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Algebraic properties of some varieties of central loops

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Abstract

Isotopes of C-loops with a unique non-identity squares are studied. It is proved that such loops are C-loops and A-loops. The relationship between C-loops and Steiner loops is further studied. Central loops with the weak and cross inverse properties are also investigated.

1. Introduction

C-loops are one of the least studied loops. Few publications that have considered C-loops include Fenyves [14], [15], Beg [7], [8], Phillips et. al. [24], [26], [21], [20], Chein [10] and Solarin et. al. [2], [30], [28], [27]. The difficulty in studying them is as a result of the nature of their identities when compared with other Bol-Moufang identities (the element occurring twice on both sides has no other element separating it from itself). Latest publications on the study of C-loops which has attracted fresh interest on the structure include [24], [26], and [21].

LC-loops, RC-loops and *C-loops* are loops that satisfies the identities

$$(xx)(yz) = (x(xy))z, \quad (zy)(xx) = z((yx)x), \quad x(y(yz)) = ((xy)y)z,$$

respectively. Fenyves' work in [15] was completed in [24]. Fenyves proved that LC-loops and RC-loops are defined by three equivalent identities. In [24] and [25], it was shown that LC-loops and RC-loops are defined by four equivalent identities. Solarin [28] named the fourth identities the *left middle* (LM) and *right middle* (RM) *identities* and loops that obey them are called

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LM-loops and *RM-loops*, respectively. These terminologies were also used in [29]. Their basic properties are found in [26], [15] and [13].

The *right* and *left translation* on a loop $(L, \cdot)$ are bijections $R_x : L \to L$ and $L_x : L \to L$ defined as $yR_x = yx$.

Definition 1.1. Let $(L, \cdot)$ be a loop. The *left nucleus* of L is the set

$$N_\lambda(L, \cdot) = \{a \in L : ax \cdot y = a \cdot xy \ \forall \ x, y \in L\}.$$

The *right nucleus* of L is the set

$$N_\rho(L, \cdot) = \{a \in L : y \cdot xa = yx \cdot a \ \forall \ x, y \in L\}.$$

The *middle nucleus* of L is the set

$$N_\mu(L, \cdot) = \{a \in L : ya \cdot x = y \cdot ax \ \forall \ x, y \in L\}.$$

The *nucleus* of L is the set

$$N(L, \cdot) = N_\lambda(L, \cdot) \cap N_\rho(L, \cdot) \cap N_\mu(L, \cdot).$$

The *centrum* of L is the set

$$C(L, \cdot) = \{a \in L : ax = xa \ \forall \ x \in L\}.$$

The *center* of L is the set

$$Z(L, \cdot) = N(L, \cdot) \cap C(L, \cdot).$$

L is said to be a *centrum square loop* if $x^2 \in C(L, \cdot)$ for all $x \in L$. L is said to be a *central square loop* if $x^2 \in Z(L, \cdot)$ for all $x \in L$. L is said to be *left alternative* if for all $x, y \in L$, $x \cdot xy = x^2y$ and is said to right alternative if for all $x, y \in L$, $yx \cdot x = yx^2$. Thus, L is said to be *alternative* if it is both left and right alternative. The triple (U, V, W) such that $U, V, W \in SYM(L, \cdot)$ is called an *autotopism* of L if and only if

$$xU \cdot yV = (x \cdot y)W \quad \forall \ x, y \in L.$$

$SYM(L, \cdot)$ is called the *permutation group* of the loop $(L, \cdot)$. The group of autotopisms of L is denoted by $AUT(L, \cdot)$. Let $(L, \cdot)$ and $(G, \circ)$ be two distinct loops. The triple $(U, V, W) : (L, \cdot) \to (G, \circ)$ such that $U, V, W : L \to G$ are bijections is called a *loop isotopism* if and only if

$$xU \circ yV = (x \cdot y)W \quad \forall \ x, y \in L.$$

We investigate central loops with the unique non-identity commutators, associators and squares. The relationship between C-loops and Steiner loops is studied. Central loops with the weak and cross inverse properties are also investigated.

For definition of concepts in theory of loops readers may consult [9], [29] and [23].

2. Preliminaries

Definition 2.1. (cf. [16]) Let a, b and c be three elements of a loop L. The *loop commutator* of a and b is the unique element (a, b) of L such that $ab = (ba)(a, b)$. The *loop associator* of a, b and c is the unique element (a, b, c) of L such that $(ab)c = \{a(bc)\}(a, b, c)$.

If X, Y, and Z are subsets of a loop L, we denote by (X, Y) and (X, Y, Z), respectively, the set of all commutators of the form (x, y) and all the associators of the form (x, y, z), where $x \in X, y \in Y, z \in Z$.

Definition 2.2. (cf. [16]) A *unique non-identity commutator* is an element $s \neq e$ (e is the identity element) in a loop L with the property that every commutator in L is e or s. A *unique non-identity commutator associator* is an element $s \neq e$ in a loop L with the property that every commutator in L is e or s and every associator is e or s. A *unique non-identity square* or *non-trivial square* is an element $s \neq e$ in a loop L with the property that every square in L is e or s.

Definition 2.3. A loop $(L, \cdot)$ is called a *weak inverse property loop* (W.I.P.L.) if and only if it satisfies the weak inverse property (W.I.P.): $y(xy)^\rho = x^\rho$ for all $x, y \in L$. L is called a *cross inverse property loop* (C.I.P.L.) if and only if it satisfies the cross inverse property (C.I.P.): $xy \cdot x^\rho = y$. $(L, \cdot)$ is a *left* (*right*) *inverse property loop* (L.I.P.L.) (resp. (R.I.P.L.)) if and only if it has the left (resp. right) inverse property (L.I.P) (resp. (R.I.P)): $x^\lambda(xy) = y$ (resp$(yx)x^\rho = y$. It is an *inverse property loop* (I.P.L.) if and only if it has the inverse property (I.P.) i.e., it has L.I.P. and R.I.P. property.

Most of our results and proofs, are written in dual form relative to RC-loops and LC-loops. That is, a statement like 'LC(RC)-loop... A(B)' where 'A' and 'B' are some equations or expressions means that 'A' is for LC-loops and 'B' is for RC-loops.

