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Prime bi-ideals in ternary semigroups

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Abstract

We introduced the notions of prime, semiprime and strongly prime bi-ideals in ternary semigroups. The space of strongly prime bi-ideals is topologized. We characterize different classes of ternary semigroups by the properties of their quasi-ideals and bi-ideals. We also characterize those ternary semigroups for which each bi-ideal is strongly prime.

1. Introduction

Ternary algebraic operations and cubic relations were considered in the 19th century by several mathematicians such as Cayley and Sylvester. Ternary structures and their generalization, the so called n-ary structures, raise certain hopes in view of their possible applications in Physics. Some significant physical applications are given in [1, 2, 11, 10]. Ternary semigroups provide natural examples of ternary algebras.

In [8], Good and Hughes introduced the notion of bi-ideals and in [15], Steinfeld introduced the notion of quasi-ideals in semigroups. In [13] the concepts of prime bi-ideals, strongly prime bi-ideals and semiprime bi-ideals in semigroups is introduced. In [14], Sioson studied some properties of quasi-ideals of ternary semigroups. In [4], Dixit and Dewan studied about the quasi-ideals and bi-ideals of ternary semigroups. Connections of some types of ideals in ternary and n-ary semigroups with the regularity of these semigroups are described in [6]. Applications of ideals to the divisibility theory in ternary and n-ary semigroups and rings one can find in [5].

In this paper we characterized some classes of ternary semigroups by the properties of their quasi-ideals and bi-ideals.

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2. Preliminaries

A ternary semigroup is an algebraic structure $(S, [\ \ ])$ such that S is a non-empty set and $[\ \ ] : S \times S \times S \longrightarrow S$ is a ternary operation satisfying the following associative law:

$$[[abc]de] = [a[bcd]e] = [ab[cde]].$$

For simplicity we will write $[abc]$ as abc.

It is clear that any ordinary semigroup $(S, *)$ induces a ternary semigroup $(S, [\,])$ by putting $[abc] = a * b * c$. But there are ternary semigroups which are not of this form. Connections between ternary semigroups and some ordinary semigroups are described in [3]. Criterion when ternary semigroup has the above form is proved in [7].

An element e in a ternary semigroup S is called *idempotent* if $eee = e$.

If a ternary semigroup S contains an element 0 such that $0ab = a0b = ab0 = 0$ for all $a, b \in S$, then 0 is called a *zero element* of S. If S has no zero then it is easy to adjoin an extra element 0 to form a ternary semigroup with zero. In this case we define $0ab = a0b = ab0 = 0$ for all $a, b \in S$ and $000 = 0$. In this case $S \cup \{0\}$ becomes a ternary semigroup with zero. A non-empty subset T of a ternary semigroup S is called a *ternary subsemigroup* of S if and only if $TTT = T^3 \subseteq T$. A subset T satisfying the identity $TTT = T$ is called an *idempotent subset*. By a *left* (*right, middle*) *ideal* of a ternary semigroup S we mean a non-empty subset A of S such that $SSA \subseteq A$ ($ASS \subseteq A$, $SAS \subseteq A$). By a *two sided ideal*, we mean a subset of S which is both a left and a right ideal of S. If a non-empty subset of S is a left, right and middle ideal of S, then it is called an *ideal* of S. It is clear that every one-sided ideal, middle ideal and two-sided ideal is a ternary subsemigroup. Let X be a non-empty subset of a ternary semigroup S. Then intersection of all left ideals of S containing X is a left ideal of S containing X, furthermore it is the smallest left ideal of S containing X and is called the *left ideal of S generated by X*. It is denoted by $\langle X \rangle_l$. Clearly,

$$\langle X \rangle_l = X \cup SSX,$$

Similarly,

$$\begin{aligned}
\langle X \rangle_r &= X \cup XSS, \\
\langle X \rangle_m &= X \cup SXS \cup SSXSS, \\
\langle X \rangle_t &= X \cup SSX \cup XSS \cup SSXSS, \\
\langle X \rangle &= X \cup XSS \cup SSX \cup SXS \cup SSXSS,
\end{aligned}$$

are the right, middle, two sided ideals , and an ideal of S generated by X, respectively.

An element a in a ternary semigroup S is called *regular* if there exists an element $x \in S$ such that $axa = a$, that is $a \in aSa$. A ternary semigroup S is called *regular* if all its elements are regular.

Definition 1. (cf. [14]) A non-empty subset Q of a ternary semigroup S is called a it quasi-ideal of S if

(i) $(QSS) \cap (SQS) \cap (SSQ) \subseteq Q$,

(ii) $(QSS) \cap (SSQSS) \cap (SSQ) \subseteq Q$.

Every right, left and middle ideal in a ternary semigroup is a quasi-ideal but the converse is not true in general. Every quasi-ideal of a ternary semigroup S is a ternary subsemigroup of S.

Definition 2. (cf. [4]) By a *bi-ideal* of a ternary semigroup S we mean a ternary subsemigroup B of S such that $BSBSB \subseteq B$.

Proposition 1. (cf. [4]) *The intersection of a family of quasi-ideals (bi-ideals) in a ternary semigroup is either empty or a quasi-ideal (bi-ideal).* □

Corollary 1. (cf. [4]) *The intersection of a right ideal R and a left ideal L of a ternary semigroup S is a quasi-ideal of S.* □

Proposition 2. (cf. [4]) *Every quasi-ideal of a ternary semigroup is a bi-ideal.* □

Proposition 3. (cf. [14]) *A ternary semigroup S is regular if and only if $R \cap M \cap L = RML$ for every right ideal R, middle ideal M and left ideal L of S.* □

2. Regular ternary semigroups

Theorem 1. *A commutative ternary semigroup is regular if and only if every its ideal is idempotent.*

Proof. Straightforward. □

Theorem 2. *If every quasi-ideal Q of S is idempotent, then S is a regular ternary semigroup.*

