K$, $p \neq 0$, then

$$V_K(m-n) \geqslant \min\{V_K(m), V_K(n)\} = \min\{[s, s], [s, s]\} = [s, s],$$

$$V_K(mp^{-1}) \geqslant \min\{V_K(m), V_K(p)\} = \min\{[s, s], [s, s]\} = [s, s],$$

and so $m - n$, $mp^{-1} \in K$.

Conversely, suppose that K is a (crisp) subfield of X. We consider the following situations:

(i) If $m, n, p \in K$, $p \neq 0$, then $m - n, mp^{-1} \in K$. Thus

$$V_K(m-n) \geqslant [s, s] = \min\{V_K(m), V_K(n)\},$$

$$V_K(mp^{-1}) \geqslant [s, s] = \min\{V_K(m), V_K(p)\}.$$

(ii) If $m \notin K$ or $n, p \notin K$, $p \neq 0$, then

$$V_K(m-n) \geqslant [t, t] = \min\{V_K(m), V_K(n)\},$$

$$V_K(mp^{-1}) \geqslant [t, t] = \min\{V_K(m), V_K(p)\}.$$

This shows that K is a vague subfield of X. □

4. Vague Lie subalgebras over a vague field

Definition 4.1. A *vague set* $A = (t_A,\, f_A)$ of L is called a *vague Lie subalgebra over a vague field* $F = (t_F, f_F)$ (briefly, *vague Lie* $\mathbb{F}$*-subalgebra*) of L if the following conditions are satisfied

(a) $V_A(x+y) \geqslant \min\{V_A(x), V_A(y)\}$,

(b) $V_A(mx) \geqslant \min\{V_F(m), V_A(x)\}$,

(c) $V_A([x,y]) \geqslant \min\{V_A(x), V_A(y)\}$

for all $x, y \in L$ and $m \in X$.

In other words,

(d) $\begin{cases} t_A(x+y) \geqslant \min\{t_A(x), t_A(y)\}, \\ 1 - f_A(x+y) \geqslant \min\{1 - f_A(x), 1 - f_A(y)\}, \end{cases}$

(e) $\begin{cases} t_A(mx) \geqslant \min\{t_F(m), t_A(x)\}, \\ 1 - f_A(mx) \geqslant \min\{1 - f_F(m), 1 - f_A(x)\}, \end{cases}$

(f) $\begin{cases} t_A([x,y]) \geqslant \min\{t_A(x), t_A(y)\}, \\ 1 - f_A([x,y]) \geqslant \min\{1 - f_A(x), 1 - f_A(y)\}. \end{cases}$

From (b), it follows that $V_A(0) \geqslant V_F(0)$.

Example 4.2. Let $\Re^2 = \{(x,y) : x, y \in R\}$ be the set of all 2-dimensional real vectors. Then $\Re^2$ with $[x,y] = x \times y$ form a real Lie algebra. Define a vague set $A = (t_A, f_A) : \Re^2 \to [0,1]$ by

$$t_A(x,y) = \begin{cases} 0.4 & \text{if } x = y = 0, \\ 0.3 & \text{otherwise}, \end{cases} \qquad f_A(x,y) = \begin{cases} 0.3 & \text{if } x = y = 0, \\ 0.4 & \text{otherwise}, \end{cases}$$

and define $F = (t_F, f_F) : \mathbb{R} \to [0,1]$ for all $m \in \mathbb{R}$ by

$$t_F(m) = \begin{cases} 0.3 & \text{if } m \in \mathbb{Q}, \\ 0.2 & \text{if } 0\ m \in \mathbb{R} - \mathbb{Q}(\sqrt{3}), \end{cases}$$

$$f_F(m) = \begin{cases} 0.2 & \text{if } m \in \mathbb{Q}, \\ 0.4 & \text{if } 0\ m \in \mathbb{R} - \mathbb{Q}(\sqrt{3}). \end{cases}$$

By routine verification, we can easily check that A is a vague Lie $\mathbb{F}$-subalgebra.