3. Inner mappings

Lemma 3.1. *Let L be a C-loop. Then for each $(A,B,C) \in AUT(L)$, there exists a unique pair $(S_1,T_1,R_1),(S_2,T_2,R_2) \in AUT(L,\cdot)$ such that $L_x^2 = S_2^{-1}S_1$, $R_x^2 = T_1^{-1}T_2$, $R_x^{-2}L_x^2 = R_2^{-1}R_1$, $R_1^{-1}R_2T_2^{-1}T_1S_2^{-1}S_1 = I$ for all $x \in L$.*

Proof. If L is a C-loop, then $(L_x^2, I, L_x^2), (I, R_x^2, R_x^2) \in AUT(L)$ for all $x \in L$. So, there exist $(S_1,T_1,R_1),(S_2,T_2,R_2) \in AUT(L)$ such that

$$(S_1,T_1,R_1) = (A,B,C)(L_x^2, I, L_x^2) \in AUT(L)$$
$$(S_2,T_2,R_2) = (A,B,C)(I, R_x^2, R_x^2) \in AUT(L).$$

Hence, the conditions hold although the identities do not depend on (A,B,C), but the uniqueness does. □

Theorem 3.1. *Let L be a C-loop and let there exist a unique pair of autotopisms $(S_1,T_1,R_1),(S_2,T_2,R_2)$ such that the conditions $L_x^2 = S_2^{-1}S_1$, $R_x^2 = T_1^{-1}T_2$ and $R_x^{-2}L_x^2 = R_2^{-1}R_1$ hold for each $x \in L$. If $\alpha_1 = S_1^{-1}$, $\alpha_2 = S_2^{-1}$, $\beta_1 = T_1^{-1}$, $\beta_2 = T_2^{-1}$, $\gamma_1 = R_1^{-1}$ and $\gamma_2 = R_2^{-1}$, then*

$$(x^2y)\alpha_1 = y\alpha_2, \qquad (yx^2)\beta_2 = y\beta_1, \qquad (x^2yx^{-2})\gamma_1 = y\gamma_2 \ \forall\ x, y \in L.$$

Proof. From Lemma 3.1 we have $L_x^2 = S_2^{-1}S_1$, $R_x^2 = T_1^{-1}T_2$, $R_x^{-2}L_x^2 = R_2^{-1}R_1$. Keeping in mind that a C-loop is power associative and nuclear square, we have the following:

1. $L_x^2 = S_2^{-1}S_1 \longleftrightarrow yL_x^2 = yS_2^{-1}S_1$ for all $y \in L \longleftrightarrow yL_{x^2} = yS_2^{-1}S_1 \longleftrightarrow x^2y = yS_2^{-1}S_1 \longleftrightarrow (x^2y)S_1^{-1} = yS_2^{-1} \longleftrightarrow x^2y\alpha_1 = y\alpha_2$.
2. $R_x^2 = T_1^{-1}T_2 \longleftrightarrow yR_x^2 = yT_1^{-1}T_2$ for all $y \in L \longleftrightarrow yx^2 = yT_1^{-1}T_2 \longleftrightarrow yx^2T_2^{-1} = yT_1^{-1} \longleftrightarrow yx^2\beta = y\beta_1$.
3. $R_x^{-2}L_x^2 = R_2^{-1}R_1 \longleftrightarrow yR_x^{-2}L_x^2 = yR_2^{-1}R_1$ for all $y \in L \longleftrightarrow x^2yx^{-2} = yR_2^{-1}R_1 \longleftrightarrow (x^2yx^{-2})R_1^{-1} = yR_2^{-1} \longleftrightarrow (x^2yx^{-2})\gamma_1 = y\gamma_2$. □

Corollary 3.1. *Let L be a C-loop. An autotopism of L can be constructed if there exists at least one $x \in L$ such that $x^2 \neq e$. In this case also the inverse can be constructed.*

Proof. We need Lemma 3.1 and Theorem 3.1. If $x^2 = e$, then the autotopism is trivial. Since L is a C-loop, using Lemma 3.1 and Theorem 3.1, it will be noticed that $(\alpha_1S_2, \beta_1T_2, \gamma_1R_2) \in AUT(L)$ and $(\alpha_2S_1, \beta_2T_1, \gamma_2R_1) = (\alpha_1S_2, \beta_1T_2, \gamma_1R_2)^{-1}$. Hence the proof. □

Lemma 3.2. *For a C-loop L the mapping $\gamma_2 R_1 : L \to L$ used in the autotopism $(\alpha_2 S_1, \beta_2 T_1, \gamma_2 R_1) \in AUT(L)$ and defined by the identity $y\gamma_2 R_1 = x^2 y x^{-2}$ for all $x \in L$ is:*

1. *an automorphism,*
2. *a semi-automorphism,*
3. *a middle inner mapping,*
4. *a pseudo-automorphism with companion x^2.*

Proof. 1. The map $\gamma_2 R_1$ is a bijection by the construction of the autotopism $(\alpha_2 S_1, \beta_2 T_1, \gamma_2 R_1) \in AUT(L)$. So we need only to show that it is an homomorphism. Let $y_1, y_2 \in L$, then: $(y_1 y_2)\gamma_2 R_1 = (x^2 y_1 x^{-2})(x^2 y_2 x^{-2}) = y_1\gamma_2 R_1 \cdot y_2 \gamma_2 R_1$. Whence, $\gamma_2 R_1$ is an automorphism.

2. We have $e\gamma_1 = e\gamma_2$, hence $e\gamma_2 R_1 = e$. Thus $(zy \cdot z)\gamma_2 R_1 = x^2(zy \cdot z)x^{-2} = x^2((zy \cdot z)x^{-2}) = (x^2 z x^{-2})(x^2 y x^{-2}) \cdot z\gamma_2 R_1 = (z\gamma_2 R_1 \cdot y\gamma_2 R_1) \cdot z\gamma_2 R_1$. So, $\gamma_2 R_1$ is a semi-automorphism.

3. Since $e\gamma_2 R_1 = e$, we hawe $y\gamma_2 R_1 = yR_{x^{-2}} L_{(x^{-2})^{-1}} = yT(x^{-2})$ for all $y \in L$, which implies $\gamma_2 R_1 = T(x^{-2}) \in Inn(L)$. Hence $\gamma_2 R_1$ is a middle inner mapping.

4. It is a consequence of the first property and the fact that any automorphism in a C-loop L is a pseudo-automorphism with companion x^2 for all $x \in L$. □

Lemma 3.3. *Let $(L, \cdot)$ be a C-loop. Then:*

1. $T(x^{-1}) = R_x T(x^{-2}) L_x^{-1}$, $T(x)^2 = R_x T(x^{-1})^{-1} L_x^{-1}$,
2. $T(x^n) = R_x^{n-1} T(x) L_x^{1-n}$, $T(x^{-n}) = R_x^{1-n} T(x^{-1}) L_x^{n-1}$ *for* $n \in \boldsymbol{Z}^+$,
3. $R(x, x) = I$, $L(x, x) = I$.

