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Simple hyper K-algebras

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Abstract

In this note we define the notion of simple hyper K-algebras and give some examples of simple hyper K-algebras. Then we investigate (weak) hyper K-ideals, normal hyper K-algebras and commutative hyper K-ideals.

1. Introduction

The study of BCK-algebra was initiated by K. Iséki [3] in 1966 as a generalization of concept of the set-theoretic difference and propositional calculus. Since the many researches worked in this area. Hyper structures (called also multialgebras) were introduced in 1934 by F. Marty [5] at the 8^{th} congress of Scandinavian Mathematicians. Around the 40 years several authors worked on hyper groups, specially in France and United States, but also in Italy, Russia, Japan and Iran.

Hyper structures have many applications to several sectors of both pure and applied sciences. Recently Y. B. Jun et al. [4] introduced and studied hyper BCK-algebras which are generalization of BCK-algebras. R. A. Borzooei and M. M. Zahedi [1, 10] constructed the hyper K-algebras , (weak) hyper K-ideals and defined simple hyper K-algebras of order 3. T. Roodbari and M. M. Zahedi [8] defined 9 different types of commutative hyper K-ideals. In this paper we define the notion of simple hyper K-algebras and give some examples of simple hyper K-algebras. Then we investigate (weak) hyper K-ideals, normal hyper K-algebras and commutative hyper K-ideals.

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2. Preliminaries

Definition 2.1. Let H be a nonempty set and " $\circ$ " be a *hyperoperation* on H, that is " $\circ$ " is a function from $H \times H$ to the family $\mathcal{P}^*(H)$ of all nonempty subsets of H. Then H is called a *hyper K-algebra* if it contains a constant "0" and satisfies the following axioms:

(HK1) $(x \circ z) \circ (y \circ z) < x \circ y$,

(HK2) $(x \circ y) \circ z = (x \circ z) \circ y$,

(HK3) $x < x$,

(HK4) $x < y, \;\; y < x \longrightarrow x = y$,

(HK5) $0 < x$,

where $x < y$ is defined by $0 \in x \circ y$ and for every $A, B \subseteq H$, $A < B$ means that there are $a \in A$ and $b \in B$ such that $a < b$. By $A \circ B$ we denote the union of all $a \circ b$ such that $a \in A$, $b \in B$.

Theorem 2.2. *Let $(H, \circ, 0)$ be a hyper K-algebra. Then for all $x, y, z \in H$ and for all nonempty subsets A, B and C of H the following hold*:

(i) $x \circ y < z \longleftrightarrow x \circ z < y$, (ii) $(x \circ z) \circ (x \circ y) < y \circ z$,

(iii) $x \circ (x \circ y) < y$, (iv) $x \circ y < x$,

(v) $A \subseteq B \longrightarrow A < B$, (vi) $x \in x \circ 0$,

(vii) $(A \circ C) \circ (A \circ B) < B \circ C$, $(viii)$ $(A \circ C) \circ (B \circ C) < A \circ B$,

(ix) $A \circ B < C \Leftrightarrow A \circ C < B$, (x) $A \circ B < A$.

Definition 2.3. Let I be a nonempty subset of a hyper K-algebra $(H, \circ, 0)$ and $0 \in I$. Then I is called

(i) a *weak hyper K-ideal* of H if $x \circ y \subseteq I$ and $y \in I$ imply $x \in I$,

(ii) a *hyper K-ideal* of H if $x \circ y < I$ and $y \in I$ imply that $x \in I$.

