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Quasi union hyper K-algebras

Mohammad A. Nasr-Azadani and Mohammad M. Zahedi

Abstract

We give a method of construction of a hyper K-algebra on a set of order α, where α is a fixed cardinal number. Then we introduce the notion of quasi union hyper K-algebra and prove that any quasi union hyper K-algebra is implicative and whenever $0 \circ 0 = \{0\}$, it is strong implicative hyper K-algebra. Also a quasi union hyper K-algebra is positive implicative if and only if it is a hyper BCK-algebra. Finally we prove that any hyper K-algebra $H \overset{\mathrm{C}}{=} \oplus_{i\in\Lambda} A_i$ (closed set), where $|A_i| = 2$ under some conditions is a quasi union hyper K-algebra or a quasi union hyper BCK-algebra.

1. Introduction

The study of BCK-algebra was initiated by Imai and Iséki [6] in 1966 as a generalization of the concept of set-theoretic difference and propositional calculi. The hyper structure theory (called also multi algebras) was introduced in 1934 by Marty [8] at the 8th congress of Scandinavian Mathematicians. Hyper structures have many applications to several sectors of both pure and applied sciences. Borzooei, et.al. [4, 7] applied the hyper structure to BCK-algebras and introduced the concept of hyper BCK-algebra and hyper K-algebra in which, each of them is a generalization of BCK-algebra. Borzooei and Harizavi [3] introduced a decomposition for a hyper BCK-algebra. Nasr-Azadani and Zahedi [9] study S-absorbing (P)-decomposable hyper K-algebras as a generalization of decomposition for hyper BCK-algebras. Now, we follow [9] and obtain some results as mentioned in the abstract.

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2. Preliminaries

Let H be a non-empty set, the set of all non-empty subset of H is denoted by $\mathcal{P}^*(H)$. A *hyperoperation* on H is a map $\circ : H \times H \to \mathcal{P}^*(H)$, where $(a, b) \to a \circ b$ for all $a, b \in H$. A set H, endowed with a hyperoperation, "$\circ$", is called a *hyperstructure*. If $A, B \subseteq H$, then $A \circ B = \bigcup_{a \in A, b \in B} a \circ b$.

Definition 1. [4, 7] Let H be a non-empty set containing a constant "0" and "$\circ$" be a hyperoperation on H. Then H is called a *hyper K-algebra* (*hyper BCK-algebra*) if it satisfies K1 – K5 (respectively: HK1 – HK4).

K1: $(x \circ z) \circ (y \circ z) < x \circ y$, HK1: $(x \circ z) \circ (y \circ z) \ll x \circ y$,
K2: $(x \circ y) \circ z = (x \circ z) \circ y$, HK2: $(x \circ y) \circ z = (x \circ z) \circ y$,
K3: $x < x$, HK3: $x \circ H \ll x$,
K4: $x < y,\ y < x$, then $x = y$, HK4: $x \ll y,\ y \ll x$, then $x = y$,
K5: $0 < x$

for all $x, y, z \in H$, where $x < y$ ($x \ll y$) means $0 \in x \circ y$. Moreover for any $A, B \subseteq H$, $A < B$ if $\exists\, a \in A$, $\exists\, b \in B$ such that $a < b$ and $A \ll B$ if $\forall\, a \in A$, $\exists\, b \in B$ such that $a \ll b$.

For briefly the readers could see some definitions and results about hyper K-algebra and hyper BCK-algebra in [4, 7]. In the sequel H always denotes a hyper K-algebra. If $I \subset H$, then $I' = H \setminus I$ and $I^* = I' \cup \{0\}$.

Definition 2. [5] An element $b \in H$ is called a *left* (*right*) *scalar* if $|b \circ x| = 1$ ($|x \circ b| = 1$) for all $x \in H$. An element is called *scalar* if it is a left and a right scalar.