The proofs of the following propositions are obvious.

Proposition 4.3. *A vague set $A = (t_A, f_A)$ of L is a vague Lie $\mathbb{F}$-subalgebra of L if and only if t_A and $1 - f_A$ are fuzzy Lie $\mathbb{F}$-subalgebras over a fuzzy field.*

Proposition 4.4. *Let $\{A_i : i \in \Lambda\}$ be a family of vague Lie $\mathbb{F}$-subalgebras of L. Then $\cap_{i\in\Lambda} A_i$ is a vague Lie $\mathbb{F}$-subalgebra of L.*

Proposition 4.5. *The zero vague set, unit vague set and α-vague set are vague Lie $\mathbb{F}$-subalgebras of L.*

Theorem 4.6. *Let A be a vague Lie $\mathbb{F}$-subalgebra of L. Then for any α, β $\in [0, 1]$, the vague-cut $A_{(\alpha,\beta)}$ is a crisp Lie subalgebra of L.*

Proof. Suppose that $A = (t_A, f_A)$ is a vague Lie subalgebra of L over a vague field $F = (t_F, f_F)$. Let $x, y, m \in A_{(\alpha,\beta)}$, $x, y \in L$, $m \in X$. Then

$$t_A(x) \geqslant \alpha, \quad 1 - f_A(x) \geqslant \beta, \quad t_A(y) \geqslant \alpha, \quad 1 - f_A(y) \geqslant \beta, \quad t_F(m) \geqslant \alpha$$

and $1 - f_F(m) \geqslant \beta$.

From Definition 4.1, it follows that

$$\begin{aligned}
t_A(x+y) &\geqslant \min\{t_A(x), t_A(y)\} \geqslant \min\{\alpha, \alpha\} = \alpha,\\
1 - f_A(x+y) &\geqslant \min\{1 - f_A(x), 1 - f_A(y)\} \geqslant \min\{\beta, \beta\} = \beta,\\
t_A(mx) &\geqslant \min\{t_F(m), t_A(x)\} \geqslant \min\{\alpha, \alpha\} = \alpha,\\
1 - f_A(mx) &\geqslant \min\{1 - t_F(m), 1 - t_A(x)\} \geqslant \min\{\beta, \beta\} = \beta,\\
t_A([x, y]) &\geqslant \min\{t_A(x), t_A(y)\} \geqslant \min\{\alpha, \alpha\} = \alpha,\\
1 - f_A([x, y]) &\geqslant \min\{1 - f_A(x), 1 - f_A(y)\} \geqslant \min\{\beta, \beta\} = \beta.
\end{aligned}$$

This implies that $x + y$, mx, $[x, y] \in A_{(\alpha,\beta)}$. Hence $A_{(\alpha,\beta)}$ is a crisp Lie subalgebra of L. □

Corollary 4.7. *Let A be a vague Lie $\mathbb{F}$-subalgebra of L. Then for $\alpha \in [0, 1]$, the vague-cut A_α is a crisp Lie subalgebra of L.*

The proofs of the following propositions are obvious.

Proposition 4.8.

(i) *Let $f : L_1 \to L_2$ be an onto homomorphism of Lie algebras. If $B = (t_B, f_B)$ is a vague Lie $\mathbb{F}$-subalgebra of L_2, then the preimage $f^{-1}(B)$ of B under f is a vague Lie $\mathbb{F}$-subalgebra L_1.*

(ii) *Let $f : L_1 \to L_2$ be an epimorphism of Lie algebras. If $A = (t_A, f_A)$ is a vague Lie $\mathbb{F}$-subalgebra of L_2, then $f^{-1}(A^c) = (f^{-1}(A))^c$.*

(iii) *Let $f : L_1 \to L_2$ be an epimorphism of Lie algebras. If $A = (t_A, f_A)$ is a vague Lie $\mathbb{F}$-subalgebra of L_2 and $B = (t_B, f_B)$ is the preimage of $A = (\mu_A, \lambda_A)$ under f. Then $B = (t_B, f_B)$ is a vague Lie $\mathbb{F}$-subalgebra of L_1.*

