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Semigroup, monoid and group models of groupoid identities

Nick C. Fiala

Abstract

In this note, we characterize those groupoid identities that have a (finite) non-trivial (semigroup, monoid, group) model.

1. Introduction

Definition 1.1. A *groupoid* consists of a non-empty set equipped with a binary operation, which we simply denote by juxtaposition. A groupoid G is *non-trivial* if $|G| > 1$, otherwise it is *trivial*. A *semigroup* is a groupoid S that is *associative* ($(xy)z = x(yz)$ for all $x, y, z \in S$). A *monoid* is a semigroup M possessing a *neutral element* $e \in M$ such that $ex = xe = x$ for all $x \in M$ (the letter e will always denote the neutral element of a monoid). A *group* is a monoid G such that for all $x \in G$ there exists an *inverse* x^{-1} such that $x^{-1}x = xx^{-1} = e$. A *quasigroup* is a groupoid Q such that for all $a, b \in Q$, there exist unique $x, y \in Q$ such that $ax = b$ and $ya = b$. A *loop* is a quasigroup possessing a neutral element.

A *groupoid term* is a product of universally quantified variables. A *groupoid identity* is an equation, the left-hand side and right-hand side of which are groupoid terms. By the words *term* and *identity*, we shall always mean groupoid term and groupoid identity, respectively. The letters s and t will always denote terms. We will say that an identity $s = t$ has a (finite) non-trivial *model* if there exists a (finite) non-trivial groupoid G such that $s = t$ is valid in G. We will say that an identity $s = t$ has a (finite) non-trivial (semigroup, monoid, group, quasigroup, loop) model if $s = t$ has a

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(finite) non-trivial model that is a (semigroup, monoid, group, quasigroup, loop).

The question of whether or not an identity has a (finite) non-trivial model is known to be *undecidable* (not answerable by an algorithm) [3]. In this note, we show that the question of whether or not an identity has a (finite) non-trivial (semigroup, monoid, group) model is *decidable*.

2. Results

Lemma 2.1. *An identity is valid in some non-trivial group if and only if it is valid in some non-trivial abelian group.*

Proof. Suppose that the identity $s = t$ is valid in some non-trivial group G. Let a be any non-neutral element of G. Then $s = t$ is valid in a non-trivial cyclic, and hence abelian, subgroup of G containing a. □

Given a term t and a variable x_i, we denote by $o_i(t)$ the number of occurrences of x_i in t. Given an identity $s = t$ and a variable x_i, we denote by d_i the quantity $|o_i(s) - o_i(t)|$. Given an identity $s = t$ in the variables $x_1, x_2, \ldots, x_n$, we denote by g the quantity $\gcd(d_1, d_2, \ldots, d_n)$.

Proposition 2.2. *An identity $s = t$ in the variables $x_1, x_2, \ldots, x_n$ has a non-trivial group model if and only if $g \neq 1$.*

Proof. Suppose $g = 1$. Suppose $s = t$ is valid in some non-trivial group G. By Lemma 2.1, $s = t$ is valid in some non-trivial abelian group H.

Now, in H, $s = t$ is equivalent to

$$x_1^{d_1} x_2^{d_2} \cdots x_n^{d_n} = e.$$

Let $m_1, m_2, \ldots, m_n \in \mathbb{Z}$ be such that $m_1 d_1 + m_2 d_2 + \cdots + m_n d_n = 1$. Then, in H,

$$\begin{aligned} x = x^1 &= x^{m_1 d_1 + m_2 d_2 + \cdots + m_n d_n} = x_1^{m_1 d_1} x_2^{m_2 d_2} \cdots x_n^{m_n d_n} \\ &= (x_1^{m_1})^{d_1} (x_2^{m_2})^{d_2} \cdots (x_n^{m_n})^{d_n} = e, \end{aligned}$$

a contradiction.

Finally, suppose $g \neq 1$. Then $s = t$ is valid in the non-trivial group $\mathbb{Z}_g$. □

As was mentioned before, the question of whether or not an identity has a *finite* non-trivial model is also known to be undecidable [3]. In fact, there exist identities with no non-trivial finite models but that do have infinite models, such as the identity $(((yy)y)x)(((yy)((yy)y))z) = x$ [1].

Corollary 2.3. *An identity has a non-trivial group model if and only if it has a finite non-trivial group model.*

Proof. Suppose $s = t$ has a non-trivial group model. By Proposition 2.2, $g \neq 1$. Then $s = t$ is valid in the finite non-trivial group $\mathbb{Z}_g$. □

Proposition 2.2 with "group" replaced by "loop" or "quasigroup" is false. Indeed, the identity $((xx)x)x = x(xx)$ is valid in the loop below (found with the model-generator Mace4 [2]).

$\cdot$	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	1	2	3	5	0	6	4
2	2	4	5	1	6	3	0
3	3	0	6	4	2	1	5
4	4	3	0	6	5	2	1
5	5	6	1	0	3	4	2
6	6	5	4	2	1	0	3

It seems to be unknown if Corollary 2.3 with "group" replaced by "loop" or "quasigroup" is true.

