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Left almost semigroups defined by a free algebra

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Abstract

We have constructed LA-semigroups through a free algebra, and the structural properties of such LA-semigroups have been investegated. Moreover, the isomorphism theorems for LA-groups constructed through free algebra have been proved.

1. Introduction

A left almost semigroup, abbreviated as an LA-semigroup, is an algebraic structure midway between a groupoid and a commutative semigroup. The structure was introduced by M. A. Kazim and M. Naseeruddin [3] in 1972. This structure is also known as Abel-Grassmann's groupoid, abbreviated as an AG-groupoid [6] and as an invertive groupoid [1].

A groupoid G with *left invertive law*, that is: $(ab)\,c = (cb)\,a, \forall a, b, c \in G$, is called an *LA-semigroup*.

An LA-semigroup satisfies the medial law: $(ab)\,(cd) = (ac)\,(bd)$. An LA-semigroup with left identity is called an *LA-monoid*.

An LA-semigroup in which either $(ab)\,c = b\,(ca)$ or $(ab)\,c = b\,(ac)$ holds for all $a, b, c, d \in G$, is called an AG^**-groupoid* [6].

Let G be an LA-semigroup and $a \in G$. A mapping $L_a : G \longrightarrow G$, defined by $L_a\,(x) = ax$, is called the *left translation* by a. Similarly, a mapping $R_a : G \longrightarrow G$, defined by $R_a\,(x) = xa$, is called the *right translation* by a. An LA-semigroup G is called *left* (*right*) *cancellative* if all the left (right) translations are injective. An LA-semigroup G is called *cancellative* if all translations are injective.

Let X be a non-empty set and W'_X denote the free algebra over X in the variety of algebras of the type $\{0, \alpha, +\}$, consisting of nullary, unary and

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binary operations determined by the following identities:

$$(x+y)+z = x+(y+z)\,, \quad x+y = y+x, \quad x+0 = x,$$

$$\alpha\,(x+y) = \alpha x + \alpha y, \quad \alpha 0 = 0.$$

Every element $u \in W'_X$ has the form $u = \sum_{i=1}^{r} \alpha^{n_i} x_i$, where $r \geqslant 0$, and n_i are non-negative integers. This expression is unique up to the order of the summands. Moreover $r = 0$ if and only if $u = 0$.

Let us define a multiplication on W'_X by $u \circ v = \alpha u + \alpha^2 v$. Then the set W'_X is an LA-semigroup under this binary operation. We denote it by W_X. It is easy to see that W_X is cancellative.

If n is the smallest non-negative integer such that $\alpha^n x = x$, then n is called the *order* of α. The following examples show the existence of such LA-semigroups.

Example 1. Consider a field $F_5 = \{0,1,2,3,4\}$ and define $\alpha\,(x) = 3x$ for all $x \in F_5$. Then F_5 becomes an LA-semigroup under the binary operation defined by $u \circ v = \alpha u + \alpha^2 v$, $\forall\ u, v \in F_5$.

$\circ$	0	1	2	3	4
0	0	4	3	2	1
1	3	2	1	0	4
2	1	0	4	3	2
3	4	3	2	1	0
4	2	1	0	4	3

Example 2. Let $X = \{x, y\}$ and α be defined as $\alpha\,(a) = 2a$, for all $a \in X$ and $2 \in F_3$. Then the following table illustrates an LA-semigroup W_X.

$\circ$	0	x	$2x$	y	$2y$	$x+y$	$2x+y$	$x+2y$	$2x+2y$
0	0	x	$2x$	y	$2y$	$x+y$	$2x+y$	$x+2y$	$2x+2y$
x	$2x$	0	x	$2x+y$	$2x+2y$	y	$x+y$	$2y$	$x+2y$
$2x$	x	$2x$	0	$x+y$	$x+2y$	$2x+y$	y	$x+2y$	$2y$
y	$2y$	$x+2y$	$2x+2y$	0	y	x	$2x$	$x+y$	$2x+y$
$2y$	y	$x+y$	$2x+y$	$2y$	0	$x+2y$	$2x+2y$	x	$x+y$
$x+y$	$2x+2y$	$2y$	$x+2y$	$2x$	$2x+y$	0	x	y	$x+y$
$2x+y$	$x+2y$	$2x+2y$	$2y$	x	$x+y$	$2x$	0	$2x+2y$	y
$x+2y$	$2x+y$	y	$x+y$	$2x+2y$	$2x$	$2y$	$x+2y$	0	x
$2x+2y$	$x+y$	$2x+y$	y	$x+2y$	x	$2x+2y$	$2y$	$2x+2y$	0

An LA-semigroup is called an *LA-band* [6], if all of its elements are idempotents. An LA-band can easily be constructed from a free algebra by choosing a unary operation α such that $\alpha + \alpha^2 = Id_X$, where Id_X denotes the identity map on X.