Proof. 1. For $\gamma_2 R_1$ from Lemma 3.2 we have $y\gamma_2 R_1 = x^2 y x^{-2} = yR_{x^{-2}} L_{x^2} = yR_x^{-1} R_x^{-1} L_x L_x = yR_x^{-1} T(x^{-1}) L_x$. Thus, $\gamma_2 R_1 = R_x^{-1} T(x^{-1}) L_x$. But $\gamma_2 R_1 = T(x^{-2})$ is the middle inner mapping, so, $T(x^{-2}) = R_x^{-1} T(x^{-1}) L_x$ implies $T(x^{-1}) = R_x T(x^{-2}) L_x^{-1}$. Therefore $T(x)^2 = R_x L_x^{-1} R_x L_x^{-1} = R_x (R_{x^{-1}} L_{x^{-1}}^{-1})^{-1} L_x^{-1} = R_x T(x^{-1})^{-1} L_x^{-1}$.

2. By induction.
$n = 1,\ T(x) = R_x^{1-1} T(x) L_x^{1-1} = R_{x^0} T(x) L_{x^0} = T(x)$ for $x \in L$,
$n = 2,\ T(x^2) = T(xx) = R_{x^2} L_{x^2}^{-1} = R_x R_x L_x^{-1} L_x^{-1} = R_x T(x) L_x^{-1}$ for $x \in L$,
$n = 3,\ T(x^3) = T(x^2 x) = R_{x^2 x} L_{(x^2 x)^{-1}} = R_{x^2} R_x L_{x^{-1} x^{-2}} = R_{x^2} R_x L_{x^{-1}} L_{x^{-2}}$
$= R_x^2 T(x) L_x^{-2}$ for all $x \in L$.

Let $n = k,\ T(x^k) = R_x^{k-1} T(x) L_x^{1-k}$. Then for $n = k+1$ we have

$T(x^{k+1}) = T(x^{k-1}x^2) = R_{x^{k-1}x^2}L^{-1}_{(x^{k-1}x^2)} = R_{x^{k-1}x^2}L_{x^{-2}x^{1-k}} =$
$R_{x^{k-1}}R_{x^2}L_{x^{-2}}L_{x^{1-k}} = R_{x^{k-1}}T(x^2)L_{x^{1-k}} = R_x^{k-1}R_xT(x)L_x^{-1}L_x^{1-k}$
$= R_x^kT(x)L_x^{-k}$.

Therefore $T(x^n) = R_x^{n-1}T(x)L_x^{1-n}$ for all $n \in \boldsymbol{Z}^+$. Replacing x by x^{-1} we obtain $T(x^{-n}) = T((x^{-1})^n) = R_{x^{-1}}^{n-1}T(x^{-1})L_{x^{-1}}^{1-n} = R_x^{1-n}T(x^{-1})L_x^{n-1}$. Thus, $T(x^{-n}) = R_x^{1-n}T(x^{-1})L_x^{n-1}$ for all $n \in \boldsymbol{Z}^+$.

3. $R(x,x) = R_x^2R_x^{-2} = I, L(x,x) = L_x^2L_x^{-2} = I$. □

Remark 3.1. Lemma 3.2 gives an example of a bijective mapping which is an automorphism, pseudo-automorphism, semi-automorphism and an inner mapping.

4. Relationship between C-loops and Steiner loops

For a loop $(L,\cdot)$, the bijection $J : L \to L$ is defined by $xJ = x^{-1}$. A *Steiner loop* is a loop satisfying the identities

$$x^2 = e, \quad yx\cdot x = y, \quad xy = yx.$$

Theorem 4.1. *A C-loop $(L,\cdot)$ in which $(I, L_z^2, JL_z^2J$ or (R_z^2, I, JR_z^2J) lies in $AUT(L)$ is a loop of exponent* 4.

Proof. 1. If $(I, L_z^2, JL_z^2J) \in AUT(L)$ for all $z \in L$, then $x\cdot yL_z^2 = (xy)JL_z^2J$ for all $x,y,z \in L$ implies $x\cdot z^2y = xy\cdot z^{-2}$. Whence $z^2y\cdot z^2 = y$. Then $y^4 = e$ for every $y \in L$.

2. If $(R_z^2, I, JR_z^2J) \in AUT(L)$ for all $z \in L$, then $xR_z^2\cdot y = (xy)JR_z^2J$ for all $x,y,z \in L$ implies $(xz^2)\cdot y = [(xy)^{-1}z^2]^{-1}$. Whence $(xz^2)\cdot y = z^{-2}(xy)$, consequently $(xz^2)\cdot y = z^{-2}x\cdot y$. Thus $xz^2 = z^{-2}x$ which implies $z^4 = e$ for every $z \in L$. □

Theorem 4.2. *A C-loop $(L,\cdot)$ in which (I, L_z^2, JL_z^2J) and (R_z^2, I, JR_z^2J) lies in $AUT(L)$ is a central square C-loop of exponent* 4.

Proof. 1. If $(I, L_z^2, JL_z^2J) \in AUT(L)$ for all $z \in L$, then $x\cdot yL_z^2 = (xy)JL_z^2J$ for all $x,y,z \in L$ implies $x\cdot z^2y = xy\cdot z^{-2}$.

2. If $(R_z^2, I, JR_z^2J) \in AUT(L)$ for all $z \in L$, then $xR_z^2\cdot y = (xy)JR_z^2J$ for all $x,y,z \in L$ implies $xz^2\cdot y = z^{-2}(xy)$.

Therefore $x\cdot z^2y = xz^2\cdot y$ if and only if $xy\cdot z^{-2} = z^{-2}\cdot xy$. Putting $t = xy$ we have $tz^{-2} = z^{-2}t$, i.e., $z^2t^{-1} = t^{-1}z^2$. Whence we conclude that

$z^2 \in C(L, \cdot)$ for all $z \in L$. Since C-loops are nuclear square (see [26]), we have $z^2 \in Z(L, \cdot)$. Hence L is a central square C-loop. By Theorem 4.1, $x^4 = e$. □

Corollary 4.1. *If $(I, L_z^2, JL_z^2J) \in AUT(L)$ and $(R_z^2, I, JR_z^2J) \in AUT(L)$ for a C-loop $(L, \cdot)$, then L is flexible, $(xy)^2 = (yx)^2$ for all $x, y \in L$ and $x \mapsto x^3$ is an anti-automorphism*

Proof. By Theorem 4.2, Lemma 5.1 and Corollary 5.2 of [21]. □

Theorem 4.3. *A central square C-loop of exponent 4 is a group.*

Proof. To prove this, it shall be shown that $R(x, y) = I$ for all $x, y \in L$. By Corollary 4.1, for $w \in L$ we get $wR(x, y) = wR_xR_yR_{xy}^{-1} = (wx)y \cdot (xy)^{-1} = (wx)(x^2yx^2) \cdot (xy)^{-1} = (wx^3)(yx^2) \cdot (xy)^{-1} = (w^2(w^3x^3))(yx^2) \cdot (xy)^{-1} = (w^2(xw)^3)(yx^2) \cdot (xy)^{-1} = w^2(xw)^3 \cdot (yx^2)(xy)^{-1} = w^2(xw)^3 \cdot [y \cdot x^2(xy)^{-1}] = w^2(xw)^3 \cdot [y \cdot x^2(y^{-1}x^{-1})] = w^2(xw)^3 \cdot [y(y^{-1}x^{-1} \cdot x^2)] = w^2(xw)^3 \cdot [y(y^{-1}x)] = w^2(xw)^3 \cdot x = w^2(w^3x^3) \cdot x = w^2 \cdot (w^3x^3)x = w^2 \cdot (w^3x^{-1})x = w^2w^3 = w^5 = w$. So, $R(x, y) = I$, i.e., $R_xR_yR_{xy}^{-1} = I$. Thus $R_xR_y = R_{xy}$ and $zR_xR_y = zR_{xy}$. So, $zx \cdot y = z \cdot xy$. Therefore L is a group. □