Definition 4.9. Let $g : L_1 \to L_2$ be a homomorphism of Lie algebras. For any vague fuzzy set $A = (t_A, f_A)$ in a Lie algebra L_2, we define a vague fuzzy set $A^g = (t_A^g, f_A^g)$ in L by

$$t_A^g(x) = t_A(g(x)), \quad f_A^g(x) = f_A(g(x))$$

for all $x \in L_1$. Clearly, $A^g(x_1) = A^g(x_2) = A(x)$ for all $x_1, x_2 \in g^{-1}(x)$.

Lemma 4.10. *Let $g : L_1 \to L_2$ be a homomorphism of Lie algebras. If $A = (t_A, f_A)$ is a vague Lie $\mathbb{F}$-subalgebra of L_2, then A^g is a vague Lie $\mathbb{F}$-subalgebra of L_1.*

Proof. Let $x, y \in L_1$ and $m \in X$. Then

$$\begin{aligned} t_A^g(x+y) &= t_A(g(x+y)) = t_A(g(x)+g(y)) \\ &\geqslant \min\{t_A(g(x)), t_A(g(y))\} = \min\{t_A^g(x), t_A^g(y)\}, \end{aligned}$$

$$\begin{aligned} 1 - f_A^g(x+y) &= 1 - f_A((g(x+y)) = 1 - f_A(g(x)+g(y)) \\ &\geqslant \min\{1 - f_A(g(x)), 1 - f_A(g(y))\} \\ &= \min\{1 - f_A^g(x), 1 - f_A^g(y)\}. \end{aligned}$$

The verification of the other conditions is similar. Hence, A^g is a vague Lie $\mathbb{F}$-subalgebra of L_1. □

Theorem 4.11. *Let $g : L_1 \to L_2$ be an epimorphism of Lie algebras. Then A^g is a vague Lie $\mathbb{F}$-subalgebra of L_1 if and only if A is a vague Lie $\mathbb{F}$-subalgebra of L_2.*

Proof. The sufficiency follows from Lemma 4.10. In proving the necessity, we first recall that g is a surjective mapping. Hence for any $x, y \in L_2$, there exist x_1, $y_1 \in L_1$ such that $x = g(x_1)$, $y = g(y_1)$. Thus $t_A(x) = t_A^g(x_1)$, $t_A(y) = t_A^g(y_1)$, $1 - f_A(x) = 1 - f_A^g(x_1)$, $1 - f_A(y) = 1 - f_A^g(y_1)$, whence

$$\begin{aligned} t_A(x+y) &= t_A(g(x_1)+g(y_1)) = t_A(g(x_1+y_1)) \\ &= t_A^g(x_1+y_1) \geqslant \min\{t_A^g(x_1), t_A^g(y_1)\} = \min\{t_A(x), t_A(y)\}, \end{aligned}$$

$$\begin{aligned} 1 - f_A(x+y) &= 1 - f_A(g(x_1)+g(y_1)) = 1 - f_A(g(x_1+y_1)) \\ &= 1 - f_A^g(x_1+y_1) \geqslant \min\{1 - f_A^g(x_1), 1 - f_A^g(y_1)\} \\ &= \min\{1 - f_A(x), 1 - f_A(y)\}. \end{aligned}$$

The verification of the other conditions is similar. This proves that $A = (t_A, f_A)$ is a vague Lie $\mathbb{F}$-subalgebra of L_2. □

5. Special types of vague Lie subalgebras

Definition 5.1. Let $A = (t_A, f_A)$ be a vague Lie $\mathbb{F}$-subalgebra in L. Define inductively a sequence of vague Lie $\mathbb{F}$-subalgebras in L by Lie brackets

$$A^0 = A, \quad A^1 = [A^0, A^0], \quad A^2 = [A^1, A^1], \ \ldots, \ A^n = [A^{n-1}, A^{n-1}].$$