Proof. Let R be a right ideal, M a middle ideal and L a left ideal of S, then $(R \cap M \cap L)$ is a quasi-ideal of S. Since each quasi-ideal is idempotent so,

$$(R \cap M \cap L) = (R \cap M \cap L)^3 \subseteq RML.$$

On the other hand, $RML \subseteq R \cap M \cap L$ always. Thus $RML = R \cap M \cap L$. Hence by Proposition 3, S is a regular ternary semigroup. □

Theorem 3. *For a ternary semigroup S, the following assertions are equivalent:*

(i) S *is regular,*

(ii) $R \cap L = RSL$ *for every right ideal R and every left ideal L of S,*

(iii) $\langle a\rangle_r \cap \langle b\rangle_l = \langle a\rangle_r S \langle b\rangle_l$ *for every $a, b \in S$,*

(iv) $\langle a\rangle_r \cap \langle a\rangle_l = \langle a\rangle_r S \langle a\rangle_l$ *for every $a \in S$.*

Proof. $(i) \to (ii)$ Assume that S is a regular ternary semigroup. Let R and L be right and left ideals of S, respectively. Since $RSL \subseteq RSS \subseteq R$ and $RSL \subseteq SSL \subseteq L$, therefore $RSL \subseteq R \cap L$. Let $a \in R \cap L$, then there exists $x \in S$ such that $a = axa$. As $axa \in RSL$, thus $R \cap L \subseteq RSL$. Hence $R \cap L = RSL$.

$(ii) \to (iii)$ and $(iii) \to (iv)$ are trivial.

$(iv) \to (i)$ Consider $a \in S$, then

$$\begin{aligned} a \in \langle a\rangle_r \cap \langle a\rangle_l = \langle a\rangle_r S \langle a\rangle_l &= (a \cup aSS)S(a \cup SSa) \\ &= aSa \cup aSSSa \cup aSSSa \cup aSSSSSa \subseteq aSa, \end{aligned}$$

which implies $a \in aSa$. So $a = axa$ for some $x \in S$. Hence S is regular. □

Theorem 4. *The following assertions on a ternary semigroup S are equivalent:*

(i) S *is regular,*

(ii) $B = BSB$ *for every bi-ideal of S,*

(iii) $Q = QSQ$ *for every quasi-ideal Q of S.*

Proof. $(i) \to (ii)$ Suppose S is a regular ternary semigroup and let b be any element of B. Then there exists $x \in S$ such that $b = bxb$. As $b = bxb \in BSB$, so $B \subseteq BSB$. Now let $y \in BSB$, then $y = b_1sb_2$ for some $b_1, b_2 \in B$ and $s \in S$. Since S is regular so b_1 can be written as $b_1 = b_1tb_1$ for some $t \in S$, thus $y = b_1sb_2 = b_1tb_1sb_2 \in BSBSB \subseteq B$, which implies $BSB \subseteq B$. Hence $BSB = B$.

$(ii) \to (iii)$ Since every quasi-ideal of S is a bi-ideal, so by (ii), $Q = QSQ$ for every quasi-ideal Q of S.

$(iii) \to (i)$ Suppose $Q = QSQ$ for every quasi-ideal Q of S. Let R be a right ideal, M be a middle ideal and L be a left ideal of S, then $Q = R \cap M \cap L$ is a quasi-ideal. Now $R \cap M \cap L = Q = QSQ = QSQSQ = (R \cap M \cap L)S(R \cap M \cap L)S(R \cap M \cap L) \subseteq RSMSL \subseteq RML$.

Also $RML \subseteq R \cap M \cap L$ always. Therefore $RML = R \cap M \cap L$. Hence by Proposition 3, S is regular. □

Proposition 4. *If B is a bi-ideal of a regular ternary semigroup S and T_1, T_2 are non-empty subsets of S, then BT_1T_2, T_1BT_2 and T_1T_2B are bi-ideals of S.*

Proof. Let S be a regular ternary semigroup, B a bi-ideal of S and T_1, T_2 are non-empty subsets of S. Then,

$$\begin{aligned}(BT_1T_2)(BT_1T_2)(BT_1T_2) &\subseteq B(T_1T_2B)(T_1T_2B)T_1T_2\\ &\subseteq B(SSB)(SSB)T_1T_2 = B(SSBSS)(BT_1T_2)\\ &\subseteq B(SSSSS)BT_1T_2 \subseteq B(SSS)BT_1T_2\\ &\subseteq (BSB)T_1T_2 = BT_1T_2\end{aligned}$$

because in a regular ternary semigroup $B = BSB$. Thus BT_1T_2 is a ternary subsemigroup of S. Also

$$\begin{aligned}(BT_1T_2)S(BT_1T_2)S(BT_1T_2) &= B(T_1T_2S)B(T_1T_2S)BT_1T_2\\ &\subseteq B(SSS)B(SSS)BT_1T_2\\ &\subseteq (BSBSB)T_1T_2 \subseteq BT_1T_2.\end{aligned}$$

Hence BT_1T_2 is a bi-ideal of S.

Similarly, we can show that T_1BT_2, T_1T_2B are bi-ideals of S. □

Corollary 2. *If B_1, B_2 and B_3 are bi-ideals of a regular ternary semigroup S then $B_1B_2B_3$ is a bi-ideal of S.*

Corollary 3. *If Q_1, Q_2, Q_3 are quasi-ideals of a regular ternary semigroup S then $Q_1Q_2Q_3$ is a bi-ideal.*

Theorem 5. *A ternary semigroup in which all bi-ideals are idempotent is regular.*

Proof. Let R be a right ideal, M be a middle ideal and L be a left ideal of S. Then $R \cap M \cap L$ is a bi-ideal. Therefore by the hypothesis

$$R \cap M \cap L = (R \cap M \cap L)^3 = (R \cap M \cap L)(R \cap M \cap L)(R \cap M \cap L) \subseteq RML.$$

Also, $RML \subseteq R \cap M \cap L$ always. Hence $R \cap M \cap L = RML$. Thus by Proposition 3, S is a regular ternary semigroup. □