Example 3. Define a unary operation α as $\alpha(x) = 2x$, where $x \in F_5$. Then under the binary operation $\circ$ defined as above, F_5 is an LA-band.

$\circ$	0	1	2	3	4
0	0	4	3	2	1
1	2	1	0	4	3
2	4	3	2	1	0
3	1	0	4	3	2
4	3	2	1	0	4

An LA-semigroup $(G, \cdot)$ is called an *LA-group* [5], if
(i) there exists $e \in G$ such that $ea = a$ for every $a \in G$,
(ii) for every $a \in G$ there exists $a' \in G$ such that $a'a = e$.

A subset I of an LA-semigroup $(G, \cdot)$ is called a *left* (*right*) *ideal* of G, if $GI \subseteq I$ ($IG \subseteq I$), and I is called a *two sided ideal* of G if it is left and right ideal of G. An LA-semigroup is called *left* (*right*) *simple*, if it has no proper left (right) ideals. Consequently, an LA-semigroup is *simple* if it has no proper ideals.

Theorem 1. *A cancellative LA-semigroup is simple.*

Proof. Let G be a cancellative LA-semigroup. Suppose that G has a proper left ideal I. Then by definition $GI \subseteq I$ and so I being its proper ideal, is a proper LA-subsemigroup of G. If $g \in G \backslash I$, then $gi \in GI$, for all $i \in I$. But $GI \subseteq I$, so there exists an $i' \in I$, such that $gi = i'$. Since G is cancellative so is then I. This implies that all the right and left translations are bijective. Therefore there exists $i_1 \in I$, such that $L_{i_1}(i) = i'$. This implies that $gi = i_1 i$. By applying the right cancellation, we obtain $g = i_1$. This implies that $g \in I$, which contradicts our supposition. Hence G is simple. □

Corollary 1. *An LA-semigroup defined by a free algebra is simple.*

Theorem 2. *If G is a right* (*left*) *cancellative LA-semigroup, then $G^2 = G$.*

Proof. Let G be a right (left) cancellative LA-semigroup. Then all the right (left) translations are bijective. This implies that for each $x \in G$, there exist some $y, z \in G$ such that $R_y(z) = x$ $(L_y(z) = x)$. Hence $G^2 = G$. □

Corollary 2. *An AG^*-groupoid cannot be defined by a free algebra.*

Proof. It has been proved in [6], that if G is an AG*-groupoid then G^2 is a commutative semigroup. Moreover, if G is a right (left) cancellative LA-semigroup, then $G^2 = G$. □

We now define a subset T_x of W_X such that $T_x = \{\sum_{i=1}^{r} \alpha^{n_i} x \mid x \in X\}$.

Theorem 3. *T_x is an LA-subsemigroup of W_X.*

Proof. It is sufficient to show that T_x is closed under the operation $\circ$. Let $u, v \in T_x$. Then $u = \sum_{i=1}^{n} \alpha^{n_i} x$, $v = \sum_{i=1}^{m} \alpha^{n_i} x$, and so

$$\begin{aligned} u \circ v &= \alpha(u) + \alpha^2(v) = \alpha\left(\textstyle\sum_{i=1}^{n} \alpha^{n_i} x\right) + \alpha^2\left(\textstyle\sum_{i=1}^{m} \alpha^{n_i} x\right) \\ &= \left(\textstyle\sum_{i=1}^{n} \alpha^{n_i+1} + \textstyle\sum_{i=1}^{m} \alpha^{n_i+2}\right) x = \textstyle\sum_{i=1}^{r} \alpha^{m_i} x, \end{aligned}$$

where $r = n + m$, $m_i = n_i + 1$ for $i \leqslant n$ and $m_i = n_i + 2$ for $i > n$. □

