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The action of G_2^2 on $PL(F_p)$

Qaiser Mushtaq and Nasir Siddiqui

Abstract

Γ_3 is a copy of unique circuit-free connected graph all of whose vertices have degree 3, called cubic tree. The group $G_2^2 = \langle x, y, t : x^2 = t,\ y^3 = t^2 = (yt)^2 = 1\rangle$, is one of the seven finitely presented isomorphism types of subgroups of the full automorphism group $\mathrm{Aut}(\Gamma_3)$ of Γ_3. These seven groups act arc-transitively on the arcs of Γ_3 with a finite vertex stabilizer. In this paper we have found a condition on p such that the action of G_2^2 on the projective line over the finite field, $PL(F_p)$, always yields the subgroups of the alternating groups of degree $p+1$. We have shown also that the action of G_2^2 on $PL(F_p)$ is transitive.

1. Introduction

A cubic tree Γ_3 is a copy of unique circuit-free connected graph all of whose vertices have degree 3. Djoković and Miller [1] have proved that there are seven groups act arc-transitively on the arcs of Γ_3 with a finite vertex stabilizer. The group

$$G_2^2 = \langle x, y, t : x^2 = t,\ y^3 = t^2 = (yt)^2 = 1\rangle$$

is one of these seven finitely presented isomorphism types of subgroups of the full automorphism group $\mathrm{Aut}(\Gamma_3)$ of Γ_3.

Γ_3 can be constructed by the group G_2^2 as follows.
Let $\Omega = \{gH : g \in G_2^2\}$ be the collection of all distinct left cosets of the subgroup

$$H = \langle y, t : y^3 = t^2 = (yt)^2 = 1\rangle$$

of G_2^2 in G_2^2. Two cosets g_1H and g_2H can be joined by an edge if and only if $g_1^{-1}g_2 \in HxH$. Thus vertex H is joined to xH, yxH and y^2xH, whereas xH is joined to H, $xyxH$ and xy^2xH and so on as shown in the Figure 1.

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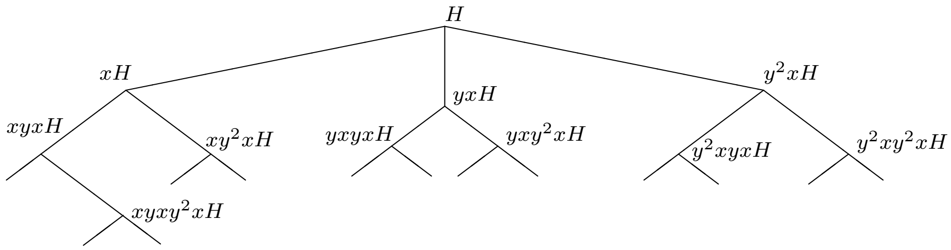

Figure 1.

In fact there is a one to one correspondence between the vertices of Γ_3 and all the reduced words in x and y (and y^2), which are different from identity, which end in x. The elements of G_2^2 induce automorphisms of Γ_3 by left multiplication. For example, the multiplication of y fixes vertex H and rotate other neighbours of vertex H, whereas multiplication of x interchanges H by xH, and the other neighbours of H with the other neighbours of xH and so on as shown in the Figure 2.

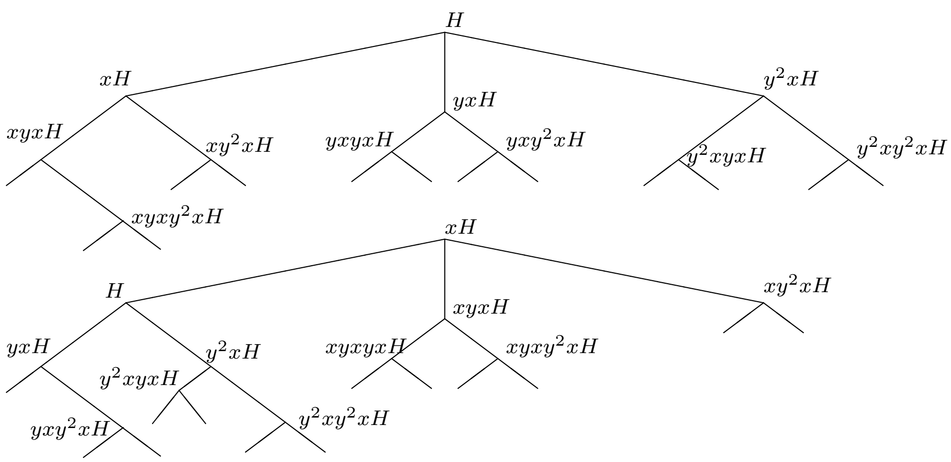

Figure 2.