Theorem 6. *The following assertions are equivalent for a ternary semigroup S:*

(*i*) *S is regular,*

(*ii*) *$I \cap B = BIB$ every middle ideal I and for every bi-ideal B,*

(*iii*) *$I \cap Q = QIQ$ for every middle ideal I and every quasi-ideal Q.*

Proof. $(i) \to (ii)$ Suppose S is a regular ternary semigroup, I a middle ideal and B a bi-ideal of S. Since $BIB \subseteq SIS \subseteq I$ and by Theorem 4, $BIB \subseteq BSB = B$. Therefore $BIB \subseteq I \cap B$. Now let $a \in I \cap B$. Since S is regular, so there exists $x \in S$ such that $a = axa$. Thus we have $a = axa = (axa)xa = a(xax)a \in BIB$ which shows that $I \cap B \subseteq BIB$. Hence $BIB = I \cap B$.

$(ii) \to (iii)$ Since every quasi-ideal of a ternary semigroup S is also a bi-ideal, so by (ii), we have $I \cap Q = QIQ$.

$(iii) \to (i)$ Let Q be a quasi-ideal of S. Then by (iii), we can write $Q = S \cap Q = QSQ$. Hence by Theorem 5, S is regular. □

Theorem 7. *For a ternary semigroup S, the following conditions are equivalent:*

(*i*) *S is regular,*

(*ii*) *$B \cap L \subseteq BSL$ for every bi-ideal B and every left ideal L,*

(*iii*) *$Q \cap L \subseteq QSL$ for every quasi-ideal Q and every left ideal L,*

(*iv*) *$B \cap R \subseteq RSB$ for every bi-ideal B and every right ideal R,*

(*v*) *$Q \cap R \subseteq RSQ$ for every quasi-ideal Q and every right ideal R.*

Proof. $(i) \to (ii)$ Let $a \in B \cap L$. Since S is regular, so there exists $x \in S$ such that $a = axa$. As $a = axa \in BSL$, therefore $B \cap L \subseteq BSL$.

$(ii) \to (iii)$ Since every quasi-ideal of S is a bi-ideal, so by (ii), we have $Q \cap L \subseteq QSL$.

$(iii) \to (i)$ Assume that $Q \cap L \subseteq QSL$, for every quasi-ideal Q and every left ideal L of S. We show that S is regular. Let R be any right ideal of S. Take $Q = R$ the by (iii) $R \cap L \subseteq RSL$, but $RSL \subseteq R \cap L$ always. Hence $RSL = R \cap L$. Thus by Theorem 3, S is regular.

Similarly we can show that $(i) \to (iv) \to (v) \to (i)$. □

Theorem 8. *For a ternary semigroup S, the following conditions are equivalent:*

(i) *S is regular,*

(ii) *$B_1 \cap B_2 \subseteq (B_1SB_2) \cap (B_2SB_1)$ for any bi-ideals B_1 and B_2,*

(iii) *$B \cap Q \subseteq (BSQ) \cap (QSB)$ for any bi-ideal B and quasi-ideal Q,*

(iv) *$B \cap L \subseteq (BSL) \cap (LSB)$ any bi-ideal B and for any left ideal L,*

(v) *$Q \cap L \subseteq (QSL) \cap (LSQ)$ for any left ideal L and quasi-ideal Q,*

(vi) *$R \cap L \subseteq (RSL) \cap (LSR)$ any right ideal R and for any left ideal L,*

(vii) *$B \cap R \subseteq (BSR) \cap (RSB)$ any bi-ideal B and for any right ideal R,*

$(viii)$ *$Q \cap R \subseteq (QSR) \cap (RSQ)$ for any right ideal R and any quasi-ideal Q.*

Proof. $(i) \to (ii)$ Suppose S is a regular ternary semigroup and B_1, B_2 are bi-ideals of S. Let $a \in B_1 \cap B_2$. Then there exists $x \in S$ such that $a = axa$. As $a = axa \in (B_1SB_2)$ and $a = axa \in (B_2SB_1)$, thus $B_1 \cap B_2 \subseteq (B_1SB_2) \cap (B_2SB_1)$.

$(ii) \to (iii)$ Since every quasi-ideal of S is a bi-ideal, therefore by (ii), we have $B \cap Q \subseteq (BSQ) \cap (QSB)$ for any bi-ideal B and for any quasi-ideal Q of S.

$(iii) \to (iv)$ Since every one-sided ideal of S is a quasi-ideal, therefore by (iii), we have $B \cap L \subseteq (BSL) \cap (LSB)$ for any bi-ideal B and for any left ideal L of S.

$(iv) \to (v)$ As every quasi-ideal of S is also a bi-ideal, therefore by (iv), we have $Q \cap L \subseteq (QSL) \cap (LSQ)$, for any left ideal L and for any quasi-ideal Q of S.

$(v) \to (vi)$ Since every one-sides ideal of S is a quasi-ideal, therefore by (v), we have $R \cap L \subseteq (RSL) \cap (LSR)$, for any right ideal R and for any left ideal L of S.

$(vi) \to (i)$ Suppose $R \cap L \subseteq (RSL) \cap (LSR)$, for any right ideal R and for any left ideal L of S. Now (vi) implies $R \cap L \subseteq (RSL) \cap (LSR) \subseteq RSL$. On the other hand, $RSL \subseteq R \cap L$ always. Thus $RSL = R \cap L$. Thus by Theorem 3, S is regular.

Similarly we can show that $(i) \longleftrightarrow (vii) \longleftrightarrow (viii)$. □

3. Weakly regular ternary semigroups

Definition 3. A ternary semigroup S is said to be *right* (resp. *left*) *weakly regular*, if for each $x \in S$, $x \in (xSS)^3$ (resp. $x \in (SSx)^3$).

Every regular ternary semigroup is right (left) weakly regular but the converse is not true.