Then, A^n is said to be the *nth derived vague Lie* $\mathbb{F}$*-subalgebra* of L. Moreover, a series

$$A^0 \supseteq A^1 \supseteq A^2 \supseteq \cdots \supseteq A^n \supseteq \cdots$$

is said to be a *derived series* of a vague Lie $\mathbb{F}$-subalgebra A in L. A vague Lie $\mathbb{F}$-subalgebra A in L is called a *solvable vague Lie* $\mathbb{F}$*-subalgebra* if there exists a positive integer n such that $A^n = \mathbf{0}$.

Definition 5.2. Let $A = (t_A, f_A)$ be a vague Lie $\mathbb{F}$-subalgebra in L. We define inductively a sequence of vague Lie $\mathbb{F}$-subalgebras in L by Lie brackets

$$A_0 = A, \quad A_1 = [A, A_0], \quad A_2 = [A, A_1], \ \ldots, \ A_n = [A, A_{n-1}].$$

Then we call the series

$$A_0 \supseteq A_1 \supseteq A_2 \supseteq \cdots \supseteq A_n \supseteq \cdots$$

the *descending central series* of a vague Lie $\mathbb{F}$-subalgebra A in L. An vague vague Lie $\mathbb{F}$-subalgebra A in L is called a *nilpotent vague Lie* $\mathbb{F}$*-subalgebra* if there exists a positive integer n such that $A_n = \mathbf{0}$.

By using similar arguments as in the proof of Theorem 4.7 in [2], we obtain the following theorem.

Theorem 5.3.

(I) *The homomorphic image of a solvable vague Lie* $\mathbb{F}$*-subalgebra is a solvable vague Lie* $\mathbb{F}$*-subalgebra.*

(II) *The homomorphic image of a nilpotent vague Lie $\mathbb{F}$-subalgebra is a nilpotent vague Lie $\mathbb{F}$-subalgebra.*

(III) *If A is a nilpotent vague Lie $\mathbb{F}$-subalgebra, then it is solvable.*

Definition 5.4. A vague Lie $\mathbb{F}$-subalgebra $A = (t_A, f_A)$ of a Lie algebra L is said to be *normal* if there exists an element $x_0 \in L$ such that $V_A(x_0) = \mathbf{1}$, i.e., $t_A(x_0) = 1$ and $f_A(x_0) = 0$.

The following Lemma is easy to prove and we hence omit the proof.

Lemma 5.5. *Let $A = (t_A, f_A)$ be a vague Lie $\mathbb{F}$-subalgebra of L such that $t_A(x) + f_A(x) \leqslant t_A(0) + f_A(0)$ for all $x \in L$. Define $A^+ = (t_A^+, f_A^+)$, where $t_A^+(x) = t_A(x) + 1 - t_A(0)$, $f_A^+(x) = f_A(x) - f_A(0)$ for all $x \in L$. Then A^+ is normal vague set.*

By using the above lemma, we deduce the following theorem.

Theorem 5.6. *Let $A = (t_A, f_A)$ be a vague Lie $\mathbb{F}$-subalgebra of a Lie algebra L. Then the vague set A^+ is a normal vague Lie $\mathbb{F}$-subalgebra of L containing A.*

Proof. Let $x, y \in L$ and $m \in X$. Then

$$\begin{aligned}\min\{V_A^+(x), V_A^+(y)\} &= \min\{V_A(x) + 1 - V_A(0), V_A(y) + 1 - V_A(0)\}\\ &= \min\{V_A(x), V_A(y)\} + 1 - V_A(0)\}\\ &\leqslant V_A(x+y) + 1 - V_A(0) = V_{A^+}(x+y),\end{aligned}$$