Corollary 4.2. *A C-loop $(L, \cdot)$ in which for all $z \in L$ (I, L_z^2, JL_z^2J) and (R_z^2, I, JR_z^2J) are in $AUT(L)$ is a group.*

Proof. This follows from Theorem 4.2 and Theorem 4.3. □

Remark 4.1. Central square C-loops of exponent 4 are A-loops.

Theorem 4.4. *A C-loop is a central square loop if and only if $\gamma_2 R_1 = I$.*

Proof. $\gamma_2 R_1 = I \longleftrightarrow T(x^{-2}) = I$ for all $x \in L \longleftrightarrow R_{x^{-2}}L_{x^2} = I \longleftrightarrow yx^2 = x^2y \longleftrightarrow L$ is central square. □

Theorem 4.5. *Let L be a C-loop such that the mapping $x \mapsto T(x)$ is a bijection, then L is of exponent 2 if and only if $\gamma_2 R_1 = I$.*

Proof. Indeed, $\gamma_2 R_1 = I \longleftrightarrow T(x^{-2}) = I$ for all $x \in L \longleftrightarrow T(x^{-2}) = I = R_x^{-1}T(x^{-1})L_x \longleftrightarrow T(x^{-1}) = T(x) \longleftrightarrow x^{-1} = x$. Since $x \mapsto T(x)$ is a bijection L is a loop of exponent 2. □

Corollary 4.3. *A C-loop in which $x \mapsto T(x)$ is a bijection is a loop of exponent 2 if and only if it is central square.*

Proof. By Theorem 4.4 and Theorem 4.5. □

Corollary 4.4. *A central square C-loop in which the map $x \mapsto T(x)$ is a bijection is a Steiner loop.*

Proof. By the converse of Corollary 4.3, a C-loop in which $x \mapsto T(x)$ is a bijection, is of exponent 2 if it is central square. By the result of [26], an inverse property loop of exponent 2 is a Steiner loop. By the fact that C-loops are inverse property loops [26], it is a Steiner loop. □

Corollary 4.5. *A C-loop $(L,\cdot)$ in which $x \mapsto T(x)$ is a bijection and (I, L_z^2, JL_z^2J), (R_z^2, I, JR_z^2J) are in $AUT(L)$ for every $z \in L$, is a Steiner loop of exponent* 4.

Proof. According to Theorem 4.2, L is a central square loop. Since $x \mapsto T(x)$ is a bijection, by Corollary 4.4, L is a Steiner loop. By Theorem 4.1, it has a an exponent of 4. □

Corollary 4.6. *A C-loop L in which the mapping $x \mapsto T(x)$ is a bijection is a Steiner loop if and only if L is a central square C-loop.*

Proof. A Steiner loop L is a C-loop [26]. Steiner loops are loops of exponent two, hence by Corollary 4.3, L is central square since in L, the mapping $x \mapsto T(x)$ is a bijection. Conversely, by Corollary 4.3, a central square C-loop L in which the mapping $x \mapsto T(x)$ is a bijection is a loop of of exponent two. The fact that an inverse property loop of exponent two is a Steiner loop [26], completes the proof. □

4.1. Flexibility in C-loops

Lemma 4.1. *A C-loop is flexible if the mapping $x \mapsto x^2$ is onto.*

Proof. Let L be a C-loop . Then $yx^2 \cdot y = y \cdot x^2y$ for all $x, y \in L$. Thus, L is square flexible, hence by [12], it is flexible since the mapping $x \mapsto x^2$ is onto. □

Theorem 4.6. *A C-loop L is flexible if (I, L_z^2, JL_z^2J) and (R_z^2, I, JR_z^2J) are in $AUT(L)$ for all $z \in L$ and the middle inner mappings are of order* 2.

Proof. By Lemma 3.3, for every $x \in L$ we have $T(x)^2 = R_xT(x^{-1})^{-1}L_x^{-1} = R_x(R_xT(x^{-2})L_x^{-1})^{-1}L_x^{-1} = R_x(L_x(R_xT(x^{-2}))^{-1})L_x^{-1} = R_x(L_xT(x^{-2})^{-1}$ $R_x^{-1})L_x^{-1} = R_xL_xT(x^{-2})^{-1}R_x^{-1}L_x^{-1} = R_xL_xT(x^{-2})^{-1}(L_xR_x)^{-1}$. Therefore

$T(x)^2 = R_xL_xT(x^{-2})^{-1}(L_xR_x)^{-1} \longleftrightarrow T(x)^2L_xR_x = R_xL_xT(x^{-2})^{-1} = R_xL_x(\gamma_2R_1)^{-1} = R_xL_x\gamma_1R_2 \longleftrightarrow T(x)^2L_xR_x = R_xL_x\gamma_1R_2$. If $|T(x)| = 2$, $T(x)^2 = I$ and if $\gamma_1R_2 = I \longleftrightarrow L$ is central square, then $L_xR_x = R_xL_x \longleftrightarrow xy \cdot x = x \cdot yx$ is a flexible loop. □

Philips and Vojtěchovský [26] studied the close relationship between C-loops and Steiner loops. In [23], it is shown that Steiner loops are exactly commutative inverse property loops of exponent 2. But in [26], this fact was improved, so that commutativity is not a sufficient condition for an inverse property loop of exponent 2 to be a Steiner loop. So they said 'Steiner loops are exactly inverse property loops of exponent 2'. This result is general for inverse property loops among which are C-loops. They also proved that Steiner loops are C-loops.

The flexibility is possible in a C-loop if the loop is commutative or diassociative [23]. But C-loops naturally do not even satisfy the latter. Apart from the condition stated in Lemma 4.1, Theorem 4.6 when compared with Corollary 5.2 of [21] shows that some middle inner-mappings do not need to be of exponent of 2 for a C-loop to be flexible.