Lemma 1. *A ternary semigroup S is right weakly regular if and only if $R \cap I = RII$, for every right ideal R and for every two-sided ideal I of S.*

Proof. Suppose S is right weakly regular and $x \in J \cap I$. Since S is right weakly regular, therefore $x \in (xSS)^3$, that is $x = (xs_1t_1)(xs_1t_2)(xs_3t_3)$ for some $s_1, s_2, s_3, t_1, t_2, t_3 \in S$. Thus $x = (xs_1t_1)(xs_1t_2)(xs_3t_3) \in JII$, hence $J \cap I \subseteq JII$. On the other hand, $JII \subseteq J \cap I$ always. So, $J \cap I = JII$.

Conversely, assume that $J \cap I = JII$, for all right ideals J and for all two-sided ideals I of S. We show that S is right weakly regular. Suppose $x \in S$. Let J be the right and I be the two-sided ideal of S generated by x, that is $J = x \cup xSS$, $I = x \cup SSx \cup xSS \cup SSxSS$. Then

$$\begin{aligned}
x \in J \cap I &= JII \\
&= (x \cup xSS)(x \cup SSx \cup xSS \cup SSxSS)(x \cup SSx \cup xSS \cup SSxSS) \\
&= (xx \cup xSSx \cup xxSS \cup xSSxSS \cup xSSx \cup xSSSSx \cup xSSxSS \\
&\qquad \cup xSSSSxSS)(x \cup SSx \cup xSS \cup SSxSS) \\
&= (xx \cup xSSx \cup xxSS \cup xSSxSS)(x \cup SSx \cup xSS \cup SSxSS) \\
&\subseteq x^3 \cup xxSSx \cup x^3SS \cup xxSSxSS \cup xSSxx \cup xSSxSSx \cup xSSxxSS \\
&\qquad \cup xSSxSSxSS.
\end{aligned}$$

Simple calculations shows that in any case $x \in (xSS)^3$. Hence S is right weakly regular. □

Theorem 9. *For a ternary semigroup S, the following conditions are equivalent:*

(i) *S is right weakly regular,*

(ii) *$B \cap I \cap R \subseteq BIR$ for every bi-ideal B, every two-sided ideal I and every right ideal R of S,*

(iii) *$Q \cap I \cap R \subseteq QIR$ for every quasi-ideal Q, every two-sided ideal I and every right ideal R of S.*

Proof. $(i) \rightarrow (ii)$ Let S be a right weakly regular ternary semigroup and $x \in B \cap I \cap J$. Since S is right weakly regular, therefore $x \in (xSS)^3$, that is $x = (xs_1t_1)(xs_2t_2)(xs_3t_3)$ for some $s_1, s_2, s_3, t_1, t_2, t_3 \in S$. Thus

$$x = (xs_1t_1)(xs_2t_2)(xs_3t_3) = x(s_1t_1xs_2t_2)(xs_3t_3) \in BIJ.$$

Hence $B \cap I \cap J \subseteq BIJ$.

$(ii) \rightarrow (iii)$ Since every quasi-ideal of S is a bi-ideal, (ii) implies (iii).

$(iii) \rightarrow (i)$ Let R be a right ideal and I a two sided ideal of S. Take $Q = R$, and $J = I$, then we have $Q \cap I \cap J = R \cap I \cap I = R \cap I$ and $QIJ = RII$. Thus by (iii) it follows that $R \cap I \subseteq RII$. But $RII \subseteq R \cap I$ always. Hence $R \cap I = RII$ and so by Lemma 1, S is right weakly regular. □

Theorem 10. *For a ternary semigroup S the following conditions are equivalent:*

(i) *S is right weakly regular,*

(ii) *$B \cap I \subseteq BII$ for every bi-ideal B and every two-sided ideal I,*

(iii) *$Q \cap I \subseteq QII$ for every quasi-ideal Q and every two-sided ideal I.*

Proof. $(i) \rightarrow (ii)$ Let $x \in B \cap I$, where B is a bi-ideal and I is a two-sided ideal of S. Since S is right weakly regular, therefore $x \in (xSS)^3$. Consequently $x = (xs_1t_1)(xs_2t_2)(xs_3t_3)$ for some $s_1, s_2, s_3, t_1, t_2, t_3 \in S$. Thus $x = x(s_1t_1xs_2t_2)(xs_3t_3) \in BII$. Hence $B \cap I \subseteq BII$.

$(ii) \rightarrow (iii)$ Since every quasi-ideal of S is also a bi-ideal, therefore we have $Q \cap I \subseteq QII$ for every quasi-ideal Q and every two-sided ideal I of S.

$(iii) \rightarrow (i)$ Let R be a right ideal of S and I be a two sided ideal of S. Take $Q = R$, then by hypothesis $R \cap I \subseteq RII$. On the other hand $RII \subseteq R \cap I$ is always true. Thus $R \cap I = RII$, for every right ideal R and for every two-sided ideal I of S. Thus by Lemma 1, S is right weakly regular. □

4. Prime, strongly prime and semiprime bi-ideals

Throughout this section S will be considered as the ternary semigroup with zero.

Definition 4. A bi-ideal B of a ternary semigroup S is called

- *prime* if $B_1B_2B_3 \subseteq B$ implies $B_1 \subseteq B$ or $B_2 \subseteq B$ or $B_3 \subseteq B$ for any bi-ideals B_1, B_2, B_3 of S,

- *strongly prime* if $B_1B_2B_3 \cap B_2B_3B_1 \cap B_3B_1B_2 \subseteq B$ implies $B_1 \subseteq B$ or $B_2 \subseteq B$ or $B_3 \subseteq B$ for any bi-ideals B_1, B_2, B_3 of S,
- *semiprime* if $B_1^3 \subseteq B$ implies $B_1 \subseteq B$ for any bi-ideal B_1 of S.

Remark 1. Every strongly prime bi-ideal of a ternary semigroup S is a prime bi-ideal and every prime bi-ideal is a semiprime bi-ideal. A prime bi-ideal is not necessarily a strongly prime bi-ideal and a semiprime bi-ideal is not necessarily a prime bi-ideal.