Theorem 1. [10] *Let $(H_i, \circ_i, 0)$, $i \in \Omega$ be a family of hyper K-algebras such that $H_i \cap H_j = \{0\}$, $i \neq j \in \Omega$, 0 be a left scalar in each H_i, $i \in \Omega$, $H = \bigcup_{i \in \Omega} H$ and "$\circ$" on H is defined as follows:*

$$x \circ y := \begin{cases} x \circ_i y & \text{if } x, y \in H_i, \\ \{x\} & \text{if } x \in H_i,\ y \notin H_i. \end{cases}$$

Then $(H, \circ, 0)$ is hyper K-algebra denoted by $H = \oplus_{i \in \Omega} H_i$. □

Definition 3. [1, 2] A hyper K-algebra H is called

(i) *weak implicative* if $x < x \circ (y \circ x)$,
(ii) *implicative* if $x \in x \circ (y \circ x)$,
(iii) *strong implicative* if $x \circ 0 \subseteq x \circ (y \circ x)$,
(iv) *positive implicative* if $(x \circ y) \circ z = (x \circ z) \circ (y \circ z)$

holds for all $x, y, z \in H$.

Definition 4. [9, 4, 11] A non-empty subset I of H is said to be *closed* if $x < y$ and $y \in I$ imply $x \in I$, and it is said to be a *hyper K-ideal* of H if $x \circ y < I$ and $y \in I$ imply $x \in I$.

Theorem 2. [9] *Any hyper K-ideal of H is closed.* □

Definition 5. [9] Let I and S be non-empty subsets of H. Then we say that I is *S-absorbing* if $x \in I$ and $y \in S$ imply $x \circ y \subseteq I$. In the case $S = I'$ or $S = I^*$ we say that I is *C-absorbing* or *C^*-absorbing*, respectively.

Theorem 3. [9] *Let H be a hyper BCK-algebra and I be a hyper BCK-ideal or closed set. Then I is H-absorbing.* □

Definition 6. [9] A hyper K-algebra H is called *(P)-decomposable* if there exists a non-trivial family $\{A_i\}_{i\in\Lambda}$ of subsets of H with P-property such that $H \neq \{A_i\}$ for all $i \in \Lambda$, $H = \bigcup_{i\in\Lambda} A_i$ and $A_i \cap A_j = \{0\}$, $i \neq j$.

In this case, we write $H = \oplus_{i\in\Lambda} A_i(\mathrm{P})$ and say that $\{A_i\}_{i\in\Lambda}$ is a (P)-decomposition for H. If each A_i, $i \in \Lambda$, is S-absorbing we write $H \overset{\mathrm{S}}{=} \oplus_{i\in\Lambda} A_i(\mathrm{P})$. Moreover, we say that this decomposition is *closed union*, in short (P)-CUD, if $\cup_{i\in\Delta} A_i$ has P-property for any non-empty subset Δ of Λ. If there exists a (P)-CUD for H, then we say that H is a (P)-CUD.

Theorem 4. [9] *Let $H \overset{\mathrm{H}}{=} A \oplus B$. Then 0 is a left scalar element.* □

Theorem 5. [9] *Let $H \overset{\mathrm{C^*}}{=} \oplus_{i\in\Lambda} A_i$(hyper K-ideal). Then H is (hyper K-ideal)-CUD and $H \overset{\mathrm{C^*}}{=} I \oplus I^*$(hyper K-ideal), where $I = \cup_{i\in\Delta} A_i$ for any non-empty subset Δ of Λ.* □

Theorem 6. [10] *Let $(H, \circ, 0)$ be a hyper BCK-algebra. Then $H = \oplus_{i\in\Omega} H_i$ (hyper BCK-algebra) if and only if $H = \oplus_{i\in\Omega} H_i$(hyper BCK-ideal).* □

3. Quasi union hyper K-algebra

In this section we give a method to construct a hyper K-algebra of order α where α is a given cardinal number. Also we introduce the concept of quasi union hyper K-algebra and investigate some properties of it.

Remark 1. Let H be a set containing "0", $\mathcal{P}_0(H) = \{A \subseteq H : 0 \in A\}$ and $\mathcal{S} = \{f | f : H \to \mathcal{P}_0(H) \text{ is a function}\}$. For convenience we use F^x instead of $f(x)$ for any $f \in \mathcal{S}$. Clearly $\mathcal{S} \neq \emptyset$, because the functions $f, g : H \to \mathcal{P}_0(H)$, where $f(x) = \{0\}$ and $g(x) = \{0, x\}$ for all $x \in H$, are members of $\mathcal{S}$.