5. Unique non-identity commutator and associator

Lemma 5.1. *If s is a unique non-identity commutator in a C-loop L, then $|s| = 2$, $s \in C(L)$ and $s \in Z(L^2)$.*

Proof. $xy = (yx)(x, y) \longleftrightarrow (x, y) = (yx)^{-1}(xy) = (x^{-1}y^{-1})(xy)$. Therefore $(x, y)^{-1} = [(x^{-1}y^{-1})(xy)]^{-1} = (xy)^{-1}(x^{-1}y^{-1})^{-1} = (y^{-1}x^{-1})(yx) = (y, x)$. Thus, $s^{-1} = s$ or $s^{-1} = e$ implies $s^2 = e$ or $s = e$. So, $s^2 = e$.

If $xs \neq sx$, then $xs = (sx)s$ implies $x = sx$, whence $s = e$. So, $xs = sx$, i.e., $s \in C(L)$. Hence, $s \in Z(L^2)$. □

Lemma 5.2. *If s is a unique non-identity associator in a C-loop L, then $s \in N(L)$.*

Proof. If $(xy)s \neq x(ys)$, then $(xy)s = x(ys) \cdot s$ implies $xy = x \cdot ys$. Whence $y = ys$, i.e., $s = e$. So, $(xy)s = x(ys)$, that is, $s \in N(L)$. □

Lemma 5.3. *If a C-loop $(L, \cdot)$ has a unique non-identity commutator associator s, then s is a central element of order 2.*

Proof. We shall keep in mind that L as a C-loop has the inverse property. $s \in (L, L)$ implies $s^{-1} \in (L, L)$, whence $s^{-1} = s$. Since $s^{-1} \neq e$ if and only if $s \neq e$, we have $s^2 = e$. Let $xs \neq sx$ for some $x, y \in L$. Then $xs = (sx)s$ implies $x = sx$, i.e., $s = e$, which is a contradiction. Thus, $s \in C(L)$. If $(xy)s \neq x(ys)$ for some $x, y \in L$, then $(xy)s = (x \cdot ys)s$ implies $xy = x \cdot ys$. Thus $y = ys$, i.e, $s = e$, which is a contradiction. So, $s \in N(L)$. Therefore $s \in C(L)$, $s \in N(L)$ implies $s \in Z(L)$. □

Remark 5.1. The result of Lemma 5.3 is similar to the result proved in [16] for Moufang loops.

Lemma 5.4. *In $LC(RC)$-loops with a unique non-identity square s is $|s| = 2$, $|x| = 4$ or $|x| = 2$, $s \in N_\lambda$ or $s \in N_\rho$ and $s \in N_\mu$.*

Proof. For all $x \in L$ we have $x^2 = s$. Since $s^2 = s$ implies $s^{-1}s^2 = s^{-1}s$ or $s^2s^{-1} = ss^{-1}$, so $s = e$. This is a contradiction, thus $s^2 = e$ if and only if $|s| = 2$. Moreover, $x^2 = s$ implies $x^4 = x^2x^2 = s^2 = e$. Therefore $x^4 = e$ or $x^2 = e$. In any LC-loop, $x^2 \in N_\lambda$, N_μ, thus $s \in N_\lambda$, N_μ. In an RC-loop, $x^2 \in N_\rho, N_\mu$, thus $s \in N_\rho, N_\mu$. □

Lemma 5.5. *An $LC(RC)$-loop L has a unique non-identity square s if and only if $J = R_s^{-1} = R_{s^{-1}}^{-1}$ or $J = I$ (resp. $J = L_s^{-1} = L_{s^{-1}}^{-1}$ or $J = I$).*

Proof. Let L be a RC-loop. Then $x^2 = s \longleftrightarrow x^2x^{-1} = sx^{-1} \longleftrightarrow x = sx^{-1} \longleftrightarrow x = xJL_s \longleftrightarrow I = JL_s \longleftrightarrow J = L_s^{-1} = L_{s^{-1}}^{-1}$. Similarly, $x^2 = e \longleftrightarrow x = x^{-1} \longleftrightarrow x = xJ \longleftrightarrow J = I$.

For LC-loops the proof is analogous. □

Theorem 5.1. *For any L.I.P. (R.I.P.) $RC(LC)$-loop $(L, \cdot)$ with a unique non-identity square s,*

1. *$s \in Z(L, \cdot)$, i.e., L is centrum square,*
2. *$J = L_s$ (resp. $J = R_s$),*
3. *$x^2y^2 \neq (xy)^2 \neq y^2x^2$, i.e., $x \mapsto x^2$ is neither an automorphism nor an anti-automorphism,*
4. *$(a, b, c) = (bc \cdot a)(ab \cdot c)$,*
   (a) *$ab = a^{-1}b^{-1}$ if and only if $(J, J, I) \in AUT(L)$,*
   (b) *$(a, b, a) = (bs)(ab \cdot a)$ or $(a, b, a) = b(ab \cdot a)$,*
5. *L is a group or Steiner loop,*

6. *If L is a non-commutative C-loop, then s is its unique non-identity commutator.*

Proof. 1. $x^2 = s$ implies $x = sx^{-1}$, whence $x^{-1} = s^{-1}x$. This, by Lemma 2.1 from [1], gives $x^{-1} = (sx^{-1})^{-1} = (x^{-1})^{-1}s^{-1} = xs^{-1}$. Thus, $x^{-1} = s^{-1}x = xs^{-1}$, i.e., $sx = xs$. So, $s \in Z(L, \cdot)$.

2. This follows from Lemma 5.5.

3. If $(xy)^2 = x^2y^2$ or $(xy)^2 = y^2x^2$, then $s = s^2$ implies $s = e$ which is a contradiction.

4. $(a, b, c) = [a(bc)]^{-1} \cdot (ab)c = (bc)^{-1}a^{-1} \cdot (ab)c = (c^{-1}b^{-1})a^{-1} \cdot (ab \cdot c) = [s^{-1}(bc)](s^{-1}a) \cdot (ab \cdot c) = (bc \cdot s^{-1})(s^{-1}a) \cdot (ab \cdot c) = (bcs^{-2} \cdot a)(ab \cdot c) = (bc \cdot a)(ab \cdot c)$. So, $(a, b, c) = (bc \cdot a)(ab \cdot c)$.

4a. The above for $c = e$ gives $(a, b, e) = (ba)(ab) = e$, whence $ab = (ba)^{-1} = a^{-1}b^{-1}$. So, $(J, J, I) \in AUT(L)$.

4b. For $c = a$ we have $(a, b, a) = (ba \cdot a)(ab \cdot a) = (ba^2)(ab \cdot a) = (bs)(ab \cdot a)$. Thus $(a, b, a) = (bs)(ab \cdot a)$ or $(a, b, a) = b(ab \cdot a)$.

5. This follows from Lemma 5.4.

6. $(x, y) = x^{-1}y^{-1} \cdot xy = (x^{-1}y^{-1})(xy^{-1} \cdot y^2) = ((x^{-1}y^{-1})(xy^{-1}) \cdot y^2 = [x^{-2}(xy^{-1}) \cdot (xy^{-1})]y^2 = x^{-2}[(xy^{-1})(xy^{-1})]y^2 = e$ or $(x, y) = s$. Thus, L is either commutative or s is its unique non-identity commutator.