Example 1. Let $S = \{0, a, b\}$ and $abc = (a * b) * c$ for all $a, b, c \in S$, where $*$ is defined by the table:

$*$	0	a	b
0	0	0	0
a	0	a	a
b	0	b	b

Then S is a ternary semigroup.

Bi-ideals in S are: $\{0\}$, $\{0, a\}$, $\{0, b\}$ and $\{0, a, b\}$. All bi-ideals are prime and hence semiprime. The prime bi-ideal $\{0\}$ is not strongly prime, because

$$\begin{aligned}&(\{0,a\}\{0,b\}\{0,a,b\}) \cap (\{0,b\}\{0,a,b\}\{0,a\}) \cap (\{0,a,b\}\{0,a\}\{0,b\})\\ &= \{0,a\} \cap \{0,b\} \cap \{0,a,b\} = \{0\},\end{aligned}$$

but neither $\{0, a\}$ nor $\{0, b\}$ nor $\{0, a, b\}$ is contained in $\{0\}$.

Example 2. Let S be a left zero ternary semigroup, that is $xyz = x$ for all $x, y, z \in S$ and let $|S| > 1$. We extend s to $S^0 = S \cup \{0\}$, where $0 \notin S$, by putting $xyz = x$ for $x, y, z \in S$ and $xyz = 0$ in all other cases. Then all subsets B_1, B_2, B_3 of S^0 containing 0 we have $B_1S^0B_1S^0B_1 = B_1$ and $B_1B_2B_3 = B_1$. Thus every subset of S^0 containing 0 is a bi-ideal of S^0 and every bi-ideal of S^0 is prime. If B is a bi-ideal of S^0 such that $|S^0 \backslash B| \geqslant 3$, then B is not strongly prime, since for any distinct elements $a, b, c \in S^0 \backslash B$,

$$\begin{aligned}&(B \cup \{a\})(B \cup \{b\})(B \cup \{c\}) \cap (B \cup \{b\})(B \cup \{c\})(B \cup \{a\})\\ &\cap (B \cup \{c\})(B \cup \{a\})(B \cup \{b\}) = (B \cup \{a\}) \cap (B \cup \{b\}) \cap (B \cup \{c\}) = B\end{aligned}$$

but neither $B \cup \{a\}$ nor $B \cup \{b\}$ nor $B \cup \{c\}$ is contained in B. In particular, $\{0\}$ is a prime bi-ideal of S^0 which is not strongly prime.

Example 3. Let $0 \in S$ and $|S| > 3$. Then S with the ternary operation defined by

$$xyz = \begin{cases} x & \text{if } x = y = z, \\ 0 & \text{otherwise,} \end{cases}$$

is a ternary semigroup with zero. Since for all subsets B_1, B_2, B_3 of S containing 0 we have $B_1SB_1SB_1 = B_1$ and $B_1B_2B_3 = B_1 \cap B_2 \cap B_3$, all these subsets are semiprime bi-ideals.

Note that a semiprime bi-ideal B of S such that $|S \backslash B| \geqslant 3$ is not a prime bi-ideal because for distinct $a, b, c \in S \backslash B$, we have

$$(B \cup \{a\})(B \cup \{b\})(B \cup \{c\}) = (B \cup \{a\}) \cap (B \cup \{b\}) \cap (B \cup \{c\}) = B,$$

but neither $(B \cup \{a\})$ nor $(B \cup \{b\})$ nor $(B \cup \{c\})$ is contained in B. In particular, $\{0\}$ is a semiprime bi-ideal but it is not prime.

It is not difficult to verify that the following proposition is true.

Proposition 5. *The intersection of any family of prime bi-ideals of a ternary semigroup S is a semiprime bi-ideal.* □

5. Irreducible and strongly irreducible bi-ideals

Definition 5. A bi-ideal B of a ternary semigroup S is called *irreducible* (*strongly irreducible*) if $B_1 \cap B_2 \cap B_3 = B$ (resp. $B_1 \cap B_2 \cap B_3 \subseteq B$) implies $B_1 = B$ or $B_2 = B$ or $B_3 = B$ (resp. $B_1 \subseteq B$ or $B_2 \subseteq B$ or $B_3 \subseteq B$) for all bi-ideals B_1, B_2, B_3 of S.

Every strongly irreducible bi-ideal of a ternary semigroup S is an irreducible bi-ideal but the converse is not true in general.

Example 4. Let $S = \{0, 1, 2, 3, 4, 5\}$ and $abc = (a * b) * c$ for all $a, b, c \in S$, where $*$ is defined by the table

*	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	1	1	1	1
2	0	1	2	3	1	1
3	0	1	1	1	2	3
4	0	1	4	5	1	1
5	0	1	1	1	4	5

Then S is a ternary semigroup with bi-ideals: $\{0\}$, $\{0, 1\}$, $\{0, 1, 2\}$, $\{0, 1, 3\}$, $\{0, 1, 4\}$, $\{0, 1, 5\}$, $\{0, 1, 2, 4\}$, $\{0, 1, 3, 5\}$, $\{0, 1, 2, 3\}$, $\{0, 1, 4, 5\}$ and S. Bi-ideals $\{0\}$, $\{0, 1, 2, 4\}$, $\{0, 1, 3, 5\}$, $\{0, 1, 2, 3\}$, $\{0, 1, 4, 5\}$ and S are irreducible. Strongly irreducible are only $\{0\}$ and S.