Definition 2.4. A nonempty subset I of H such that $0 \in I$ is called a *commutative hyper K-ideal of*

- *type* 1, if $((x \circ y) \circ z) \bigcap I \neq \emptyset$ and $z \in I$ imply $(x \circ (y \circ (y \circ x))) \subseteq I$,
- *type* 2, if $((x \circ y) \circ z) \bigcap I \neq \emptyset$ and $z \in I$ imply $(x \circ (y \circ (y \circ x))) \bigcap I \neq \emptyset$,
- *type* 3, if $((x \circ y) \circ z) \bigcap I \neq \emptyset$ and $z \in I$ imply $(x \circ (y \circ (y \circ x))) < I$,
- *type* 4, if $((x \circ y) \circ z) \subseteq I$, $z \in I$ imply $(x \circ (y \circ (y \circ x))) \subseteq I$,
- *type* 5, if $((x \circ y) \circ z) \subseteq I$ and $z \in I$ imply $(x \circ (y \circ (y \circ x))) \bigcap I \neq \emptyset$,
- *type* 6, if $((x \circ y) \circ z) \subseteq I$ and $z \in I$ imply $(x \circ (y \circ (y \circ x))) < I$,
- *type* 7, if $((x \circ y) \circ z) < I$, $z \in I$ imply $(x \circ (y \circ (y \circ x))) \subseteq I$,
- *type* 8, if $((x \circ y) \circ z) < I$ and $z \in I$ imply $(x \circ (y \circ (y \circ x))) \bigcap I \neq \emptyset$,
- *type* 9, if $((x \circ y) \circ z) < I$ and $z \in I$ imply $(x \circ (y \circ (y \circ x))) < I$.

Definition 2.5. An element a of a hyper K-algebra $(H, \circ, 0)$ is called a *hyper atom* if $x < a$ implies $x = 0$ or $x = a$. By $A(H)$ we denote the set of all hyper atoms of H. If in H there exists an element e such that $x < e$ for all $x \in H$, then H is called a *bounded hyper K-algebra*.

Definition 2.6. A hyper K-algebra $(H, \circ, 0)$ in which for all $x, y \in H$, $x < y$ implies $x \in y \circ (y \circ x)$ is called *quasi-commutative*. A hyper K-algebra satisfying the identity $x \circ (x \circ y) = y \circ (y \circ x)$ for all $x, y \in H$ is called *commutative*.

Theorem 2.7. *If $(H, \circ, 0)$ is a quasi-commutative hyper K-algebra, then the hyper K-ideal $\{0\}$ is a commutative hyper K-ideal of type* 9 and 6.

Definition 2.8. Let $(H, \circ, 0)$ be a hyper K-algebra and S be a nonempty subset of H. Then the sets

$${}_{l1}S = \{x \in H \mid a < (a \circ x), \forall a \in S\}, \quad {}_{l2}S = \{x \in H \mid a \in (a \circ x), \forall a \in S\},$$
$$S_{r1} = \{x \in H \mid x < (x \circ a), \forall a \in S\}, \quad S_{r2} = \{x \in H \mid x \in (x \circ a), \forall a \in S\}$$

are called *left hyper stabilizers of type* 1 (type 2, respectively) and *right hyper stabilizer of type* 1 (type 2, respectively).

In the case $S = \{s\}$, for simplicity, we will write ${}_{li}s$ and s_{ri} instead of ${}_{li}\{s\}$ and ${}_{ri}\{s\}$.

Definition 2.9. A hyper K-algebra $(H, \circ, 0)$ is called a *left* (*right*) *hyper normal of type i* if ${}_{li}a$ (respectively a_{ri}) is a hyper K-ideal of H for any $a \in H$ and $i = 1, 2$. If H is both left and right hyper normal K-algebra of type i, then H is called a *hyper normal K-algebra of type i*.

3. Simple hyper K-algebra

Definition 3.1. A hyper K-algebra $(H, \circ, 0)$ is called *simple* if for all distinct elements $a, b \in H - \{0\}$ we have $a \not< b$ and $b \not< a$.

Theorem 3.2. *Let H be a nonempty set and $0 \in H$. Define a hyper operation " $\circ$ " on H by putting*

$$x \circ y = \begin{cases} \{x\} & \text{if} \quad x \neq y, \ y = 0, \\ \{x, y\} & \text{if} \quad x \neq y, \ y \neq 0, \\ \{0, x\} & \text{if} \quad x = y, \end{cases}$$

for all $x, y \in H$. Then $(H, \circ, 0)$ is a simple hyper K-algebra.