In particular, action of G_2^2 is transitive on the vertices of Γ_3 and is sharply transitive on its arcs(ordered edges). In other words, the action of G_2^2 is arc-regular on Γ_3, that is, the stabilizer of each arc in G_2^2 is the identity. Of course, the cubic tree has many more automorphisms then these. Indeed, given any path $(v_0,v_1,\dots,v_{n-1},v_n)$ of length n in Γ_3, there are automorphisms fixing each vertex v_i on this path and interchanging the other two vertices adjacent to v_n, it follows that Γ_3 is highly arc-transitive, its full automorphism group is transitive on paths of length n, for all $n \geqslant 0$.

Now clearly the stabilizer (in full automorphism group) of any given vertex is infinite. On the other hand, there are subgroups which act transitively on the arcs of Γ_3 but which have a finite vertex stabilizer, for example, in the G_2^2 the stabilizer of the vertex H is the subgroup H itself of order 6. Up to isomorphism, there are only seven such subgroups and they are:

$$G_1 = \langle x, y : x^2 = y^3 = 1\rangle,$$
$$G_2^1 = \langle x, y, t : x^2 = y^3 = t^2 = (xt)^2 = (yt)^2 = 1\rangle,$$
$$G_2^2 = \langle x, y, t : x^2 = t,\ y^3 = t^2 = (yt)^2 = 1\rangle,$$
$$G_3 = \langle x, y, t, q : x^2 = y^3 = t^2 = q^2 = 1, tq = qt, ty = yt, qyq = y^{-1}, xt = qx\rangle,$$
$$G_4^1 = \langle x, y, t, q, r : x^2 = y^3 = t^2 = q^2 = r^2 = 1,\ tq = qt,\ tr = rt,\ rq = tqr,\ y^{-1}ty = q,\ y^{-1}qy = tq,\ ryr = y^{-1},\ xt = tx,\ xq = rx\rangle,$$
$$G_4^2 = \langle x, y, t, q, r : y^3 = t^2 = q^2 = r^2 = 1,\ x^2 = t,\ tq = qt,\ tr = rt,\ rq = tqr,\ y^{-1}ty = q,\ y^{-1}qy = tq,\ ryr = y^{-1},\ xt = tx,\ xq = rx\rangle,$$
$$G_5 = \langle x, y, t, q, r, s : x^2 = y^3 = t^2 = q^2 = r^2 = s^2 = 1,\ tq = qt,\ tr = rt,\ ts = st,\ rq = qr,\ qs = sq,\ sr = tqrs,\ ty = yt,\ y^{-1}qy = r,\ y^{-1}ry = tqr,\ xt = qx,\ xr = sx\rangle.$$

The group G_2^2 is generated by the linear fractional transformations $x(z) = \frac{z+i}{iz+1}$, $y(z) = \frac{z-1}{z}$ and $t(z) = \frac{1}{z}$, which satisfy the relations $y^3 = t^2 = (yt)^2 = 1, x^2 = t$. In [4], Q. Mushtaq and I. Ali have shown that G_2^2 is generated by x, y, t and $x^2 = t, y^3 = t^2 = (yt)^2 = 1$ are the defining relations.

The group G_2^2 acts on the projective line over the finite field, $PL(F_p)$, provided p is prime and $p-1$ is a perfect square in F_p. These primes are known as Pythagorean primes. In this short note, by p we shall mean a Pythagorean prime. The action of G_2^2 on $PL(F_p)$ results into the permutation group $G = \langle \overline{x}, \overline{y} : \overline{x}^4 = \overline{y}^3 = (\overline{xy})^k = 1\rangle$, which is homomorphic image of $\Delta(3,4,k)$. When $k = 1$, G is trivial group and when $k = 2$, the group G is isomorphic to the triangle group $\Delta(3,4,2)$, which is symmetric group S_4. If $k \geqslant 3$, G is homomorphic image of an infinite triangle group $\Delta(3,4,k)$. If $p \equiv 1 (\bmod 8)$ then G is a simple subgroup of an alternating group A_{p+1}, and isomorphic to $PSL(2,p)$ because the order G is equal to $|PSL(2,p)| = \frac{p(p-1)(p+1)}{2}$. These results can be verified with the help of *GAP*. The following table gives orders of various groups corresponding to some values of the Pythagorean prime p.