Proposition 6. *Every strongly irreducible semiprime bi-ideal is strongly prime.*

Proof. Let B be a strongly irreducible semiprime bi-ideal of S. Suppose B_1, B_2 and B_3 are bi-ideals of S such that

$$B_1B_2B_3 \cap B_2B_3B_1 \cap B_3B_1B_2 \subseteq B.$$

Since

$$(B_1 \cap B_2 \cap B_3)(B_1 \cap B_2 \cap B_3)(B_1 \cap B_2 \cap B_3) \subseteq B_1B_2B_3,$$
$$(B_1 \cap B_2 \cap B_3)(B_1 \cap B_2 \cap B_3)(B_1 \cap B_2 \cap B_3) \subseteq B_2B_3B_1,$$
$$(B_1 \cap B_2 \cap B_3)(B_1 \cap B_2 \cap B_3)(B_1 \cap B_2 \cap B_3) \subseteq B_3B_1B_2,$$

we have

$$(B_1 \cap B_2 \cap B_3)^3 \subseteq B_1B_2B_3 \cap B_2B_3B_1 \cap B_3B_1B_2 \subseteq B.$$

But B is semiprime, so $(B_1 \cap B_2 \cap B_3) \subseteq B$.

Also since B is strongly irreducible, so we have either $B_1 \subseteq B$ or $B_2 \subseteq B$ or $B_3 \subseteq B$. Thus B is a strongly prime bi-ideal of S. □

Proposition 7. *Let B be a bi-ideal of S. For any $a \in S \setminus B$ there exists an irreducible bi-ideal I of S such that $B \subseteq I$ and $a \notin I$.*

Proof. Suppose $\Im = \{B_i : i \in I\}$ be the collection of all bi-ideals of S which contains B and does not contain a, then $\Im \neq \emptyset$ because $B \in \Im$. Evidently $\Im$ is partially ordered under inclusion. If Ω is any totally ordered subset of $\Im$ then $\bigcup \Omega$ is a bi-ideal of S containing B and not containing a. Hence by Zorn's lemma, there exists a maximal element I in $\Im$. We show that I is an irreducible bi-ideal of S. Let C, D and E be any three bi-ideals of S such that $I = C \cap D \cap E$. If all of three bi-ideals C, D and E properly contain I then according to the hypothesis $a \in C$, $a \in D$ and $a \in E$. Hence $a \in C \cap D \cap E = I$. This contradicts the fact that $a \notin I$. Thus either $I = C$ or $I = D$ or $I = E$. Hence I is irreducible. □

Theorem 11. *For a regular ternary semigroup S, the following assertions are equivalent:*

- (i) *every bi-ideal of S is idempotent,*
- (ii) $B_1 \cap B_2 \cap B_3 = B_1B_2B_3 \cap B_2B_3B_1 \cap B_3B_1B_2$ *for every bi-ideals of S,*
- (iii) *every bi-ideal of S is semiprime,*
- (iv) *each proper bi-ideal of S is the intersection of all irreducible semiprime bi-ideals of S which contain it.*

Proof. $(i) \to (ii)$ Let B_1, B_2 and B_3 be bi-ideals of S. Then by the hypothesis

$$B_1 \cap B_2 \cap B_3 = (B_1 \cap B_2 \cap B_3)^3 \subseteq B_1 B_2 B_3.$$

Similarly,

$$B_1 \cap B_2 \cap B_3 \subseteq B_2 B_3 B_1 \quad \text{and} \quad B_1 \cap B_2 \cap B_3 \subseteq B_3 B_1 B_2.$$

Thus

$$B_1 \cap B_2 \cap B_3 \subseteq B_1 B_2 B_3 \cap B_2 B_3 B_1 \cap B_3 B_1 B_2 \,. \tag{1}$$

By Corollary 2, $B_1B_2B_3$, $B_2B_3B_1$ and $B_3B_1B_2$ are bi-ideals. Also by Proposition 1, $B_1B_2B_3 \cap B_2B_3B_1 \cap B_3B_1B_2$ is a bi-ideal. Thus by hypothesis

$$\begin{aligned} B_1B_2B_3 \cap B_2B_3B_1 \cap B_3B_1B_2 &= (B_1B_2B_3 \cap B_2B_3B_1 \cap B_3B_1B_2)^3 \\ &\subseteq (B_1B_2B_3)\,(B_3B_1B_2)\,(B_2B_3B_1) \\ &\subseteq (B_1SS)\,(SB_1S)\,(SSB_1) \\ &= B_1(SSS)B_1(SSS)B_1 \subseteq B_1SB_1SB_1 \subseteq B_1. \end{aligned}$$

Similarly,

$$B_1B_2B_3 \cap B_2B_3B_1 \cap B_3B_1B_2 \subseteq B_2$$

and

$$B_1B_2B_3 \cap B_2B_3B_1 \cap B_3B_1B_2 \subseteq B_3.$$

Thus

$$B_1B_2B_3 \cap B_2B_3B_1 \cap B_3B_1B_2 \subseteq B_1 \cap B_2 \cap B_3. \tag{2}$$

Hence from (1) and (2),

$$B_1 \cap B_2 \cap B_3 = B_1B_2B_3 \cap B_2B_3B_1 \cap B_3B_1B_2.$$

$(ii) \to (i)$ Obvious.

$(i) \to (iii)$ Let B and B_1 be any two bi-ideals of S such that $B_1^3 \subseteq B$, then by hypothesis $B_1 = B_1^3 \subseteq B$. Hence every bi-ideal of S is semiprime.

$(iii) \to (iv)$ Let B be a proper bi-ideal of S, then B is contained in the intersection of all irreducible bi-ideals of S which contain B. Proposition 7, guarantees the existence of such irreducible bi-ideals. If $a \notin B$, then there exists an irreducible bi-ideal of S which contains B but does not contain a. Thus B is the intersection of all irreducible bi-ideals of S which contain B.

By hypothesis each bi-ideal is semiprime, so each bi-ideal is the intersection of irreducible semiprime bi-ideals of S which contains it.