Theorem 7. *Let $H = X \cup \{0\}$, where X is a non-empty set. Then for any $f \in \mathcal{S}$ we can define the hyperoperation $\circ_f : H \times H \longrightarrow \mathcal{P}^*(H)$ by putting:*

$$x \circ_f y := \begin{cases} F^x & \text{if } x = y, \\ \{x\} & \text{otherwise.} \end{cases}$$

Moreover, the following statements are equivalent

(*i*) $(H, \circ_f, 0)$ *is a hyper K-algebra,*

(*ii*) $F^x \circ_f y = F^x$ *for all* $y \neq x, y \in H$,

(*iii*) $x \neq y$ *and* $y \in F^x$ *imply* $y \in F^y$ *and* $F^y \subseteq F^x$.

Proof. By Remark 1, $u = v$ implies $f(u) = F^u = f(v) = F^v$. This yields that "$\circ_f$" is well-defined and hence it is a hyperoperation on H.

$(i) \Rightarrow (ii)$. Let $(H, \circ_f, 0)$ be a hyper K-algebra and $y \neq x, y \in H$. Then by definition of "$\circ_f$" and K2 we have:

$$F^x \circ_f y = (x \circ_f x) \circ_f y = (x \circ_f y) \circ_f x = (x \circ_f x) = F^x.$$

$(ii) \Rightarrow (i)$. To do this, we show that H satisfies K1 – K5. Since $0 \in F^x = x \circ_f x$, hence $x < x$ for all $x \in H$ and K3 holds. Moreover by definition of $\circ_f$ we have $0 \circ_f x = \{0\}$ for all $x \neq 0$, that is $0 < x$. Thus K5 holds.

To check K1, K2 and K4, we consider the following five cases:
(I) $x \neq y$, $x \neq z$ and $y \neq z$, (II) $x = y \neq z$, (III) $x = z \neq y$,
(IV) $x \neq y = z$, (V) $x = y = z$.

K1: $(x \circ_f z) \circ_f (y \circ_f z) < x \circ_f y$.
For convenience, we put $(x \circ_f z) \circ_f (y \circ_f z) = A$ and $x \circ_f y = B$. If (I) holds, then $A = \{x\} = B$ and by K3, $A < B$. If (II) holds, then $A = F^x = B$, therefore $A < B$. If (III) holds, then by (ii), $A = F^x \circ_f y = F^x$ and $B = \{x\}$. Since $0 \in F^x$ and K5 holds, then $A < B$. If (IV) holds, then $A = x \circ_f F^y$ and $B = \{x\}$. Since $0 \in F^y$ and K3 holds, thus $x \in x \circ_f 0 \subseteq x \circ_f F^y$ and it yields that $A < B$. If (V) holds, then $A = F^x \circ_f F^x$ and $B = F^x$. Since $0 \in F^x$ and K5 holds, then $A < B$. Therefore K1 holds in all cases.

K2: $(x \circ_f y) \circ_f z = (x \circ_f z) \circ_f y$.
We put $A = (x \circ_f y) \circ_f z$ and $B = (x \circ_f z) \circ_f y$ and show that $A = B$ for all cases (I) – (V). If (I) holds, then $A = \{x\} = B$. If (II) holds, then by (ii) we have $A = F^x \circ_f z = F^x$ and $B = F^x$, so $A = B$. If (III) holds, similar to the proof of case (II) we have $A = B$. If (IV) holds, then $A = \{x\} = B$. If (V) holds, then $A = B$. Finally we show that K4 holds, i.e., $x < y,\ y < x \Rightarrow x = y$. Suppose $x < y,\ y < x$ and $x \neq y$. Then we

have $0 \in x \circ_f y = \{x\}$ and $0 \in y \circ_f x = \{y\}$. Hence $x = y = 0$ which is a contradiction to $x \neq y$. Thus $(H, \circ_f, 0)$ is hyper K-algebra.

$(ii) \Rightarrow (iii)$. Let $y \neq x$ and $y \in F^x$. Then, according to the definition, $u \circ_f y = \{u\}$ where $u \neq y$. Therefore

$$F^x \circ_f y = \cup_{u \neq y, u \in F^x}(u \circ_f y) \cup y \circ y = (F^x - \{y\}) \cup F^y. \qquad (1)$$

By (ii), $F^x \circ_f y = F^x$. So equality (1) yields that $y \in F^y$ and $F^y \subseteq F^x$, that is, (iii) holds.