Theorem 4. *If $X = \{x_1, x_2, \ldots, x_n\}$, then $W_X = T_{x_1} \oplus T_{x_2} \oplus \ldots \oplus T_{x_n}$.*

Proof. Every element $u \in W_X$ is of the form $u = \sum_{i=1}^{r} \alpha^{n_i} x_i$, where r and n_i are non-negative integers. This expression is unique up to the order of the summands. This implies that $W_X = T_{x_1} + T_{x_2} + \ldots + T_{x_n}$. To complete the proof it is sufficient to show that $T_{x_i} \cap T_{x_j} = \{0\}$, for $i \neq j$. Let $u \in T_{x_i} \cap T_{x_j}$, such that $u \neq 0$. Then $u \in T_{x_i}$ and $u \in T_{x_j}$. This is possible only if $x_i = x_j$. Which is a contradiction to the fact that $x_i \neq x_j$. Hence the proof. □

Proposition 1. *The direct sum of any T_{x_i} and T_{x_j} for $i \neq j$ is an LA-subsemigroup of W_X.*

Proof. The proof is straightforward. □

Theorem 5. *The direct sum of any finite number of T_{x_i}'s is an LA-subsemigroup of W_X.*

Proof. The proof follows directly by induction. □

Theorem 6. *The set W_X / T_x of all right (left) cosets of T_x in W_X is an LA-semigroup.*

Proof. Let $W_X / T_x = \{u \circ T_x \mid u \in W_X\}$, and $u \circ T_x$, $v \circ T_x \in W_X / T_x$. Then by the medial law $(u \circ T_x) \circ (v \circ T_x) = (u \circ v) \circ T_x \circ T_x$. But $T_x \circ T_x = T_x$. Hence $(u \circ T_x) \circ (v \circ T_x) = (u \circ v) \circ T_x \in W_X / T_x$.

Let $u \circ T_x, v \circ T_x, w \circ T_x \in W_X \diagup T_x$. Then

$$\begin{aligned}((u \circ T_x) \circ (v \circ T_x)) \circ (w \circ T_x) &= ((u \circ v) \circ T_x) \circ w \circ T_x \\ &= ((u \circ v) \circ w) \circ T_x = ((w \circ v) \circ u) \circ T_x \\ &= ((w \circ T_x) \circ (v \circ T_x)) \circ (u \circ T_x)\end{aligned}$$

implies that $W_X \diagup T_x$ is an LA-simigroup. □

Remark 1. $\alpha (T_x) = T_x$.

Proposition 2. *For any $T_x \leq W_X$ and $v \in W_X$ we have*

(*a*) $T_x \circ v = (\alpha (v)) \circ T_x$,
(*b*) $T_x \circ (T_x \circ v) = \alpha^2 (T_x \circ v) = \alpha^3 (v \circ T_x)$,
(*c*) $(T_x \circ v) \circ T_x = \alpha (T_x \circ v) = \alpha^2 (v \circ T_x)$,
(*d*) $T_x \circ v = \alpha (v \circ T_x)$.

Proof. The proof is straightforward. □

Theorem 7. $W_X \diagup T_{x_i} = \{v \circ T_{x_i} : v \in W_X\}$ *forms a partition of* W_X.

Proof. We shall show that $u \circ T_{x_i} \cap v \circ T_{x_i} = \emptyset$ for $u \neq v$, and $W_X = \cup_{v \in W_X} v \circ T_{x_i}$. Let $w \in v \circ T_{x_i} \cap u \circ T_{x_i}$. Then $w \in v \circ T_{x_i}$ and $w \in u \circ T_{x_i}$. This implies that $w = v \circ t_1$ and $w = u \circ t_2$, where $t_1, t_2 \in T_{x_i}$. This implies $v \circ t_1 = u \circ t_2$. Hence $\alpha (v) + \alpha^2 (t_1) = \alpha (u) + \alpha^2 (t_2)$, which further gives $\alpha (v) = \alpha (u) + \alpha^2 (t_2) - \alpha^2 (t_1)$ where $\alpha^2 (t_2) - \alpha^2 (t_1) \in T_{x_i}$.