Proof. Since axioms (HK3), (HK4) and (HK5) are obvious, we verify only (HK1) and (HK2). For this we consider the following cases:

Case (i). $x \neq y$, $x \neq z$ and $y = z = 0$. Then
$(x \circ z) \circ (y \circ z) = \{x\} < \{x\} = x \circ y$ and $(x \circ y) \circ z = (x \circ z) \circ y = \{x\}$.

Case (ii). $x \neq y$, $x \neq z$, $z \neq 0$ and $y = 0$. Then
$(x \circ z) \circ (y \circ z) = \{x, z\} < \{x\} = x \circ y$ and $(x \circ y) \circ z = (x \circ z) \circ y = \{x, z\}$.

Case (iii). $x \neq y$, $x \neq z$, $y \neq z$, $y \neq 0$ and $z \neq 0$. Then
$(x \circ z) \circ (y \circ z) = \{0, x, y, z\} < \{x, y\} = x \circ y$ and $(x \circ y) \circ z = \{x, y, z\} = (x \circ z) \circ y$.

Case (iv). $x \neq y$, $y \neq z$, $x = z$, $y = 0$ and $z \neq 0$. Then
$(x \circ z) \circ (y \circ z) = \{0, x\} < \{x\} = x \circ y$ and $(x \circ y) \circ z = (x \circ z) \circ y = \{0, x\}$.

Case (v). $x \neq y$, $x \neq z$, $y \neq z$, $z = 0$ and $y \neq 0$. Then
$(x \circ z) \circ (y \circ z) = \{x, y\} < \{x, y\} = x \circ y$ and $(x \circ y) \circ z = (x \circ z) \circ y = \{x, y\}$.

Case (vi). $x \neq y$, $x \neq z$, $y = z$, $y \neq 0$ and $z \neq 0$. Then
$(x \circ z) \circ (y \circ z) = \{0, x, y\} < \{x, y\} = x \circ y$ and $(x \circ y) \circ z = (x \circ z) \circ y = \{0, x, y\}$.

Case (vii). $x \neq y$, $y \neq z$, $y \neq 0$ and $x = z = 0$. Then
$(x \circ z) \circ (y \circ z) = \{0, y\} < \{0, y\} = x \circ y$ and $(x \circ y) \circ z = (x \circ z) \circ y = \{0, y\}$.

Case $(viii)$. $x \neq y$, $y \neq z$, $x = z$, $y \neq 0$ and $z \neq 0$. Then
$(x \circ z) \circ (y \circ z) = \{0, x, y\} < \{x, y\} = x \circ y$ and $(x \circ y) \circ z = (x \circ z) \circ y = \{0, x, y\}$.

Case (ix). $x \neq z$, $y \neq z$, $x = y$ and $z = 0$. Then
$(x \circ z) \circ (y \circ z) = \{x\} < \{0, x\} = x \circ y$ and $(x \circ y) \circ z = (x \circ z) \circ y = \{0, x\}$.

Case (x). $x \neq z$, $y \neq z$, $x = y$ and $z \neq 0$. Then
$(x \circ z) \circ (y \circ z) = \{0, x, z\} < \{0, x\} = x \circ y$ and $(x \circ y) \circ z = (x \circ z) \circ y = \{0, x, z\}$.

Case (xi). $x = z = y$. Then
$(x \circ z) \circ (y \circ z) = \{0, x\} < \{0, x\} = x \circ y$ and $(x \circ y) \circ z = (x \circ z) \circ y = \{0, x\}$.
Therefore $(H, \circ, 0)$ is a simple hyper K-algebra. □

Corollary 3.3. *A hyper K-algebra defined in Theorem* 3.2 *is commutative and normal of types* 1 *and* 2.