$(iii) \Rightarrow (ii)$. Suppose $x \neq y$. We consider two cases (a): $y \notin F^x$ and (b): $y \in F^x$. If (a) holds , then $u \neq y$ for all $u \in F^x$. Thus by definition of $\circ_f$ we have $F^x \circ_f y = F^x$, hence (ii) holds. If (b) holds, then by equality (1) and hypothesis $(F^y \subseteq F^x)$ we get that $F^x \circ_f y = F^x$. □

Definition 7. The hyperoperation and hyper K-algebra which have been introduced in Theorem 7 are called a *quasi union hyper operation* and a *quasi union hyper K-algebra*, respectively.

Corollary 1. *For any set X such that $0 \notin X$ and $f(x) \in \{\{0\}, \{0, x\}\}$ for all $f \in \mathcal{S}$ and $x \in H$ there is a quasi union hyper K-algebra on $H = X \cup \{0\}$ with the hyperoperation defined as follows:*

$$x \circ y := \begin{cases} F^x = \{0\} \text{ or } F^x = \{0, x\} & \text{if } x = y \\ \{x\} & \text{otherwise.} \end{cases}$$

Proof. Since $F^x \circ y = F^x$, for all $x \neq y \in H$, thus by Theorem 7 (ii) and Definition 7, $(H, \circ, 0)$ is a quasi union hyper K-algebra. □

Example 1. Let $X = \{1, 2\}$. Then according to Corollary 1, each of the following tables are quasi union hyper K-algebra on $H = \{0, 1, 2\}$.

$\circ_1$	0	1	2
0	{0}	{0}	{0}
1	{1}	{0}	{1}
2	{2}	{2}	{0}

$\circ_2$	0	1	2
0	{0}	{0}	{0}
1	{1}	{0,1}	{1}
2	{2}	{2}	{0}

$\circ_3$	0	1	2
0	{0}	{0}	{0}
1	{1}	{0}	{1}
2	{2}	{2}	{0,2}

$\circ_4$	0	1	2
0	{0,1}	{0}	{0}
1	{1}	{0,1}	{1}
2	{2}	{2}	{0,1,2}

Corollary 2. *Let H be a quasi union hyper K-algebra and $x \neq y$. If $y \in F^x$ and $x \in F^y$, then $F^y = F^x$.*

The proof follows from Theorem 7 (iii).

4. Some results on quasi union hyper K-algebras

Theorem 8. *Let H be a quasi union hyper K-algebra. Then the following statements are equivalent:*

(i) *H is positive implicative hyper K-algebra,*

(ii) *$F^x = \{0\}$ or $F^x = \{0, x\}$ for all $x \in H$,*

(iii) *H is a hyper BCK-algebra.*

Proof. $(i) \Rightarrow (ii)$. Let H be positive implicative, i.e., $(x \circ y) \circ z = (x \circ z) \circ (y \circ z)$ for all $x, y, z \in H$ and $u \in F^x$. If $u \neq x$, since $(u \circ x) \circ x = (u \circ x) \circ (x \circ x)$ we get that $\{u\} = \{u\} \circ (x \circ x)$. From $u \in F^x = x \circ x$, we conclude that $0 \in \{u\} \circ (x \circ x) = \{u\}$. So $u = 0$ and $F^x = \{0\}$ or $F^x = \{0, x\}$ for all $x \in H$.

$(ii) \Rightarrow (i)$. Suppose $F^x = \{0\}$ or $F^x = \{0, x\}$ for all $x \in H$. We show that H is a positive implicative hyper K-algebra, i.e., H satisfies the following identity:

$$(x \circ y) \circ z = (x \circ z) \circ (y \circ z). \tag{2}$$

We prove it by considering the following cases: (I) $x \circ x = \{0\}$, (II) $x \circ x = \{0, x\}$.

Case 1. $x \neq y,\ x \neq z,\ y \neq z$.

$$(x \circ y) \circ z = \{x\} \circ z = \{x\} \text{ and } (x \circ z) \circ (y \circ z) = \{x\} \circ \{y\} = \{x\}.$$

Case 2. $x = y \neq z$. If (I) holds, then

$$(x \circ y) \circ z = \{0\} \circ z = \{0\} \text{ and } (x \circ z) \circ (y \circ z) = \{x\} \circ \{x\} = \{0\}.$$

If (II) holds, then

$$(x \circ y) \circ z = \{0, x\} \circ z = \{0, x\} \text{ and } (x \circ z) \circ (y \circ z) = \{x\} \circ \{x\} = \{0, x\}.$$

Case 3. $x = z \neq y$. By K2 and the proof of Case 2, (2) holds.