$$\begin{aligned}\min\{V_F^+(m), V_A^+(x)\} &= \min\{V_F(m) + 1 - V_F(0), V_A(x) + 1 - V_A(0)\}\\ &= \min\{V_F(m), V_A(x)\} + 1 - (V_F(0) + V_A(0))\}\\ &\leqslant V_A(mx) + 1 - (V_F(0) + V_A(0)) = V_{A^+}(mx),\end{aligned}$$

$$\begin{aligned}\min\{V_A^+(x), V_A^+(y)\} &= \min\{V_A(x) + 1 - V_A(0), V_A(y) + 1 - V_A(0)\}\\ &= \min\{V_A(x), V_A(y)\} + 1 - V_A(0)\}\\ &\leqslant V_A([x,y]) + 1 - V_A(0) = V_{A^+}([x,y]).\end{aligned}$$

Thus, A^+ is a normal vague Lie $\mathbb{F}$-subalgebra of L. Clearly $A \subseteq A^+$. □

The following theorems are obvious.

Theorem 5.7. *A vague Lie $\mathbb{F}$-subalgebra A of a Lie algebra L is normal if and only if $A^+ = A$.*

Theorem 5.8. *If $A = (t_A, f_A)$ is a vague Lie $\mathbb{F}$-subalgebra of a Lie algebra L, then $(A^+)^+ = A^+$.*

Corollary 5.9. *If A is normal vague Lie $\mathbb{F}$-subalgebra of a Lie algebra L, then $(A^+)^+ = A$.*

Theorem 5.10. *Let A and B be vague Lie $\mathbb{F}$-subalgebras of a Lie algebra L. Then $(A \cup B)^+ = A^+ \cup B^+$.*

Proof. Let $A = (t_A, f_A)$ and $B = (t_B, f_B)$ be two vague Lie $\mathbb{F}$-subalgebras of a Lie algebra L. Then $A \cup B = (t_{A\cup B}, f_{A\cup B})$, where

$$t_{A\cup B}(x) = \max\{t_A(x), t_B(x)\}, \quad f_{A\cup B}(x) = \min\{f_A(x), f_B(x)\}, \quad \forall x \in L.$$

Thus $(A \cup B)^+ = (t_{(A\cup B)^+}(x), f_{(A\cup B)^+}(x))$, where

$$\begin{aligned} t_{(A\cup B)^+}(x) &= t_{(A\cup B)}(x) + 1 - t_{(A\cup B)}(0) \\ &= \max\{t_A(x), t_B(x)\} + 1 - \max\{t_A(0), t_B(0)\} \\ &= \max\{t_A(x) + 1 - t_A(0), t_B(x) + 1 - t_B(0)\} \\ &= \max\{t_{A^+}(x), t_{B^+}(x)\} = t_{A^+\cup B^+}(x). \end{aligned}$$

Similarly, we can prove that $f_{(A\cup B)^+}(x) = f_{A^+\cup B^+}(x)$ for $x \in L$. Hence, $(A \cup B)^+ = A^+ \cup B^+$. □

The proof of the following theorem is obvious.

Theorem 5.11. *Let A be a vague Lie $\mathbb{F}$-subalgebra of a Lie algebra L. If there exist a vague Lie $\mathbb{F}$-subalgebra B of L satisfying $B \subset A^+$, then A is normal.*

Corollary 5.12. *Let A be a vague Lie $\mathbb{F}$-subalgebra of a Lie algebra L. If there exists a vague Lie $\mathbb{F}$-subalgebra B of L satisfying $B^+ \subset A$, then $A^+ = A$.*

Denote the family of all vague Lie $\mathbb{F}$-subalgebras of a Lie algebra L by $VLS(L)$, and the set of all normal vague Lie $\mathbb{F}$-subalgebra of L by $\mathcal{N}(L)$. It is clear that $\mathcal{N}(L)$ is a poset under set inclusion.