Proof. The commutativity is obvious. Also $a_{ri} =_{li} a = H$ for all $a \in H$ and $i = 1, 2$. □

Example 3.4. Consider the following two hyper K-algebras defined on $H = \{0, 1, 2, 3\}$:

$\circ$	0	1	2	3
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0,2,3\}$
1	$\{1\}$	$\{0,1\}$	$\{1,2,3\}$	$\{1,2,3\}$
2	$\{2\}$	$\{2\}$	$\{0,2,3\}$	$\{2\}$
3	$\{3\}$	$\{3\}$	$\{3\}$	$\{0,3\}$

$\circ$	0	1	2	3
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{0\}$	$\{0\}$
2	$\{2\}$	$\{2\}$	$\{0\}$	$\{2\}$
3	$\{3\}$	$\{1,2\}$	$\{0,1\}$	$\{0,2\}$

The first hyper K-algebra is simple, the second is not simple, because $3 < 2$.

It is not difficult to see that the following theorem is true.

Theorem 3.5. *A hyper K-algebra is simple if and only if it contains only hyper atoms.* □

Theorem 3.6. *For a simple hyper K-algebra the following statements hold.*

(i) $a \circ 0 = \{a\}$ *for all* $a \in H - \{0\}$,

(ii) $a \in a \circ b$ *for all distinct elements* $a, b \in H$,

(iii) $H - \{a\} \subseteq H \circ a$ *for all* $a \in H$,

(iv) $a \in b \circ c \longleftrightarrow c \in b \circ a$ *for distinct elements* $a, c \in H$ *and* $b \in H - \{0\}$,

(v) $x < x \circ a \longleftrightarrow x \in x \circ a$ *for all* $a, x \in H$,

(vi) $A < A \circ b \longleftrightarrow A \cap (A \circ b) \neq \emptyset$ *for all* $b \in H$ *and* $\emptyset \neq A \subseteq H$,

(vii) $(x \circ y) \circ z < x \circ (y \circ z)$ *for all* $x, y, z \in H$,

$(viii)$ *If* $0 \in I \subseteq H$, *then* $A \circ B < I \longleftrightarrow (A \circ B) \cap I \neq \emptyset$ *for all nonempty subsets* A *and* B *of* H.

Proof. (i) We have $a \in a \circ 0$. Now let $b \in a \circ 0$. Then $0 \in (a \circ 0) \circ b = (a \circ b) \circ 0$. Thus there is $t \in a \circ b$ such that $0 \in t \circ 0$ i.e., $t < 0$. Hence $t = 0$ and so $a < b$. Since H is simple and $a \in H - \{0\}$, then $a = b$. Therefore $a \circ 0 = \{a\}$.

(ii) If $a = 0$, then it is clear that $0 \in 0 \circ b$, for all $b \in H$. Now let $a, b \in H, a \neq 0$ and $a \neq b$. Since by Theorem 2.2(iv) $a \circ b < a$, then there is $t \in a \circ b$ such that $t < a$. Thus $t = 0$ or $t = a$. Hence $a \neq b$ and $a \neq 0$ imply that $t \neq 0$. Therefore $t = a$ and so $a \in a \circ b$.

(iii) Let $x \in H - \{a\}$. Then $x \neq a$ and so by (ii) we have $x \in x \circ a$. Therefore $x \in H \circ a$.

(iv) Let $a \in b \circ c$. Then $0 \in (b \circ c) \circ a = (b \circ a) \circ c$. Thus there exists $t \in b \circ a$ such that $0 \in t \circ c$ and so $t < c$. Hence $t = 0$ or $t = c$. Since $b \neq a$ and $b \neq 0$, then $t \neq 0$. So $t = c$. Therefore $c \in b \circ a$. The proof of the converse statement is similar.

(v) Let $x < x \circ a$. Then there exists $t \in x \circ a$ such that $x < t$. Thus $x = 0$ or $x = t$. If $x = 0$, then by (HK5), $0 \in 0 \circ a$. If $x = t$, then $x \in x \circ a$. Conversely, let $x \in x \circ a$. Then by Theorem 2.2(v), $x < x \circ a$.