Case 4. $x \neq y = z$. By considering $F^0 = 0 \circ 0 = \{0\}$, if (I) holds then

$$(x \circ y) \circ z = \{x\} \circ z = \{x\} \text{ and } (x \circ z) \circ (y \circ z) = \{x\} \circ \{0\} = \{x\}.$$

If (II) holds, then

$$(x \circ y) \circ z = \{x\} \circ z = \{x\} \text{ and } (x \circ z) \circ (y \circ z) = \{x\} \circ \{0, y\} = \{x\}.$$

Case 5. $x = y = z$. By considering $F^0 = 0 \circ 0 = \{0\}$, if (I) holds then

$$(x \circ y) \circ z = \{0\} \circ x = \{0\} \text{ and } (x \circ z) \circ (y \circ z) = \{0\} \circ \{0\} = \{0\}.$$

If (II) holds, then $(x \circ y) \circ z = \{0, x\} \circ x = \{0, x\}$ and $(x \circ z) \circ (y \circ z) = \{0, x\} \circ \{0, x\} = \{0, x\}$. These cases imply that the identity (2) is satisfied, thus H is a positive implicative hyper K-algebra.

$(ii) \Rightarrow (iii)$. Let $F^x = \{0\}$ or $F^x = \{0, x\}$ for all $x \in H$. We show that H is a hyper BCK-algebra. To do this, since each hyper K-algebra satisfies HK2 and HK4, it is sufficient to prove H satisfies HK1 and HK3. Now we show that HK1 holds, i.e., $(x \circ z) \circ (y \circ z) \ll x \circ y$ for all $x, y \in H$. We prove it by considering the following cases:

$$\text{(I) } x \circ x = \{0\}, \qquad \text{(II) } x \circ x = \{0, x\}.$$

Case 1. $x \neq y,\ x \neq z,\ y \neq z$.

$$(x \circ z) \circ (y \circ z) = \{x\} \ll x \circ y = \{x\}.$$

Case 2. $x = y \neq z$.

$$(x \circ z) \circ (y \circ z) = \{x\} \circ \{x\} = x \circ x \ll x \circ y = x \circ x.$$

Case 3. $x = z \neq y$. By considering $F^0 = 0 \circ 0 = \{0\}$, if (I) holds then

$$(x \circ z) \circ (y \circ z) = \{0\} \circ \{y\} = \{0\} \ll x \circ y = \{x\}.$$

If (II) holds, then $(x \circ z) \circ (y \circ z) = \{0, x\} \circ \{y\} = \{0, x\} \ll x \circ y = \{x\}$.

Case 4. $x \neq y = z$. If (I) holds, then $(x \circ z) \circ (y \circ z) = \{x\} \circ \{0\} = \{x\} \ll \{x\}$. If (II) holds, then $(x \circ z) \circ (y \circ z) = \{x\} \circ \{0, y\} = \{x\} \ll x \circ y = \{x\}$.

Case 5. $x = y = z$. If (I) holds, then $(x \circ z) \circ (y \circ z) = \{0\} \ll x \circ y = \{0\}$. If (II) holds, then $(x \circ z) \circ (y \circ z) = \{0, x\} \ll x \circ y = \{0, x\}$.

Therefore HK1 holds. Finally since $0 \ll x$, $x \ll x$, hence $\{0, x\} \ll x$. Therefore by considering "$\circ$" of H we have $x \circ y \ll x$ for all $x, y \in H$, i.e., HK3 holds. Thus H is a hyper BCK-algebra.

$(iii) \Rightarrow (ii)$. Let H be a quasi union hyper BCK-algebra. Then $F^0 = 0 \circ 0 = \{0\}$. So, let $u \in F^x$ and $u \neq x$. Then, since $x \circ x \ll x$, we have $u \ll x$ or $0 \in u \circ x = \{u\}$. This implies that $u = 0$, hence $F^x = \{0\}$ or $F^x = \{0, x\}$ for all $x \in H$. □

Theorem 9. *Any quasi union hyper K-algebra H is implicative.*

Proof. Let H be a quasi union hyper K-algebra. By considering Definition 3, it is enough to show that $x \in x \circ (y \circ x)$ for all $x, y \in H$. Let $x, y \in H$. Then if $x \neq y$, we have $x \circ (y \circ x) = \{x\}$ and if $x = y$, then $x \in x \circ (x \circ x)$. Because $0 \in x \circ x$. Hence we have $x \in x \circ (y \circ x)$, for any $x, y \in H$. □