$p \equiv 1(\text{mod}8)$	k	$\text{Order}(G) = \frac{p(p-1)(p+1)}{2}$
17	9	2448
41	21	34440
73	37	194472
89	15	352440
97	49	456288
113	56	721392
137	68	1285608
193	48	3594432

If p is not congruent to $1(\text{mod}\,8)$ then G is a subgroup of symmetric group S_{p+1} and the order G is $p(p-1)(p+1)$.

$p \not\equiv 1(\text{mod}\,8)$	k	$\text{Order}(G) = p(p-1)(p+1)$
5	6	120
13	14	2184
29	28	24360
37	36	50616
53	52	148824
61	62	226920
101	34	1030200
109	108	1294920
149	148	3307800
157	158	3869736

Theorem 1. *The action of G_2^2 on $PL(F_p)$, where p is the Pythagorean prime, gives a permutation group G. If $p \equiv 1(\text{mod}\,8)$ then G is a subgroup of A_{p+1}.*

Proof. Note that the group G is generated by permutations $\overline{x}$ and $\overline{y}$ where $\overline{x}$ is a product of cycles each of length 4 and $\overline{y}$ is a product of cycles each of length 3. Also since $\overline{y}$ is a product of cycles of length 3, each cycle can be decomposed into an even number of transpositions. Thus implying that $\overline{y}$ is an even permutation. In the decomposition of the permutation $\overline{x}$, each cycle can be reduced into odd number of transpositions. Let N represent number of cycles in the permutation $\overline{x}$. If N is even then $\overline{x}$ is even also. Since $\overline{x}$ has $\frac{p-1}{4}$ cycles, so $N = \frac{p-1}{4}$. Now if $p \equiv 1(\text{mod}\,8)$ then there exists an integer m such that $p = 8m + 1$, and therefore $N = 2m$. Thus $\overline{x}$ is even, implies that G is generated by two even permutations $\overline{x}$ and $\overline{y}$. Hence G is always a subgroup of A_{p+1}. □

2. Higman's Coset diagrams

The idea of coset diagrams for modular group has been propounded and used by G. Higman and Q. Mushtaq in [2] and the transitivity has been discussed in [3].

An action of G_2^2 on $PL(F_p)$ can be represented by a coset diagram. The group G_2^2 is generated by the linear fractional transformations $x(z) = \frac{z+i}{iz+1}$, $y(z) = \frac{z-1}{z}$, and $t(z) = \frac{1}{z}$, which satisfy the relations $y^3 = t^2 = (yt)^2 = 1$, $x^2 = t$. A coset diagram for the group G_2^2 is defined as follows. Since the generator x has order 4, so the 4-cycles of x are represented by twisted squares, with the convention that x permutes their vertices counter-clockwise. The generator y has order 3, so the 3-cycles of y are denoted by doted edges permuting counter-clockwise. Fixed points of x and y, if they exist, are denoted by heavy dots. The generator t is an involution and therefore it is represented by symmetry along a vertical line of axis passing through the coset diagram.

For example, the action of G_2^2 on $PL(F_{17})$, is depicted by the following coset diagram.

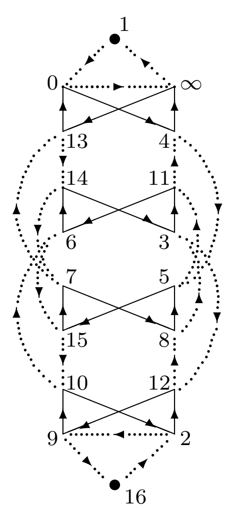

Figure 3.

According to Figure 3, in the coset diagram we begin walking along the path by starting from the vertex labelled as 1. The path $y^2x^{-1}yx^{-1}y^2x^{-1}y^2$ ends at $p-1 = 16$. Thus there exists a word $y^2x^{-1}yx^{-1}y^2x^{-1}y^2$ which connects 1 with the vertex $p-1$, that is $(1)(y^2x^{-1}yx^{-1}y^2x^{-1}y^2) = 16$. Similarly we can connect any two vertices of this coset diagram by a word. Hence the action of G_2^2 on $PL(F_{17})$ is transitive.