$(iv) \to (i)$ Let B be a bi-ideal of a ternary semigroup S. If $B^3 = S$, then clearly B is idempotent. If $B^3 \neq S$, then B^3 is a proper bi-ideal of S containing B^3, so by the hypothesis,

$$B^3 = \bigcap \left\{ B_\alpha : B_\alpha \text{ is irreducible semiprime bi-ideal of } S \text{ containing } B^3 \right\}.$$

Since each B_α is semiprime bi-ideal, $B^3 \subseteq B_\alpha$ implies $B \subseteq B_\alpha$. Therefore $B \subseteq \bigcap B_\alpha = B^3$ implies $B \subseteq B^3$, but $B^3 \subseteq B$. Hence $B^3 = B$. □

Proposition 8. *If each bi-ideal of a ternary semigroup S is idempotent, then a bi-ideal B of S is strongly irreducible if and only if B is strongly prime.*

Proof. Suppose that a bi-ideal B is strongly irreducible and let B_1, B_2, B_3 are bi-ideals of S such that

$$B_1B_2B_3 \cap B_2B_3B_1 \cap B_3B_1B_2 \subseteq B.$$

By Theorem 11,

$$B_1 \cap B_2 \cap B_3 = B_1B_2B_3 \cap B_2B_3B_1 \cap B_3B_1B_2,$$

so we have

$$B_1 \cap B_2 \cap B_3 \subseteq B.$$

Since B is strongly irreducible so, either $B_1 \subseteq B$ or $B_2 \subseteq B$ or $B_3 \subseteq B$. Thus B is strongly prime.

On the other hand, if B is strongly prime and $B_1 \cap B_2 \cap B_3 \subseteq B$ for some bi-ideals B_1, B_2 and B_3 of S, then, in view of Theorem 11,

$$B_1B_2B_3 \cap B_2B_3B_1 \cap B_3B_1B_2 \subseteq B,$$

whence we conclude either $B_1 \subseteq B$ or $B_2 \subseteq B$ or $B_3 \subseteq B$. Therefore B is strongly irreducible. □

Next we characterize those ternary semigroups for which each bi-ideal is strongly irreducible and also those ternary semigroups in which each bi-ideal is strongly prime.

Theorem 12. *Each bi-ideal of a regular ternary semigroup S is strongly prime if and only if every bi-ideal of S is idempotent and the set of bi-ideals of S is totally ordered by inclusion.*

Proof. Suppose that each bi-ideal of S is strongly prime, then each bi-ideal of S is semiprime. Thus by Theorem 11, every bi-ideal of S is idempotent. We show that the set of bi-ideals of S is totally ordered by inclusion. Let B_1 and B_2 be any two bi-ideals of S, then by Theorem 11,

$$B_1 \cap B_2 = B_1 \cap B_2 \cap S = B_1 B_2 S \cap B_2 S B_1 \cap S B_1 B_2.$$

Thus

$$B_1 B_2 S \cap B_2 S B_1 \cap S B_1 B_2 \subseteq B_1 \cap B_2.$$

As each bi-ideal is strongly prime, therefore $B_1 \cap B_2$ is strongly prime bi-ideal. Hence either $B_1 \subseteq B_1 \cap B_2$ or $B_2 \subseteq B_1 \cap B_2$ or $S \subseteq B_1 \cap B_2$. Now, if $B_1 \subseteq B_1 \cap B_2$, then $B_1 \subseteq B_2$; if $B_2 \subseteq B_1 \cap B_2$, then $B_2 \subseteq B_1$; if $S \subseteq B_1 \cap B_2$, then $B_1 = S = B_2$. Thus set of bi-ideals of S is totally ordered under inclusion.

Conversely, assume that every bi-ideal of S is idempotent and the set of bi-ideals of S is totally ordered under inclusion. We show that each bi-ideal of S is strongly prime. Let B, B_1, B_2 and B_3 be bi-ideals of S such that

$$B_1 B_2 B_3 \cap B_2 B_3 B_1 \cap B_3 B_1 B_2 \subseteq B.$$

Since every bi-ideal of S is idempotent so by Theorem 11,

$$B_1 \cap B_2 \cap B_3 \subseteq B.$$

Since the set of all bi-ideals of S is totally ordered under inclusion so for B_1, B_2, B_3 we have the following six possibilities:

(1) $B_1 \subseteq B_2,\ B_2 \subseteq B_3,\ B_1 \subseteq B_3$, (2) $B_1 \subseteq B_2,\ B_3 \subseteq B_2,\ B_1 \subseteq B_3$,
(3) $B_1 \subseteq B_2,\ B_3 \subseteq B_2,\ B_3 \subseteq B_1$, (4) $B_2 \subseteq B_1,\ B_2 \subseteq B_3,\ B_1 \subseteq B_3$,
(5) $B_2 \subseteq B_1,\ B_3 \subseteq B_2,\ B_3 \subseteq B_1$, (6) $B_2 \subseteq B_1,\ B_3 \subseteq B_1,\ B_2 \subseteq B_3$.

In these cases we have

(1) $B_1 \cap B_2 \cap B_3 = B_1$, (2) $B_1 \cap B_2 \cap B_3 = B_1$; (3) $B_1 \cap B_2 \cap B_3 = B_3$,
(4) $B_1 \cap B_2 \cap B_3 = B_2$, (5) $B_1 \cap B_2 \cap B_3 = B_3$, (6) $B_1 \cap B_2 \cap B_3 = B_2$.

Thus either $B_1 \subseteq B$ or $B_2 \subseteq B$ or $B_3 \subseteq B$, which proves that B is strongly prime. □

Theorem 13. *If the set of bi-ideals of a regular ternary semigroup S is totally ordered, then every bi-ideal of S is idempotent if and only if each bi-ideal of S is prime.*

Proof. Suppose every bi-ideal of S is idempotent. Let B, B_1, B_2, B_3 be bi-ideals of S such that
$$B_1B_2B_3 \subseteq B.$$
As in the proof of the previous theorem we obtain $B_1 \subseteq B_2$, $B_2 \subseteq B_3$, $B_1 \subseteq B_3$, whence we conclude $B_1B_1B_1 \subseteq B_1B_2B_3 \subseteq B$, i.e., $B_1^3 \subseteq B$. By Theorem 11, B is a semiprime bi-ideal, so $B_1 \subseteq B$. Similarly for other cases we have $B_2 \subseteq B$ or $B_3 \subseteq B$.