Theorem 10. *Let H be a quasi union hyper K-algebra. Then H is strong implicative if and only if $F^0 = \{0\}$.*

Proof. Let H be a strong implicative quasi union hyper K-algebra. Then $x \circ 0 \subseteq x \circ (y \circ x)$ for all $x, y \in H$. If $x = 0$ and $y \neq 0$ we have $0 \circ 0 \subseteq 0 \circ (y \circ 0) = \{0\}$. Hence $0 \circ 0 = F^0 = \{0\}$. Conversely, suppose $F^0 = \{0\}$. We prove that $x \circ 0 \subseteq x \circ (y \circ x)$ for all $x, y \in H$. By considering $F^0 = 0 \circ 0 = \{0\}$, if $x \neq y$, then we have $x \circ 0 = \{x\} = x \circ (y \circ x)$. If $x = y$, then we have $x \circ 0 = \{x\} \subseteq x \circ (x \circ x)$, because $0 \in x \circ x$ and $x \circ 0 = \{x\}$. Therefore H is a strong implicative hyper K-algebra. □

Theorem 11. *If $(H, \circ, 0)$ is a quasi union hyper K-algebra, then for any $x \in H \setminus \{0\}$, $A_x = \{0, x\}$ is a hyper K-ideal of H.*

Proof. Suppose $v \circ y < A_x$ and $y \in A_x$. We show that $v \in A_x$. If $v \in \{0, x\}$, then we are done. Otherwise, we have $v \circ y = \{v\} < \{0, x\}$. This implies that $v < 0$ or $v < x$. Since $v \neq 0, x$, from these we conclude that $0 \in v \circ 0 = \{v\}$ or $0 \in v \circ x = \{v\}$. Hence $v = 0$, which is a contradiction. Therefore $v \in \{0, x\}$ and hence A_x is a hyper K-ideal of H. □

Theorem 12. *Let H be a quasi union hyper K-algebra. Then $H \overset{\mathrm{C}}{=} \oplus_{x \in H \setminus \{0\}} A_x$ (hyper K-ideal).*

Proof. By considering Definition 6 and Theorem 11, it is enough to show that for all $x \in H \setminus \{0\}$, $A_x = \{0, x\}$ is C-absorbing. Suppose $t \notin \{0, x\}$, since $u \circ t = \{u\} \subseteq A_x$ for all $u \in \{0, x\}$, we conclude that A_x is C-absorbing. □

Corollary 3. *Let H be a quasi union hyper K-algebra and $0 \circ 0 = \{0\}$. Then $H \overset{\mathrm{C}^*}{=} \oplus_{x \in H \setminus \{0\}} A_x$(hyper K-ideal).*

Proof. The proof follows from Definition 5 and Theorem 12. □

By the following example we show that there is a quasi union hyper K-algebra such that $A_x = \{0, x\}$ is not C^*-absorbing.

Example 2. Consider $H = \{0, 1, 2\}$ with the following structure:

$\circ$	0	1	2
0	{0,1}	{0}	{0}
1	{1}	{0,1}	{1}
2	{2}	{2}	{0,1,2}

Then $(H, \circ, 0)$ is a quasi hyper K-algebra and $A_2 = \{0, 2\}$ is not C^*-absorbing, because $0 \circ 0 = \{0, 1\} \nsubseteq A_2$.

Corollary 4. *Let H be a quasi union hyper K-algebra. Then $H \overset{\text{C}}{=} \oplus_{x \in H \setminus \{0\}} A_x$ (closed set).*

Proof. Since any hyper K-ideal is closed set, the proof follows from Theorem 12. □

Lemma 1. *Any hyper K-ideal I of hyper BCK-algebra H is a hyper BCK-ideal too.*

Proof. Let $x \circ y \ll I$ and $y \in I$. Then $x \circ y < I$. Since I is a hyper K-ideal and $y \in H$, we conclude that $x \in I$. Hence I is a hyper BCK-ideal of H. □

Corollary 5. *Let H be a quasi union hyper BCK-algebra. Then $H \overset{\text{H}}{=} \oplus_{x \in H \setminus \{0\}} A_x$(hyper BCK-ideal).*