For $(x, s) = x^{-1}s^{-1} \cdot xs = s$ we have $x^{-1}R_s \cdot xR_s = s$, whence $xJ^2 \cdot x^{-1} = s$. Thus $xx^{-1} = s$, i.e., $s = e$, which is a contradiction. So. $(x, s) = e$ implies $s \in C(L, \cdot)$. □

Corollary 5.1. *A C-loop with a unique non-trivial square is a group.*

Proof. By Lemma 5.4 and Theorem 5.1, it is central square of exponent 4. By Theorem 4.3, it is a group. □

Remark 5.2. A C-loop with a unique non-trivial square is an A-loop.

Theorem 5.2. *Let $(G, \cdot)$ and $(H, \circ)$ be two distinct loop such that the triple $\alpha = (A, B, C)$ is an isotopism of G onto H.*

1. *If G is a central square C-loop of exponent 4, then H is a C-loop and an A-loop.*

2. *If G is a C-loop with a unique non-identity square, then H is a C-loop and an A-loop.*

Proof. 1. By Theorem 4.3, G is a group and since groups are G-loops, H is a group, i.e., it is a C-loop and an A-loop.

2. By Corollary 5.1. □

Remark 5.3. Some results for isotopes of central loops of the type (A, B, B) and (A, B, A) are obtained in [18].

Corollary 5.2. *Let $(G, \cdot)$ and $(H, \circ)$ be distinct loops. If the triple (A, B, C) is an isotopism of G onto H such that for every $z \in G$ (I, L_z^2, JL_z^2J) and (R_z^2, I, JR_z^2J) are in $AUT(G, \cdot)$, then H is a C-loop and an A-loop.*

Proof. It follows from Theorem 4.2 and Theorem 5.2. □

Theorem 5.3. *An isotopism (A, A, C) saves the property "unique non-identity square".*

Proof. Let $(A, A, C) : (G, \cdot) \to (H, \circ)$, where G and H are two distinct loops, be an isotopism. Then $xA \circ yA = (x \cdot y)C$. For $y = x$ we have $xA \circ xA = (xA)^2 = (x \cdot x)C = x^2C$. If s is the unique non-identity square in G, i.e $x^2 = s$ or $x^2 = e$ for all $x \in G$ then $s' = sC = (xA)^2 = y'^2$ or $y'^2 = (xA)^2 = x^2C = eC = e'$ for all $y' \in H$ with e' as the identity element in H. So, s' is the unique non-identity square element in H. □

Corollary 5.3. *Central loops with unique non-identity square are isotopic invariant.* □

6. Cross inverse property in central loops

According to [5], the W.I.P. is a generalization of the C.I.P. The latter was introduced and studied by R. Artzy [3] and [4], but from the formal point of view this property was introduced by J. M. Osborn [22]. Huthnance Jr. [17], proved that the holomorph of a W.I.P.L. is a W.I.P.L. A loop property is called *universal* (or universal relative to a given property) if every loop isotope of this loop is a loop with this property. A universal W.I.P.L. is called an *Osborn loop*. Huthnance Jr. [17] investigated the structure of some holomorph of Osborn loops. Basarab [6] studied Osborn loops which are G-loops.

Theorem 6.1. *An $LC(RC)$-loop of exponent 3 is centrum square if and only if it is a C.I.P.L.*

Proof. Let L be a LC-loop. Then $x^2y = yx^2 \longleftrightarrow x^{-1}y = yx^{-1} \longleftrightarrow x(x^{-1}y) = x(yx^{-1}) \longleftrightarrow y = x(yx^{-1})$, which holds if and only if the C.I.P. holds in L.

For RC-loops the proof is analogous. □

Corollary 6.1. *If L is a centrum square $LC(RC)$-loop of exponent 3, then*

1. *L has the A.I.P. and A.A.I.P.,*
2. *L has the W.I.P.,*
3. *$N = N_\lambda = N_\rho = N_\mu$,*
4. *$n \in N$ implies $n \in Z(L)$,*
5. *L is a commutative group.*

Proof. 1. By Theorem 6.1, L is a C.I.P.L. According to [4] and [5], a C.I.P.L. has the A.I.P. Thus, the first part is true. The second part follows from the fact that $x^2 = x^{-1}$.

2. This follows from the the fact that W.I.P. is a generalization of C.I.P. [23].

3. and 4. follows from [5] and [4]. The last statement is obvious. □

Lemma 6.1. *Any $LC(RC,C)$-loop of exponent 3 is a group.* □

Corollary 6.2. *A central square C-loop of exponent 3 has the W.I.P. and C.I.P. and it a commutative group.* □

The fact that central loops of exponent 3 are groups it will be interesting to study non-commutative central loops of exponent 3 with the C.I.P. since there exist groups that do not have the C.I.P. From Theorem 6.1, it follows that the study of LC(RC)-loops of exponent 3 with C.I.P. is equivalent to the study of centrum square LC(RC)-loops of exponent 3.

The existence of central loops of exponent 3 can be deduced from [15], [26] and [27]. According to [26] and [27], the order of every element in a finite LC(RC)-loop divides the order of the loop. Since $|x| = 3$ for all $x \in L$, then

$|L| = 2m$, $m \geqslant 3$ if L is a non-left (right) Bol LC(RC)-loop, or

$|L| = 4k$, $k > 2$ if L is a non-Moufang but both left (right)-Bol and LC(RC)-loop.

The possible orders of finite RC-loops were calculated in [27].

6.1. Osborn central-loops

Theorem 6.2. *An $LC(RC)$-loop has the R.I.P. (L.I.P.) if and only if has the W.I.P.*

Proof. Let $(L,\cdot)$ be a LC-loop with the W.I.P. Then for all $x,y\in L$, $y(xy)^\rho = x^\rho$. Let $xy=z$, then $x^\lambda(xy)=x^\lambda z$ implies $y=x^\lambda z$, thus $(x^\lambda z)z^\rho = x^\rho$ implies $(x^{-1}z)z^\rho = x^{-1}$. Replacing x^{-1} by x, we obtain $(xz)z^\rho = x$. So, L has the R.I.P.