Conversely, assume that every bi-ideal of S is prime. Since the set of bi-ideals of S is totally ordered under inclusion, so the concepts of primeness and strongly primeness coincide. Hence by Theorem 13, every bi-ideal of S is idempotent. □

Theorem 14. *For a ternary semigroup S the following are equivalent:*

(i) *the set of bi-ideals of S is totally ordered under inclusion,*

(ii) *each bi-ideal of S is strongly irreducible,*

(iii) *each bi-ideal of S is irreducible.*

Proof. $(i) \to (ii)$ Let $B_1 \cap B_2 \cap B_3 \subseteq B$ for some bi-ideals of S. Since the set of bi-ideals of S is totally ordered under inclusion, therefore either $B_1 \cap B_2 \cap B_3 = B_1$ or B_2 or B_3. Thus either $B_1 \subseteq B$ or $B_2 \subseteq B$ or $B_3 \subseteq B$. Hence B is strongly irreducible.

$(ii) \to (iii)$ If $B_1 \cap B_2 \cap B_3 = B$ for some bi-ideals of S, then $B \subseteq B_1$, $B \subseteq B_2$ and $B \subseteq B_3$. On the other hand by hypothesis we have, $B_1 \subseteq B$ or $B_2 \subseteq B$ or $B_3 \subseteq B$. Thus $B_1 = B$ or $B_2 = B$ or $B_3 = B$. Hence B is irreducible.

$(iii) \to (i)$ Suppose each bi-ideal of S is irreducible. Let B_1, B_2 be bi-ideals of S, then $B_1 \cap B_2$ is also a bi-ideal of S. Since $B_1 \cap B_2 \cap S = B_1 \cap B_2$, the irreducibility of $B_1 \cap B_2$ implies that either $B_1 = B_1 \cap B_2$ or $B_2 = B_1 \cap B_2$ or $S = B_1 \cap B_2$, i.e., either $B_1 \subseteq B_2$ or $B_2 \subseteq B_1$ or $B_1 = B_2$. Hence the set of bi-ideals of S is totally ordered under inclusion. □

Let $\mathcal{B}$ be the family of all bi-ideals of S and $\mathcal{P}$ – the family of all proper strongly prime bi-ideals of S. For each $B \in \mathcal{B}$ we define
$$\Theta_B = \{J \in \mathcal{P} : B \nsubseteq J\},$$
$$\Im(\mathcal{P}) = \{\Theta_B : B \text{ is a bi-ideal of } S\}.$$

Theorem 15. *If S is ternary semigroup with the property that every bi-ideal of S is idempotent then $\Im(\mathcal{P})$ forms a topology on the set $\mathcal{P}$.*

Proof. As $\{0\}$ is a bi-ideal of S, so $\Theta_{\{0\}} = \{J \in \mathcal{P} : \{0\} \nsubseteq J\} = \emptyset$ because 0 belong to every bi-ideal. Since S is a bi-ideal of S, we have $\Theta_S = \{J \in \mathcal{P} : S \nsubseteq J\} = \mathcal{P}$ because $\mathcal{P}$ is the collection of all proper strongly prime bi-ideals in S. Thus $\emptyset$ and $\mathcal{P}$ belongs to $\Im(\mathcal{P})$.

Let $\{\Theta_{B_\alpha} : \alpha \in I\} \subseteq \Im(\mathcal{P})$. Then

$$\bigcup_{\alpha\in I} \Theta_{B\alpha} = \{J \in \mathcal{P} : B_\alpha \nsubseteq J \text{ for some } \alpha \in I\} = \{J \in \mathcal{P} : \widehat{\bigcup_{\alpha\in I} B_\alpha} \nsubseteq J\},$$

which is equal to $\Theta_{\widehat{\bigcup_{\alpha\in I} B_\alpha}} \in \Im(\mathcal{P})$, where $\widehat{\bigcup_{\alpha\in I} B_\alpha}$ means the bi-ideal of S generated by $\bigcup_{\alpha\in I} B_\alpha$.

Let Θ_{B_1} and Θ_{B_2} be arbitrary two elements from $\Im(\mathcal{P})$. We show that $\Theta_{B_1} \cap \Theta_{B_2} \in \Im(\mathcal{P})$. If $J \in \Theta_{B_1} \cap \Theta_{B_2}$, then $J \in \mathcal{P}$, $B_1 \nsubseteq J$ and $B_2 \nsubseteq J$. Suppose that $B_1 \cap B_2 = B_1 \cap B_2 \cap S \subseteq J$. By Theorem 11, we have $B_1B_2S \cap B_2SB_1 \cap SB_1B_2 \subseteq J$. Since J is a strongly prime bi-ideal, therefore either $B_1 \subseteq J$ or $B_2 \subseteq J$ ($S \nsubseteq J$ because J is a proper bi-ideal of S), which is a contradiction. Hence $B_1 \cap B_2 \nsubseteq J$, i.e., $J \in \Theta_{B_1\cap B_2}$. Thus $\Theta_{B_1} \cap \Theta_{B_2} \subseteq \Theta_{B_1\cap B_2}$.

On the other hand if $J \in \Theta_{B_1\cap B_2}$, then $J \in \mathcal{P}$ and $B_1 \cap B_2 \nsubseteq J$, which means that $B_1 \nsubseteq J$ and $B_2 \nsubseteq J$. Therefore, $J \in \Theta_{B_1}$ and $J \in \Theta_{B_2}$, i.e., $J \in \Theta_{B_1} \cap \Theta_{B_2}$. Hence $\Theta_{B_1\cap B_2} \subseteq \Theta_{B_1} \cap \Theta_{B_2}$. Thus $\Theta_{B_1\cap B_2} = \Theta_{B_1} \cap \Theta_{B_2}$, so $\Theta_{B_1} \cap \Theta_{B_2} \in \Im(\mathcal{P})$. This proves that $\Im(\mathcal{P})$ is a topology on $\mathcal{P}$. □

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