(vi) Let $A \neq \emptyset$ and $A < A \circ b$. Then there exists $a \in A$ and $t \in A \circ b$ such that $a < t$. Thus $a = 0$ or $a = t$. If $a = 0$, then $0 \in A \cap A \circ b$. If $a = t$, then $a \in A \cap A \circ b$. Therefore $A \cap A \circ b \neq \emptyset$. The proof of the converse statement is obvious.

(vii) If $x = y$ or $x = z$, then $0 \in (x \circ y) \circ z$. So $(x \circ y) \circ z < x \circ (y \circ z)$. Now let $x \neq y$ and $x \neq z$. Then by (ii), $x \in (x \circ y) \cap (x \circ z)$. Thus $x \in x \circ z \subseteq (x \circ y) \circ z$. If $y = z$, then $0 \in y \circ z$ and so $x \in x \circ (y \circ z)$. Hence $(x \circ y) \circ z < x \circ (y \circ z)$. If $y \neq z$, then by (ii), $y \in y \circ z$, so $x \in x \circ y \subseteq x \circ (y \circ z)$. Therefore $(x \circ y) \circ z < x \circ (y \circ z)$.

$(viii)$ Let $0 \in I$ and $A \circ B < I$. Then there exists $t \in A \circ B$ and $i \in I$ such that $t < i$. So $t = 0$ or $t = i$. If $t = 0$, then $0 \in (A \circ B) \cap I$. If $t = i$, then $i \in (A \circ B) \cap I$. Therefore $(A \circ B) \cap I \neq \emptyset$. The converse statement is clear. □

Corollary 3.7. *A simple hyper K-algebra is normal of type* 1 *if and only if it is normal of type* 2. □

Theorem 3.8. *In simple hyper K algebras every subset containing* 0 *is a weak hyper K-ideal.*

Proof. Let $0 \in A \subseteq H$, $x \circ y \subseteq A$ and $y \in A$. If $x = y$, then $x \in A$. If $x \neq y$, then by Theorem 3.6(ii), $x \in x \circ y \subseteq A$ and so $x \in A$. □

Corollary 3.9. *Every hyper K-subalgebra of a simple hyper K-algebra is a weak hyper K-ideal.* □

Since by Theorem 3.6(v), we have ${}_{l1}A =_{l2} A$ and $A_{r1} = A_{r2}$, for all nonempty subset $A \subseteq H$, in the sequel we will write ${}_lA$ instead of ${}_{l1}A$ and A_r instead of A_{r1}.

Corollary 3.10. *In simple hyper K-algebras A_r and ${}_lA$ are weak hyper K-ideals for any nonempty subset A of H.* □

Definition 3.11. A hyper K-algebra H is called *left* (*right*) *weak normal of type i* if ${}_{il}a$ (respectively a_{ri}) is a weak hyper K-ideal of H for any $a \in H$.

Theorem 3.12. *Every simple hyper K-algebra is a left* (*right*) *weak normal K-algebra of type $i = 1, 2$.* □

Theorem 3.13. *Let H be a simple hyper K-algebra and let $a \neq 0$. Then $H - \{a\}$ is a hyper K-ideal of H if and only if $|a \circ x| = 1$ for all $x \in H - \{a\}$.*

Proof. Let $H-\{a\}$ be a hyper K-ideal and on the contrary, let there exists $x \in H-\{a\}$ such that $|a \circ x| > 1$. Since $a \in a \circ x$, then there is $z \in H-\{a\}$ such that $z \in a \circ x$. Thus $a \circ x < H-\{a\}$. Since $H-\{a\}$ is a hyper K-ideal, then $a \in H-\{a\}$, which is a contradiction. Therefore $|a \circ x| = 1$, for all $x \in H-\{a\}$.