Theorem 5.13. *A non-constant maximal element of $(\mathcal{N}(L), \subseteq)$ takes only the values $\mathbf{0}$ and $\mathbf{1}$.*

Proof. Let $A \in \mathcal{N}(L)$ be a non-constant maximal element of $(\mathcal{N}(L), \subseteq)$. Then $t_A(x_0) = 1$ and $f_A(x_0) = 0$ for some $x_0 \in L$. Let $x \in L$ be such that $V_A(x) \neq \mathbf{1}$. We claim that $V_A(x) = \mathbf{0}$. If not, then there exists $a \in L$ such that $\mathbf{0} < V_A(a) < \mathbf{1}$. Let B be a vague set in L over vague field K defined by $V_B(x) := \frac{1}{2}\{V_A(x) + V_A(a)\}$, $V_K(x) := \frac{1}{2}\{V_F(x) + V_F(a)\}$ for all $x \in L$. For x, $y \in L$ and $m \in X$, we have

$$\begin{aligned} V_B(x+y) &= \frac{1}{2}\{V_A(x+y) + V_A(a)\} \geqslant \frac{1}{2}\{\min\{V_A(x), V_A(y)\} + V_A(a)\} \\ &= \{\min\{\frac{1}{2}(V_A(x) + V_A(a)), \frac{1}{2}(V_A(y) + V_A(a))\} \\ &= \min\{V_B(x), V_B(y)\}, \end{aligned}$$

$$\begin{aligned} V_B(mx) &= \frac{1}{2}\{V_A(mx) + V_A(a)\} \geqslant \frac{1}{2}\{\min\{V_F(m), V_A(x)\} + V_A(a)\} \\ &= \min\{\frac{1}{2}(V_F(m) + V_F(a)), \frac{1}{2}(V_A(x) + V_A(a))\} \\ &= \min\{V_K(m), V_B(x)\}, \\ V_B([x,y]) &= \frac{1}{2}\{V_A([x,y]) + V_A(a)\} \geqslant \frac{1}{2}\{\min\{V_A(x), V_A(y)\} + V_A(a)\} \\ &= \min\{\frac{1}{2}(V_A(x) + V_A(a)), \frac{1}{2}(V_A(y) + V_A(a))\} \\ &= \min\{V_B(x), V_B(y)\}. \end{aligned}$$

This proves that B is a vague Lie $\mathbb{F}$-subalgebra of L. Now we have

$$\begin{aligned} V_{B^+}(x) &= V_B(x) + 1 - V_B(0) \\ &= \frac{1}{2}\{\min\{V_A(x), V_A(a)\} + 1 - \frac{1}{2}\{\min\{V_A(0), V_A(a)\} \\ &= V_A(x) + 1, \end{aligned}$$

which implies that $V_{B^+}(0) = \frac{1}{2}\{V_A(0) + 1\} = 1$. Thus B^+ forms a normal vague Lie $\mathbb{F}$-subalgebra of L. But $V_{B^+}(0) = \mathbf{1} > V_{B^+}(a) = \frac{1}{2}\{V_A(a) + 1\} > V_A(a)$, so B^+ is a non-constant normal vague Lie $\mathbb{F}$-subalgebra of L and $V_{B^+}(a) > V_A(a)$, which is a contradiction. Hence, a non-constant maximal element of $(\mathcal{N}(L), \subseteq)$ takes only two values: $\mathbf{0}$ and $\mathbf{1}$. □

Definition 5.14. A non-constant vague Lie $\mathbb{F}$-subalgebra $A \in VLS(L)$ is called *maximal* if A^+ is a maximal element of the poset $(\mathcal{N}(L), \subseteq)$.

Theorem 5.15. *A maximal vague Lie $\mathbb{F}$-subalgebra $A \in VLS(L)$ is normal and takes only two values:* $\mathbf{0}$ *and* $\mathbf{1}$.