Given the existence of a non-trivial idempotent ($x^2 = x$) monoid, Proposition 2.2 with "group" replaced by "monoid" is false. However, we now show that the question of whether or not an identity has a (finite) non-trivial monoid model is decidable.

Proposition 2.4. *An identity $s = t$ in the variables $x_1, x_2, \ldots, x_n$ has a non-trivial monoid model if and only if every variable occurs on both sides or $g \neq 1$.*

Proof. Suppose that there exists a variable x that occurs $n > 0$ times on one side of $s = t$ but not at all on the other side. Suppose $g = 1$. Suppose that $s = t$ is valid in some non-trivial monoid M. Substituting e for every variable in $s = t$ besides x results in $x^n = e$. Therefore, every element of M has an inverse and hence M is a group. By Proposition 2.2, M must be trivial, a contradiction.

Suppose that every variable in $s = t$ occurs on both sides. Then $s = t$ is valid in the non-trivial commutative idempotent monoid $(G, \cdot)$, where $G = \{0, 1\}$, $0 \cdot 0 = 0$ and $0 \cdot 1 = 1 \cdot 0 = 1 \cdot 1 = 1$.

Finally, suppose $g \neq 1$. Then $s = t$ is valid in the non-trivial group, and hence monoid, $\mathbb{Z}_g$. □

Corollary 2.5. *An identity has a non-trivial monoid model if and only if it has a non-trivial finite monoid model.*

Proof. Suppose $s = t$ has a non-trivial monoid model. By Proposition 2.4, every variable that occurs in $s = t$ occurs on both sides or $g \neq 1$. If every variable that occurs in $s = t$ occurs on both sides, then $s = t$ is valid in the non-trivial commutative idempotent monoid above. If $g \neq 1$, then $s = t$ is valid in the finite non-trivial group, and hence monoid, $\mathbb{Z}_g$. □

Proposition 2.4 with "monoid" replaced by "semigroup" is false. Indeed, $xy = zu$ is valid in a non-trivial zero semigroup and $xy = x$ ($xy = y$) is valid in a non-trivial left-zero (right-zero) semigroup. Nevertheless, we now show that the question of whether or not an identity has a (finite) non-trivial semigroup model is decidable.

Proposition 2.6. *An identity $s = t$ in the variables $x_1, x_2, \ldots, x_n$ has a non-trivial semigroup model if and only if there are at least two variable occurrences on each side, one side is a variable which is also the left-most or right-most variable on the other side, or $g \neq 1$.*

Proof. Suppose one side of $s = t$ is a variable y. Suppose y is not the left-most or right-most variable on the other side. Suppose $g = 1$. Suppose $s = t$ is valid in some non-trivial semigroup S. Substituting x for every variable in $s = t$ besides y results in $xt(x, y)x = y$ for some (possibly empty) term $t(x, y)$ in the variables x and y.

Now, in S,

$$yt(y, x)(yt(y, z)y) = (yt(y, x)y)t(y, z)y.$$

Therefore,

$$yt(y, x)z = xt(y, z)y.$$

Substituting x for y in the above results in

$$xt(x, x)z = xt(x, z)x = z.$$

Thus, S is a monoid. By Proposition 2.4, S must be trivial, a contradiction.

Suppose that there are at least two variable occurrences on each side of $s = t$. Then $s = t$ is valid in the non-trivial zero semigroup $(G, \cdot)$, where $G = \{0, 1\}$ and $x \cdot y = 0$.

Suppose one side of $s = t$ is a variable which is also the left-most (right-most) variable on the other side. Then $s = t$ is valid in the non-trivial left-zero (right-zero) semigroup below.

$\cdot$	0	1
0	0	0
1	1	1

(

$\cdot$	0	1
0	0	1
1	0	1

)

Suppose $g \neq 1$. Then $s = t$ is valid in the non-trivial group, and hence semigroup, $\mathbb{Z}_g$. □

Corollary 2.7. *An identity has a non-trivial semigroup model if and only if it has a finite non-trivial semigroup model.*

Proof. Suppose $s = t$ has a non-trivial semigroup model. By Proposition 2.6, there are at least two variable occurrences on each side of $s = t$, one side of $s = t$ is a variable which is also the left-most or right-most variable on the other side, or $g \neq 1$. If there are at least two variable occurrences on each side of $s = t$, then $s = t$ is valid in the finite non-trivial zero semigroup above. If one side of $s = t$ is a variable which is also the left-most (right-most) variable on the other side, then $s = t$ is valid in the finite non-trivial left-zero (right-zero) semigroup above. If $g \neq 1$, then $s = t$ is valid in the finite non-trivial group, and hence semigroup, $\mathbb{Z}_g$. □

References

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Department of Mathematics, St. Cloud State University, St. Cloud, MN 56301
E-mail: ncfiala@stcloudstate.edu