Now $\alpha (v) \in \alpha (u) + T_{x_i}$ yields $\alpha (v) + T_{x_i} \subseteq \alpha (u) + T_{x_i}$, i.e., $v \circ T_{x_i} \subseteq u \circ T_{x_i}$. Similarly, $u \circ T_{x_i} = v \circ T_{x_i}$. Hence $v \circ T_{x_i} \cap u \circ T_{x_i} = \emptyset$. Obviously, $\cup_{v \in W_X} v \circ T_{x_i} \subseteq W_X$.

Conversely, let $t \in W_X$. Then $t = \sum_{i=1}^{r} \alpha^{n_i} x_i$ implies that

$$\begin{aligned}t &= \alpha^{n_1} x_1 + \alpha^{n_2} x_2 + \ldots + \alpha^{n_r} x_r \\ &= \alpha^{n_i} x_i + \alpha^{n_1} x_1 + \alpha^{n_2} x_2 + \ldots + \alpha^{n_{i-1}} x_{i-1} + \alpha^{n_{i+1}} x_{i+1} + \ldots + \alpha^{n_r} x_r.\end{aligned}$$

If $\alpha^{n_1} x_1 + \alpha^{n_2} x_2 + \ldots + \alpha^{n_{i-1}} x_{i-1} + \alpha^{n_{i+1}} x_{i+1} + ... + \alpha^{n_r} x_r = u$, then $t = \alpha^{n_i} x_i + u$, $\alpha^{n_i} x_i \in T_{x_i}$. Now $t = \alpha^{n_i} x_i + u \in T_{x_i} + u = \alpha(u) + T_{x_i} = \alpha(u) + \alpha^2(T_{x_i}) = u \circ T_{x_i} \in \cup_{v \in W_X} v \circ T_{x_i}$ implies $W_X \subseteq \cup_{v \in W_X} v \circ T_{x_i}$. Hence $W_X = \cup_{v \in W_X} v \circ T_x$. □

Theorem 8. *The order of* T_{x_i} *divides the order of* W_X.

Proof. If X is a finite non-empty set then W_X is also finite. This implies that the set of all the right (left) cosets of T_{x_i} in W_X is finite.

Let $W_X \diagup T_{x_i} = \{v_1 \circ T_{x_i}, v_2 \circ T_{x_i}, \ldots, v_r \circ T_{x_i}\}$. Then by virtue of Theorem 7, $W_X = v_1 \circ T_{x_i} \cup v_2 \circ T_{x_i} \cup \ldots \cup v_r \circ T_{x_i}$. This implies that $|W_X| = |v_1 \circ T_{x_i}| + |v_2 \circ T_{x_i}| + \ldots + |v_r \circ T_{x_i}|$. Thus $|W_X| = r\,|T_{x_i}|$. Hence $|W_X| = [T_{x_i}, W_X]\,|T_{x_i}|$, where $[T_{x_i}, W_X]$ denotes the number of cosets of T_{x_i} in W_X. □

Theorem 9. *If X is a non-empty finite set having r number of elements and the order of T_{x_i} is n, then $|W_X| = n^r$.*

Proof. Since it has already been proved that $W_X = T_{x_1} \oplus T_{x_2} \oplus \ldots \oplus T_{x_r}$ for $X = \{x_1, x_2, \ldots, x_r\}$, it is sufficient to show that $|T_{x_1} \oplus T_{x_2} \oplus \ldots \oplus T_{x_r}| = n^r$. We apply induction on r. Let $r = 2$, that is, $W_X = T_{x_1} \oplus T_{x_2}$. Construct the multiplication table of T_{x_1} and write all the elements of T_{x_2} except 0 in the index row and in the index column. Then the number of elements in the index row or column row is $2n-1$. We see from the multiplication table that when the elements of T_{x_1} are multiplied by the elements of T_{x_2} some new elements appear in the table, which are of the form $u \circ v = \alpha(u) + \alpha^2(v)$, where $u \in T_{x_1}$ and $v \in T_{x_2}$ and they are $(n-1)^2$ in number. We write all such elements in index row and column and complete the multiplication table of $T_{x_1} \oplus T_{x_2}$. We see that no new element appear in the table. Then the number of elements in the index row or column is $2n-1+(n-1)^2 = n^2$. We now consider $n = 3$. Take the multiplication table of $T_{x_1} \oplus T_{x_2}$, and write all elements of T_{x_3} except 0 in the index row and column. The number of elements in the index row and column are $n^2 + n - 1$. Multiply the elements of $T_{x_1} \oplus T_{x_2}$ and T_{x_3}. Then in the table, some new elements of the form $t \circ w = \alpha(t) + \alpha^2(w)$ appear, where $t \in T_{x_1} \oplus T_{x_2}$, $w \in T_{x_3}$ which are $n^2(n-1)$ in number. Now we write all these elements in the index row and column of the table of $T_{x_1} \oplus T_{x_2} \oplus T_{x_3}$. We see that no new element appears in the table. The number of elements in the index row or column is $n^2 + n^2(n-1) = n^3$. Continuing in this way we finally get $|T_{x_1} \oplus T_{x_2} \oplus \ldots \oplus T_{x_r}| = n^r$. □