Conversely, if L has the I.P., then $y(xy)^\rho = y(xy)^{-1} = y(y^{-1}x^{-1}) = x^{-1} = x^\rho$ hence it has the W. I. P. Let L be a RC-loop with the W.I.P. Then for all $x,y\in L$, $y(xy)^\rho = x^\rho$ if and only if $(xy)^\lambda\cdot x = y^\lambda$. Let $xy=z$, then $(xy)y^\rho = zy^\rho$ implies $x = zy^\rho$. Thus, $z^\lambda(zy^\rho)=y^\lambda$ implies $z^\lambda(zy^{-1}) = y^{-1}$. Replacing y^{-1} by y, we get $z^\lambda(zy)=y$. Thus, L has the L.I.P. □

Corollary 6.3. *Let $(L,\cdot)$ be an $LC(RC)$-loop with R.I.P. (L.I.P.). Then*

1. $N(L)=N_\lambda(L)=N_\rho(L)=N_\mu(L)$,
2. $(I,R_{x^2},R_{x^2})\in AUT(L)$ *(resp.* $(L_{x^2},I,L_{x^2})\in AUT(L)$*),*
3. $(L_x^2,R_{x^2},R_{x^2}L_x^2)\in AUT(L)$ *(resp.* $(L_{x^2},R_x^2,L_{x^2}R_x^2)\in AUT(L)$*).*

Proof. By Theorem 6.2, L has the W.I.P. According to [22], in a W.I.P.L., the three nuclei coincide, so the firs statement is true. Thus for an LC-loop, $x^2\in N_\rho$ and for an RC-loop, $x^2\in N_\lambda$. Hence for an LC-loop L, $(L_x^2,I,L_x^2),(I,R_{x^2},R_{x^2})\in AUT(L)$ implies that $(L_x^2,R_{x^2},L_x^2R_{x^2}) = (L_x^2,R_{x^2},R_{x^2}L_x^2)\in AUT(L)$. For an RC-loop L, $(I,R_x^2,R_x^2),(L_{x^2},I,L_{x^2})\in AUT(L)$ implies $(L_{x^2},R_x^2,R_x^2L_{x^2}) = (L_{x^2},R_x^2,L_{x^2}R_x^2)\in AUT(L)$. So, the last two statement are true, too. □

Remark 6.1. Corollary 6.3 is true for left (right) Bol loops (i.e., LB(RB)-loops). It follows from the fact that a RB(LB)-loop has the L.I.P. (R.I.P.) if and only if it is a Moufang loop [23], which is obviously a W.I.P.L. [19].

Theorem 6.3. *An $LC(RC)$-loop L is a C-loop if and only if one of the following equivalent statements holds:*

1. *L has the R.I.P. (L.I.P.),*
2. *L has the R.A.P. (L.A.P.),*
3. *L is a $RC(LC)$-loop,*
4. *L has the A.A.I.P. (i.e., $(xy)^{-1}=y^{-1}x^{-1}$),*

5. *L has the W.I.P.*

Proof. A C-loop satisfies 1 and 2. Conversely, if L is an LC-loop, then $(x{\cdot}xy)z = x(x{\cdot}yz)$, whence $[(x{\cdot}xy)z]^{-1} = [x(x{\cdot}yz)]^{-1}$. Thus $z^{-1}(x{\cdot}xy)^{-1} = (x \cdot yz)^{-1}x^{-1}$ and consequently $z^{-1}((xy)^{-1} \cdot x^{-1}) = ((yz)^{-1} \cdot x^{-1})x^{-1}$, i.e., $z^{-1}(y^{-1}x^{-1}{\cdot}x^{-1}) = (z^{-1}y^{-1}{\cdot}x^{-1})x^{-1}$, which means that $z(yx{\cdot}x) = (zy{\cdot}x)x$ for all $x, y, z \in L$. So, a RC-loop. Hence, L is a C-loop.

If L is an LC-loop, then according to [26], $x \cdot (y \cdot yz) = (x \cdot yy)z$ for all $x, y, z \in L$, while L is an RC-loop if and only if $(zy \cdot y)x = z(yy \cdot x)$ for all $x, y, z \in L$. Thus $x \cdot (y \cdot yz) = (x \cdot yy)z$, or equivalently $x \cdot zL_y^2 = xR_{y^2} \cdot z$. So, $(R_{y^2}, L_y^{-2}, I) \in AUT(L)$ for all $y \in L$. For $(zy \cdot y)x = z(yy \cdot x)$ we have $zR^2 \cdot x = z \cdot xL_{y^2}$, i.e., $(R_y^2, L_{y^2}^{-1}, I) \in AUT(L)$ for all $y \in L$.

If L has the right (left) alternative property, $(R_y^2, L_y^{-2}, I) \in AUT(L)$ for all $y \in L$ if and only if L is a C-loop.

3. This is shown in [15].

4. This is equivalent to 1. Indeed, if L has the L.I.P. (R.I.P.), then L has the R.I.P. (L.I.P.). so, L has the A.A.I.P. Conversely, if L.I.P. holds, then for $z = xy$, we have $y = x^{-1}z$ which gives $z^{-1} = (x^{-1}z)^{-1}x^{-1}$, whence $z^{-1} = (z^{-1}x)x^{-1}$. So, $z = (zx)x^{-1}$.

Similarly, if L has the R.I.P. (L.I.P.) then L has the L.I.P. (R.I.P.), i.e., it has the A.A.I.P. Conversely, if R.I.P. holds, then for $z = xy$, we have $x = zy^{-1}$. Thus, $z^{-1} = y^{-1}(zy^{-1})^{-1} = y^{-1}(yz^{-1})$, which proves the L.I.P.

5. This follows from 1 and Theorem 6.2. □

Theorem 6.4. (cf. [19]) *The following equivalent conditions define an Osborn loop* $(L, \cdot)$.

1. $x(yz \cdot x) = (x \cdot yE_x) \cdot zx$,
2. $(x \cdot yz)x = xy \cdot (zE_x^{-1} \cdot x)$,
3. $(A_x, R_x, R_xL_x) \in AUT(L)$,
4. $(L_x, B_x, L_xR_x) \in AUT(L)$,

where $A_x = E_xL_x$, $B_x = E_x^{-1}R_x$ *and* $E_x = R_xL_xR_x^{-1}L_x^{-1}$. □

Theorem 6.5. *If a RC(LC)-loop has the L.I.P. (R.I.P.), then it is an Osborn loop if every its element is a square.*

Proof. Let L be an RC-loop with L.I.P. Then, by Theorem 6.2, L has the W.I.P. Therefore $(A_{x^2}, I, L_{x^2}) \in AUT(L) \longleftrightarrow yA_{x^2} \cdot z = (yz)L_{x^2}$. But

$(yz)L_{x^2} = yE_{x^2}L_{x^2} \cdot z = yR_{x^2}L_{x^2}R_{x^2}^{-1}L_{x^2}^{-1}L_{x^2} \cdot z = yR_{x^2}L_{x^2}R_{x^2}^{-1} \cdot z = yR_x^2L_{x^2}R_{x^2}^{-1} \cdot z = yL_{x^2}R_x^2R_{x^2}^{-1} \cdot z = yL_{x^2} \cdot z$. This is equivalent to the fact that $(L_{x^2}, I, L_{x^2}) \in AUT(L)$ for all $x \in L$, which is true by Corollary 6.3.