Theorem 2. *Let p be the Pythagorean prime. Then G_2^2 acts transitively on $PL(F_p)$.*

Proof. Since the action of G_2^2 on $PL(F_p)$ yields a permutation group G generated by $\overline{x}$ and $\overline{y}$ in whose coset diagram we can always start our walk from the vertex labelled by 1 and end at the vertex labelled by $p-1$ as shown in the Figure 4. In this coset diagram, 4-cycles of x are represented by the four sides of a twisted square, the 3-cycles of y are represented by a triangle with broken edges, whose vertices are permuted counter-clockwise. The fixed points of x and y are represented by heavy dots.

Next we wish to show that the action of G_2^2 on $PL(F_p)$ is transitive for all Pythagorean prime p. Let w be a word connecting 1 with $p-1$, that is, for:

p	$(1)w = p-1$
5	$y^2x^{-1}y^2$
13	$y^2x^{-1}y^2x^{-1}y^2$
17	$y^2x^{-1}yx^{-1}y^2x^{-1}y^2$
29	$y^2x^{-1}yx^{-1}yx^{-1}y^2$
37	$y^2x^{-1}y^2x^{-1}yx^{-1}y^2x^{-1}y^2$
41	$y^2x^{-1}yx^{-1}yx^{-1}y^2x^{-1}y^2x^{-1}y^2$
53	$y^2x^{-1}yx^{-1}yx^{-1}y^2x^{-1}yx^{-1}y^2$
61	$y^2x^{-1}yx^{-1}y^2x^{-1}yx^{-1}y^2$
73	$y^2x^{-1}yx^{-1}yx^{-1}y^2x^{-1}yx^{-1}y^2x^{-1}y^2$
89	$y^2x^{-1}yx^{-1}yx^{-1}y^2x^{-1}yx^{-1}y^2x^{-1}y^2$
97	$y^2x^{-1}yx^{-1}yx^{-1}yx^{-1}y^2x^{-1}y^2x^{-1}y^2x^{-1}yx^{-1}yx^{-1}yx^{-1}y^2$

For, we show that there exists a path between 1 and $p-1$. We begin from 1 and apply y^2 on it to reach ∞. Next we apply x^{-1} on ∞ to reach $k = \sqrt{p-1}$, which is the right top vertex of first twisted square. Similarly, we apply a suitable y^ϵ on $\sqrt{p-1}$, where $\epsilon = \pm 1$, to reach the right top vertex of another twisted square. We again apply x^{-1} and a suitable y^ϵ to reach the right top vertex of any other twisted square. We continue in this way so that after a finite number of steps eventually we reach the vertex $p-1$. That is $(1)y^2x^{-1}y^{\epsilon}x^{-1}y^{\epsilon}x^{-1}y^{\epsilon}\ldots x^{-1}y^{\epsilon} = p-1$.

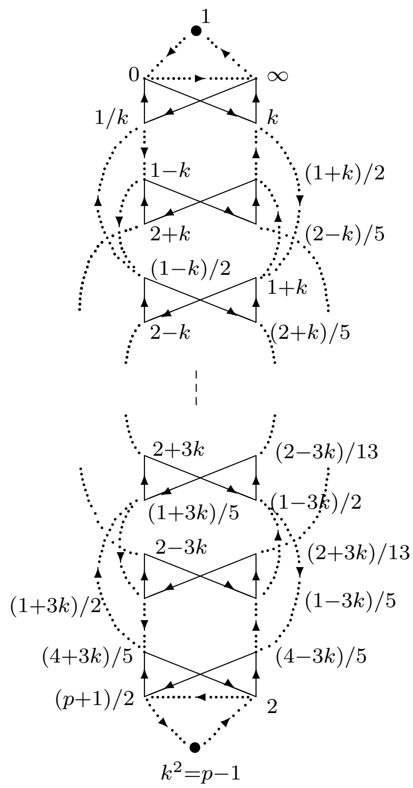

Figure 4.

This shows that the coset diagram is connected. Hence the action is transitive. □

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