Theorem 10. *Let p be prime and F_P a finite field. Let E denote the r-th extension of F_P. Then there exists a unique epimorphism between LA-semigroups formed by E and F_p.*

Proof. Let α be a unary operation. Suppose that β is a root of an irreducible polynomial with respect to F_p. It is not difficult to prove that the mapping

$\varphi : E \to F_P$ defined by $\varphi\left(a_0 + a_1\beta + ... + a_{r-1}\beta^{r-1}\right) = a_0 + a_1 + ... + a_r$ is a unique epimorphism. □

Theorem 11. *T_x is simple.*

Proof. Suppose that T_x has a proper left (right) ideal of S. Then by definition $ST_x \subseteq S$ $(T_xS \subseteq S)$ and S is proper LA-subsemigroup of T_x. We know that the order of T_x is either prime or power of a prime. So, if it has a proper LA-subsemigroup S, then the order of S will be prime. Since S is embedded into T_x, so there exists a monomorphism between T_x and S. But by Theorem 10, there exists a unique epimorphism between T_x and S. This implies that there exists an isomorphism between T_x and S. This is a contradiction. Hence the proof. □

Theorem 12. *If K is a kernel of a homomorphism h between LA-groups W and W', then*

(a) $K \leq W$,
(b) W/K *is an LA-group,*
(c) $W/K \cong Im(h)$.

Proof. (a) and (b) are obvious. For (c) define a mapping $\varphi : W/K \to Im(h)$ by $\varphi(u \circ K) = h(u)$ for $u \in W$. Then φ is an isomorphism. □

Theorem 13. *If $T_1 = T_{x_1} \oplus T_{x_2} \oplus \ldots \oplus T_{x_n}$, $T_2 = T_{x_1} \oplus T_{x_2} \oplus \ldots \oplus T_{x_m}$, where $n \neq m$, then*

(1) $T_1 \leq T_1 \oplus T_2$ *and* $T_1 \cap T_2 \leq T_2$,
(2) $T_1 \oplus T_2/T_1$ *and* $T_2/T_1 \cap T_2$ *are LA-semigroups,*
(3) $T_1 \oplus T_2/T_1 \cong T_2/T_1 \cap T_2$.

Proof. (1) and (2) are obvious. For (3) define a mapping $\varphi : T_2/T_1 \cap T_2 \longrightarrow T_1 \oplus T_2/T_1$ by $\varphi(v \circ (T_1 \cap T_2)) = v \circ T_1$ for all $v \in T_1 \cap T_2$. Then ϕ is an isomorphism. □

Theorem 14. *If W_X is an LA-group, and $T = T_{x_1} \oplus T_{x_2} \oplus \ldots \oplus T_{x_n}$, then $(W_X/T_{x_i})/(T/T_{x_i})$ is isomorphic to W_X/T, where $1 \leqslant i \leqslant n$.*

Proof. Define a mapping $\varphi : W_X/T_{x_i} \longrightarrow W_X/T$, by $\varphi(v \circ T_{x_i}) = v \circ T$, where $v \in W_X$. Then φ is an epimorphism. By Theorem 12,

$$(W_X/T_{x_i})/(Ker\,\varphi) \cong W_X/T$$

and $Ker\varphi = T/T_{x_i}$. Hence the proof. □

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