Conversely, let $|a \circ x| = 1$, for all $x \in H-\{a\}$. Since by Theorem 3.6(ii), $a \in a \circ x$, for all $x \in H-\{a\}$, then $a \circ x = \{a\}$. Thus $a \circ x \not< H-\{a\}$, for all $x \in H-\{a\}$. Therefore $H-\{a\}$ is a hyper K-ideal. □

Theorem 3.14. *Let $\emptyset \neq A \subseteq H$ and $T = \{a \in A \,|\, a \notin a \circ a\}$.*

(1) *If $T = \emptyset$, then A_r and ${}_lA$ are hyper K-ideals of H.*

(2) *If $T \neq \emptyset$ and $|a \circ x| = 1$ for all $a \in T$ and $x \in H-\{a\}$, then A_r and ${}_lA$ are hyper K-ideals of H.*

Proof. (1) By Theorem 3.6(ii) $A_r = \{x \in H \,|\, x \in x \circ a \ \forall a \in A\} = H$. Thus A_r is a hyper K-ideal.

(2) $A_r = H - T = \bigcap_{a \in T}(H-\{a\})$. So, by Theorem 3.13, A_r is a hyper K-ideal. □

The following example shows that the converse of Theorem 3.14(2) is not true in general. The condition "$|a \circ x| = 1$ for all $x \in H-\{a\}$" in Theorem 3.14(ii) is necessary.

Example 3.15. Consider the hyper K-algebra

$\circ$	0	1	2	3
0	$\{0\}$	$\{0\}$	$\{0,2\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1,2\}$	$\{1\}$
2	$\{2\}$	$\{2\}$	$\{0\}$	$\{2\}$
3	$\{3\}$	$\{3\}$	$\{2,3\}$	$\{0\}$

and $A = \{1,2\}$. Then $T = \{1,2\}$ and $A_r = \{0,3\}$ is a hyper K-ideal, but $2 \in H-\{1\}$, $|1 \circ 2| = 2$. For $A = \{1\}$, we see that $T = \{1\}$ and $A_r = \{0,2,3\}>$ But A_r is not a hyper K-ideal, because $|1 \circ 2| = 2 \neq 1$.

As a consequence of Theorems 3.13 and 3.14. we obtain

Corollary 3.16. *Let $a \neq 0$ be an element of a simple hyper K-algebra H.*

(a) *If $a \in a \circ a$, then a_r and ${}_la$ are a hyper K-ideals of H.*

(b) *If $a \notin a \circ a$, then a_r and ${}_la$ are hyper K-ideals of H if and only if $|a \circ x| = 1$ for all $x \in H-\{a\}$.*

As a consequence of the above results we obtain

Corollary 3.17. *A simple hyper K-algebra H such that $a \in a \circ a$ for every $a \in H$ is right (left) normal of type $i = 1, 2$.* □

Corollary 3.18. *In a simple hyper K-algebra all sets of the form $\{0, a\}$ are hyper K-ideals.* □

Corollary 3.19. *A bounded simple hyper K-algebra has at most two elements.*

4. Commutative hyper K-ideals

Directly from the definition of commutative hyper K-ideals and Theorem 3.6 it follows that in simple hyper K-algebras commutative hyper K-ideals of types 1 and 7 coincides. Similarly, commutative hyper K-ideals of types 2, 3, 8 and 9. Also 5 and 6.

Theorem 4.1. *A simple hyper K-algebra is quasi-commutative.*

Proof. Let $x < y$. Then $x = 0$ or $x = y$. If $x = 0$, then $0 \in y \circ y \subseteq y \circ (y \circ 0)$. If $x = y$, then $y \in y \circ 0 \subseteq y \circ (y \circ y)$. Therefore $x \in y \circ (y \circ x)$. □

Corollary 4.2. *In any simple hyper K-algebra, $I = \{0\}$ is a commutative hyper K-ideal of type $i = 2, 3, 5, 6, 8, 9$.*