Thus, $(I, R_x^2, R_x^2)(A_{x^2}, I, L_{x^2}) = (A_{x^2}, R_{x^2}, R_{x^2}L_{x^2}) \in AUT(L)$. Using Theorem 6.4, we see that L is an Osborn loop if every element in L is a square.

Now, let L be an LC-loop. If L has the R.I.P., then, by Theorem 6.2, L has the W.I.P. So, $(I, B_{x^2}, R_{x^2}) \in AUT(L)$ if and only if $y \cdot zB_{x^2} = (yz)R_{x^2}$. But $(yz)R_{x^2} = y \cdot zE_{x^2}^{-1}R_{x^2} = y \cdot z(R_{x^2}L_{x^2}R_{x^2}^{-1}L_{x^2}^{-1})^{-1}R_{x^2} = y \cdot zL_{x^2}R_{x^2}L_{x^2}^{-1}R_{x^2}^{-1}R_{x^2} = y \cdot zL_{x^2}R_{x^2}L_{x^2}^{-1} = y \cdot zR_{x^2}L_x^2L_{x^2}^{-1} = y \cdot zR_{x^2}$. This is equivalent to the fact that $(I, R_{x^2}, R_{x^2}) \in AUT(L)$ for all $x \in L$, which is true by Corollary 6.3.

Thus, $(L_x^2, I, L_x^2)(I, B_{x^2}, R_{x^2}) = (L_{x^2}, B_{x^2}, L_{x^2}R_{x^2}) \in AUT(L)$. Whence, as in previous case, we conclude that L is an Osborn loop if every element in L is a square. □

Corollary 6.4. *An LC(RC)-loop with R.I.P. (L.I.P.) is an Osborn loop if every its element is a square. Hence, this loop is a group.*

Proof. This follows from Theorem 6.5. The last conclusion is as a consequence of the fact that $x^2 \in N(L)$. □

Corollary 6.5. *A C-loop is an Osborn loop if every its element is a square. Hence, this loop is a group.* □

Question. *Does there exist a C-loop which is an Osborn loop but it is non-associative, non Moufang and non-conjugacy closed?*

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References

[1] **J. O. Adéníran and A. R. T. Solarin:** *A note on automorphic inverse property loops*, Collections of Scientific Papers of the Faculty of Science Krag., **20** (1998), 47 – 52.

[2] **J. O. Adéníran and A. R. T. Solarin:** *A note on generalized Bol identity*, Scientific Annals of Al.I.Cuza. Univ. **45** (1999), 99 – 102.

[3] **R. Artzy:** *On loops with special property*, Proc. Amer. Math. Soc. **6** (1955), 448 – 453.

[4] **R. Artzy:** *Crossed inverse and related loops*, Trans. Amer. Math. Soc. **91** (1959), 480 – 492.

[5] **R. Artzy:** *Relations between loops identities*, Proc. Amer. Math. Soc. **11** (1960), 847 – 851.

[6] **A. S. Basarab:** *Osborn's G-loop*, Quasigroups and Related Systems **1** (1994), 51 – 56.

[7] **A. Beg:** *A theorem on C-loops*, Kyungpook Math. J. **17** (1977), 91 – 94.

[8] **A. Beg:** *On LC-, RC-, and C-loops*, Kyungpook Math. J. **20** (1980), 211 – 215.

[9] **R. H. Bruck:** *A survey of binary systems*, Springer-Verlag, Berlin-Göttingen-Heidelberg, 1966.

[10] **O. Chein:** *A short note on supernuclear (central) elements of inverse property loops*, Arch. Math. **33** (1979), 131 – 132.

[11] **O. Chein, H. O. Pflugfelder and J. D. H. Smith:** *Quasigroups and Loops: Theory and Applications*, Heldermann Verlag, 1990.

[12] **V. O. Chiboka:** *Conjugacy closed loops of Moufang type*, Riv. Mat. Pura Appl. **10** (1991), 89 – 93.

[13] **J. Dénes and A. D. Keedwell:** *Latin Squares and their Applications*, the English University press Lts, 1974.

[14] **F. Fenyves:** *Extra Loops I*, Publ. Math. Debrecen **15** (1968), 235 – 238.

[15] **F. Fenyves:** *Extra Loops II*, Publ. Math. Debrecen **16** (1969), 187 – 192.

[16] **E. G. Goodaire, E. Jespers and C. P. Milies:** *Alternative Loop Rings*, NHMS(184), Elsevier, 1996.

[17] **E. D. Huthnance Jr.:** *A theory of generalised Moufang loops*, Ph.D. thesis, Georgia Institute of Technology, 1968.

[18] **T. G. Jaiyéolá and J. O. Adéníran:** *On isotopic characterisation of C-loops*, Pre-Print.

[19] **M. K. Kinyon:** *A survey of Osborn loops*, Milehigh conference on loops, quasigroups and non-associative systems, University of Denver, Denver, Colorado, 2005.

[20] **M. K. Kinyon, K. Kunen and J. D. Phillips:** *A generalization of Moufang and Steiner loops*, Alg. Universalis **48** (2002), 81 – 101.

[21] **M. K. Kinyon, J. D. Phillips and P. Vojtěchovský:** *C-loops: Extensions and construction*, J. Algebra Appl. **6** (2007), 1 – 20.

[22] **J. M. Osborn:** *Loops with the weak inverse property*, Pac. J. Math. **10** (1961), 295 – 304.

[23] **H. O. Pflugfelder:** *Quasigroups and Loops: Introduction*, Sigma series in Pure Math. 7, Heldermann Verlag, Berlin, 1990.

[24] **J. D. Phillips and P. Vojtěchovský:** *The varieties of loops of Bol-Moufang type*, Alg. Universalis **53** (2005), 115 – 137.

[25] **J. D. Phillips and P. Vojtěchovský:** *The varieties of quasigroups of Bol-Moufang type: An equational reasoning approach* J. Algebra **293** (2005), 17 – 33.

[26] **J. D. Phillips and P. Vojtěchovský:** *On C-loops*, Publ. Math. Debrecen **68** (2006), 115 – 137.

[27] **V. S. Ramamurthi and A. R. T. Solarin:** *On finite right central loops*, Publ. Math. Debrecen, **35** (1988), 261 – 264.

[28] **A. R. T. Solarin:** *On the identities of Bol-Moufang type*, Koungpook Math. J., **28** (1998), 51 – 62.

[29] **A. R. T. Solarin:** *On certain Akivis algebra*, Italian J. Pure Appl. Math. **1** (1997), 85 – 90.

[30] **A. R. T. Solarin and V. O. Chiboka:** *A note on G-loops*, Collections of Scientific Papers of the Faculty of Science Krag., **17** (1995), 17 – 26.

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