Proof. Since by Theorem 3 any hyper BCK-ideal is H-absorbing, then by using Lemma 1 and Theorem 12 we get that $H \overset{\text{H}}{=} \oplus_{x \in H} \{0, x\}$(hyper BCK-ideal). □

Corollary 6. *Let H be a quasi union hyper BCK-algebra. Then $H \overset{\text{H}}{=} \oplus_{x \in H \setminus \{0\}} A_x$ (hyper BCK-algebra), i.e., it is a union of family of hyper BCK-algebras.*

Proof. The proof follows from Corollary 5 and Theorem 6. □

Theorem 13. *Any quasi union hyper K-algebra H is (hyper K-ideal)-CUD.*

Proof. By Theorem 12, $H \overset{\text{C}}{=} \oplus_{x \in H \setminus \{0\}} A_x$(hyper K-ideal). By Theorem 1, we must show that for any non-empty subset B of $H \setminus \{0\}$, $\cup_{x \in B} A_x$ is a hyper K-ideal of H. Suppose $u \circ y < \cup_{x \in B} A_x$ and $y \in \cup_{x \in B} A_x$. If $u \neq y$ then $u \circ y = \{u\} < \cup_{x \in B} A_x$. This yields that for some $x \in B$, $u < A_x$. Since A_x is a hyper K-ideal and by Theorem 2 it is a closed set, we conclude that $u \in A_x$. Therefore $u \in \cup_{x \in B} A_x$. If $u = y$, then $u \in \cup_{x \in B} A_x$. Thus $\cup_{x \in B} A_x$ is a hyper K-ideal of H, i.e., H is a (hyper K-ideal)-CUD. □

Theorem 14. *Let H be a quasi union hyper K-algebra and I be a subset of H containing 0. Then I is a hyper K-ideal of H.*

Proof. By Theorem 12 we have $H \stackrel{\mathrm{C}}{=} \oplus_{x\in H\setminus\{0\}} A_x$(hyper K-ideal). Since $I = \cup_{x\in I}\{0, x\}$, by Theorem 13, I is a hyper K-ideal of H. □

Now, we proceed to find some relations between a quasi union hyper K-algebra and a family of hyper K-algebras of type $H \stackrel{\mathrm{C}}{=} \oplus_{i\in\Lambda} A_i$(hyper K-ideal) where, $|A_i| = 2$. In particular, we show that whenever $|H| \geqslant 4$, any type of these hyper K-algebras is a quasi union hyper K-algebra.

Remark 2. Let $H \stackrel{\mathrm{C}}{=} \oplus_{i\in\Lambda} A_i$(hyper K-ideal) where, $|A_i| = 2$. Since $|A_i| = 2$, we have $A_i = \{0, x\}$ for a nonzero element $x \in H$. Hence for convenience we write A_x instead of A_i and hence $H \stackrel{\mathrm{C}}{=} \oplus_{x\in H\setminus\{0\}} A_x$(hyper K-ideal).

Theorem 15. *Let $H \stackrel{\mathrm{C}}{=} \oplus_{x\in H\setminus\{0\}} A_x$(hyper K-ideal) and $|H| \geqslant 4$. Then H is a quasi union hyper K-algebra.*

Proof. Since by K3, we have $0 \in x \circ x = F^x$, according to Theorem 7, it is sufficient to show that $x \circ y = \{x\}$ for all $x \neq y$. Suppose $u \in x \circ y$ and $x \neq y$. Then by considering the following three cases we prove $u = x$.

$$\text{(I) } y = 0, \quad \text{(II) } x \neq 0 \text{ and } y \neq 0, \quad \text{(III) } x = 0.$$

If (I) holds, then since $x \circ 0 < A_u$ and A_u is a hyper K-ideal, we conclude that $x \in A_u$. Since $x \neq y = 0$, then $x = u$. If (II) holds, since $y \notin A_x$ and A_x is C-absorbing, we get that $x \circ y \subseteq A_x$. Thus $u \in A_x$. We show that $u \neq 0$. If $u = 0$, then $x < y$ and $x \in A_y$, because any hyper K-ideal is closed set. This yields that $x = y$, which is a contradiction. Therefore $u = x$. If (III) holds, then since $|H| \geqslant 4$ we have at least two nonzero elements $t, z \in H$ different from y. Therefore $0 \circ y \subseteq A_t \cap A_z = \{0\}$, because A_x and A_t are C-absorbing. This yield that $0 \circ y = \{0\}$, or $u = x = 0$. Therefore $x \circ y = \{x\}$, where $x \neq y \in H$. □

Theorem 15 is not true in general.