Proof. Let $A \in VLS(L)$ be maximal. Then A^+ is a non-constant maximal element of the poset $(\mathcal{N}(L), \subseteq)$ and, by Theorem 5.13, the possible values of $V_A^+(x)$ are $\mathbf{0}$ and $\mathbf{1}$, that is, t_A^+ takes only two values 0 and 1. Clearly,

$t_A^+(x) = 1$ if and only if $t_A(x) = t_A(0) = 0$; $t_A^+(x) = 0$ if and only if $t_A(x) = t_A(0) = 1$. But $A \subseteq A^+$ implies $t_A(x) \leqslant t_A^+(x)$ for all $x \in L$. Hence, $t_A^+(x) = 0$ implies $t_A(x) = 0$. Consequently, $V_A(0) = 1$. □

Theorem 5.16. *A level subset of a maximal $A \in VLS(L)$ is a maximal Lie subalgebra of L.*

Proof. Let S be a level subset of a maximal $A \in VLS(L)$, i.e., $S = L = \{x \in L \mid V_A(x) = 1\}$. It is not difficult to verify that S is a Lie subalgebra of L. Obviously $S \neq L$ because V_A takes only two values. Let M be a Lie subalgebra of L containing S. Then $V_S \subseteq V_M$. Since $V_A = V_S$ and V_A takes only two values, V_M also takes only these two values. But, by our assumption, $A \in VLS(L)$ is maximal so that $V_S = V_A = V_M$ or $V_M(x) = 1$, for all $x \in L$. In the last case, we have $S = L$ which is impossible. So, we must have $V_A = V_S = V_M$ which implies that $S = M$. This means that S is a maximal Lie subalgebra of L. □

Definition 5.17. A normal vague Lie $\mathbb{F}$-subalgebra $A \in VLS(L)$ is called *completely normal* if there exists $x \in L$ such that $A(x) = \mathbf{0}$. The set of all completely normal $A \in VLS(L)$ is denoted by $\mathcal{C}(L)$. Clearly, $\mathcal{C}(L) \subseteq \mathcal{N}(L)$.

Theorem 5.18. *A non-constant maximal element of $(\mathcal{N}(L), \subseteq)$ is also a maximal element of $(\mathcal{C}(L), \subseteq)$.*

Proof. Let A be a non-constant maximal element of $(\mathcal{N}(L), \subseteq)$. Then, by Theorem 5.13, A takes only the values $\mathbf{0}$ and $\mathbf{1}$ and so $V_A(x_0) = \mathbf{1}$ and $V_A(x_1) = \mathbf{0}$, for some $x_0, x_1 \in L$. Hence $A \in \mathcal{C}(L)$. Assume that there exists $B \in \mathcal{C}(R)$ such that $A \subseteq B$. Then, it follows that $A \subseteq B$ in $\mathcal{N}(L)$. Since A is maximal in $(\mathcal{N}(L), \subseteq)$ and B is non-constant, we have $A = B$. Thus A is maximal element of $(\mathcal{C}(L), \subseteq)$. This completes the proof. □

Theorem 5.19. *Every maximal $A \in VLS(L)$ is completely normal.*

Proof. Let $A \in VLS(L)$ be maximal. Then by Theorem 5.15, A is normal and $A = A^+$ takes only two values $\mathbf{0}$ and $\mathbf{1}$. Since A is non-constant, it follows that $V_A(x_0) = \mathbf{1}$ and $V_A(x_1) = \mathbf{0}$ for some $x_0, x_1 \in L$. Hence A is completely normal, ending the proof. □

In closing this paper, we state a method of construction for a new normal vague Lie $\mathbb{F}$-subalgebra from an old one.

Theorem 5.20. *Let $f : [0,1] \to [0,1]$ be an increasing function and $A = (t_A,\, f_A)$ a vague set on a Lie algebra L. Then $A_f = (t_{A_f},\, f_{A_f})$ defined by $t_{A_f}(x) = f(t_A(x))$ and $f_{A_f}(x) = f(f_A(x))$ is an vague Lie $\mathbb{F}$-subalgebra if and only if $A = (t_A,\, f_A)$ is an vague Lie $\mathbb{F}$-subalgebra. Moreover, if $f(t_A(0)) = 1$ and $f(f_A(0)) = 0$, then A_f is normal.*

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