Proof. The proof follows from Theorems 4.1 and 2.7. □

Theorem 4.3. *If $a \circ a = \{0\}$ holds for all elements of a simple hyper K-algebra, then $I = \{0\}$ is its commutative hyper K-ideal of type 4.*

Proof. Let $(x \circ y) \circ z \subseteq I$ and $z \in I$. Then $x \circ y \subseteq (x \circ y) \circ 0 \subseteq I$ and so $x < y$. Thus $x = 0$ or $x = y$. If $x = 0$, then $x \circ (y \circ (y \circ x)) = 0 \circ (y \circ (y \circ 0)) = 0 \circ (y \circ y) = 0 \circ 0 = I$. If $x = y$, then $y \circ (y \circ (y \circ y)) = y \circ y = I$. Therefore I is a commutative hyper K-ideal of type 4. □

Remark 4.4. The hyper K-algebra defined by the table

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0,2\}$
1	$\{1\}$	$\{0,2\}$	$\{1,2\}$
2	$\{2\}$	$\{2\}$	$\{0\}$

proves that the condition "$a \circ a = \{0\}$ for all $a \in H$" in the above theorem is necessary. Indeed, $1 \circ 1 \neq \{0\}$ and $I = \{0\}$ is not a commutative hyper K-ideal of type 4, because $(0 \circ 1) \circ 0 = I$, while $0 \circ (1 \circ (1 \circ 0)) = \{0, 2\} \not\subseteq I$.

Theorem 4.5. *In a simple hyper K-algebra $I = \{0\}$ is a commutative hyper K-ideal of type 7 (and 1) if and only if $a \circ a = \{0\}$ for all $a \in H$.*

Proof. Let $I = \{0\}$ be a commutative hyper K-ideal of type 7. Then $(y \circ y) \circ 0 < I$ and $0 \in I$ imply that $y \circ y \subseteq y \circ (y \circ 0) \subseteq y \circ (y \circ (y \circ y)) \subseteq I$. Thus $y \circ y = \{0\}$, for all $y \in H$. The proof of the converse statement is similar to the proof of Theorem 4.4. □

Theorem 4.6. *In a simple hyper K-algebra H the set $I = H - \{a\}$ is a commutative hyper K-ideal of type 6 (and 5) for any $a \neq 0$.*

Proof. Let $(x \circ y) \circ z \subseteq I$ and $z \in I$. If $x = y$, then $0 \in x \circ (y \circ (y \circ x))$ and so $x \circ (y \circ (y \circ x)) < I$. If $x \neq y$, then $x \in x \circ 0 \subseteq x \circ (y \circ y) \subseteq x \circ (y \circ (y \circ x))$. Now we show that $x \neq a$. On the contrary let $x = a$. Then $x \neq z$ and so by Theorem 3.6(ii), $x \in x \circ z \subseteq (x \circ y) \circ z \subseteq I$, which is a contradiction. Hence $x \neq a$ implies that $x \circ (y \circ (y \circ x)) < I$. □

Theorem 4.7. *Let a be a non-zero element of a simple hyper K-algebra H such that $|a \circ x| = 1$ for all $x \in H - \{a\}$. Then $I = H - \{a\}$ is a commutative hyper K-ideal of type 9 (and 2, 3, 8).*

Proof. Let $(x \circ y) \circ z < I$ and $z \in I$. If $x = y$, then $x \circ (y \circ (y \circ x)) < I$. For $x \neq y$ we consider two cases: (i) $x \neq a$, (ii) $x = a$. In the first case we have $x \in x \circ (y \circ (y \circ x))$ and so $x \circ (y \circ (y \circ x)) < I$. in the second, from $|a \circ y| = |a \circ z| = 1$ it follows $\{a\} = a \circ z = (a \circ y) \circ z < I$. Thus there exists $t \in I$ such that $a < t$. So $a = 0$ or $a = t$, which is impossible. Therefore I is a commutative hyper K-ideal of type 9. □

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