Example 3. Let $H = \{0, 1, 2\}$ with the following structure:

$\circ$	0	1	2
0	{0}	{0,2}	{0,1}
1	{1}	{0,1}	{1}
2	{2}	{2}	{0,2}

Then H= $(H, \circ, 0)$ is a hyper K-algebra such that $H \stackrel{\mathrm{C}}{=} \{0, 1\} \oplus \{0, 2\}$(hyper K-ideal) and $0 \circ y \neq \{0\}$ where $y \neq 0$. Also this example shows that even if each A_x in Theorem 15 is C^*-absorbing, then H may not be a quasi union hyper K-algebra, whenever $|H| = 3$.

Lemma 2. *Let* $H \overset{\text{H}}{=} \oplus_{x\in H\setminus\{0\}} A_x$*(closed set) and* $|H| \geqslant 3$*. Then 0 is a left scalar.*

Proof. Since $|H| \geqslant 3$ the proof follows from Theorems 5 and 4. □

Theorem 16. *Let* $H \overset{\text{H}}{=} \oplus_{x\in H\setminus\{0\}} A_x$*(closed set) and* $|H| \geqslant 3$*. Then* $x \circ y$ $=\{x\}$ *for* $x \neq y$*.*

Proof. By Lemma 2 we conclude that $0 \circ y = \{0\}$ for all $y \in H$. Now let $0 \neq x \neq y$. On the contrary, suppose $x \circ y \neq \{x\}$. Since A_x is H-absorbing we have $x \circ y \subseteq A_x = \{0, x\}$. If $x \circ y = \{0, x\}$ or $\{0\}$, then $x < y$. In this case if $y = 0$ we conclude that $x = 0$, which is a contradiction. Otherwise, $y \neq 0$, we get that $x \in A_y$, because A_y is a closed set and $y \in A_y$. This yields that $x = y$ which is also a contradiction. Hence $x \circ y = \{x\}$. So, $x \neq y$. □

Theorem 16 is not true in general.

Example 4. Let $H = \{0, 1\}$ with the following structure:

$\circ$	0	1
0	{0}	{0,1}
1	{1}	{0,1}

Then $H = (H, \circ, 0)$ is a hyper K-algebra such that $0 \circ 1 \neq \{0\}$.

Theorem 17. *Let* $H \overset{\text{H}}{=} \oplus_{x\in H\setminus\{0\}} A_x$ *(closed set) and* $|H| \geqslant 3$*. Then* 0 *is a scalar and* $x \circ y = \{x\}$ *for* $x \neq y$*.*

Proof. By Theorem 16, $a \circ 0 = \{a\}$ and $0 \circ a = \{0\}$ while $a \neq 0$. Also by Lemma 2 we have $0 \circ 0 = \{0\}$. Hence 0 is scalar. The remaining of the proof follows from Theorem 16. □

Corollary 7. *Let* $H \overset{\text{H}}{=} \oplus_{x\in H\setminus\{0\}} A_x$*(hyper K-ideal) and* $|H| \geqslant 3$*. Then* 0 *is a scalar and* $x \circ y = \{x\}$ *for* $x \neq y$*.*

The proof follows from Theorems 2 and 17.

Theorem 18. *Let* $H \overset{\text{H}}{=} \oplus_{x\in H\setminus\{0\}} A_x$*(closed set) and* $|H| \geqslant 3$*. Then* H *is a positive (strong) implicative quasi union hyper BCK-algebra.*

Proof. By hypothesis and Theorem 17, we have $0 \circ 0 = \{0\}$ and $x \circ y = \{x\}$, where $x \neq y$. Since A_x is H-absorbing we have $x \circ x \subseteq A_x$, for all $x \in H$. Hence $x \circ x = \{0\}$ or $x \circ x = \{0, x\}$. Therefore these imply that

$$x \circ y = \begin{cases} \{0\} \ or \ \{0, x\} & \text{if } x = y, \\ \{x\} & \text{otherwise.} \end{cases}$$

So the proof follows from Corollary 1 and Theorems 8 and 10. □

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Department of Mathematics
Shahid Bahonar University of Kerman
Kerman
Iran
E-mail: nasr@shahed.ac.ir, zahedi_mm@mail.uk.ac.ir

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