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Fuzzy regular congruence relations on hyper BCK-algebras

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Abstract

In this manuscript, by considering the notion of fuzzy regular congruence relation on a hyper BCK-algebra, we construct a quotient hyper BCK-algebra and then we state and prove some related theorems. Finally, we state and prove isomorphism theorems on that structure.

1. Introduction

The study of BCK-algebras was initiated by Y. Imai and K. Iséki [7] in 1966, as a generalization of the concept of set-theoretic difference and propositional calculi. Since then a great deal of literature has been produced on the theory of BCK-algebras.

The hyperstructure theory (called also multialgebras) was introduced in 1934 by F. Marty [11] at the 8th congress of Scandinavian Mathematiciens. Around the 40's, several authors worked on hypergroups, especially in France and in the United States, but also in Italy, Russia and Japan. Hyperstructures have many applications to several sectors of both pure and applied sciences.

In [10] Y. B. Jun et al. applied the hyperstructures to BCK-algebras, and introduced the notion of a hyper BCK-algebra which is a generalization of BCK-algebra, and investigated some related properties. The notion of regular congruence relation on hyper BCK-algebras have been introduced by R. A. Borzooei et al [6]. In [1], [4] and [5], the authors studied the fuzzy set theory on hyper BCK-algebras and defined the notion of a fuzzy

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congruence relation on a hyper BCK-algebra. Now, in this paper, we follow the references and we obtain some results as mentioned in the abstract.

2. Preliminaries

Definition 2.1. By a *hyper BCK-algebra* we mean a non-empty set H endowed with a hyperoperation "$\circ$" and a constant 0 satisfying the following axioms:

(HK1) $(x \circ z) \circ (y \circ z) \ll x \circ y$,
(HK2) $(x \circ y) \circ z = (x \circ z) \circ y$,
(HK3) $x \circ H \ll \{x\}$,
(HK4) $x \ll y$ and $y \ll x$ imply $x = y$,

for all $x, y, z \in H$, where $x \ll y$ is defined by $0 \in x \circ y$ and for every $A, B \subseteq H$, $A \ll B$ is defined by $\forall a \in A$, $\exists b \in B$ such that $a \ll b$.

Proposition 2.2. [10] *In any hyper BCK-algebra H for all $x, y, z \in H$ the following hold:*

(i) $x \ll x$,
(ii) $0 \circ x = \{0\}$,
(iii) $x \circ y \ll x$,
(iv) $x \circ 0 = \{x\}$.

Definition 2.3. A non-empty subset I of hyper BCK-algebra H is said to be a (*weak*) *hyper BCK-ideal* if $(x \circ y \subseteq I)$ $x \circ y \ll I$ and $y \in I$ imply $x \in I$.

Definition 2.4. Let $(H_1, \circ_1)$ and $(H_2, \circ_2)$ be two hyper BCK-algebras and $f : H_1 \longrightarrow H_2$ be a function. Then, f is called

- a *homomorphism*, if $f(x \circ_1 y) = f(x) \circ_2 f(y)$, for all $x, y \in H_1$,
- an *isomorphism*, if f is a one-to-one and onto homomorphism.

Note. From now on, in this paper, H denotes a hyper BCK-algebra.

Definition 2.5. Let Θ be a binary relation on H and $A, B \subseteq H$. Then,

(i) $A \Theta B$ means that, there exist $a \in A$ and $b \in B$ such that $a \Theta b$,

(ii) $A \bar{\Theta} B$ means that, for all $a \in A$ there exists $b \in B$ and for all $b \in B$ there exists $a \in A$ such that $a \Theta b$,

(iii) Θ is called *left* (resp. *right*) *compatible* if $x\Theta y$ implies that $a \circ x \bar{\Theta} a \circ y$ $(x \circ a \bar{\Theta} y \circ a)$, for all $a, x, y \in H$,

(iv) Θ is called a *congruence* if it is left and right compatible,

(v) Θ is called *regular* if $x \circ y \Theta \{0\}$ and $y \circ x \Theta \{0\}$ imply $x \Theta y$, for all $x, y \in H$.

Theorem 2.6. [6] *Let Θ be a regular congruence relation on H, $I = [0]_\Theta$ and $H/I = \{I_x : x \in H\}$, where $I_x = [x]_\Theta$. Then H/I with hyperoperation "$\circ$" and hyperorder "$\ll$" which is defined as follows,*

$$I_x \circ I_y = \{I_z : z \in x \circ y\} \quad , \quad I_x \ll I_y \longleftrightarrow I \in I_x \circ I_y$$

is a hyper BCK-algebra which is called "quotient hyper BCK-algebra".

Definition 2.7. Let μ be a fuzzy subset of H. Then for all $t \in [0, 1]$, the *level subset* μ_t of H is defined by $\mu_t = \{x \in H : \mu(x) \geqslant t\}$. Moreover, μ satisfies the *sup property* (*inf property*), if for each non-empty subset T of X there exists $x_0 \in T$ such that $\mu(x_0) = \sup\limits_{x \in T} \mu(x)$ $\;(\mu(x_0) = \inf\limits_{x \in T} \mu(x))$.

Definition 2.8. Let $f : H_1 \longrightarrow H_2$ be a homomorphism of hyper BCK-algebras and μ be a fuzzy subset of H_2. Then fuzzy subset $f^{-1}(\mu)$ of H_1 is defined by $f^{-1}(\mu)(x) = \mu(f(x))$, for all $x \in H_1$.

Definition 2.9. Let H be a hyper BCK-algebra. A function $\rho : H \times H \to [0, 1]$ is called a *fuzzy relation* on H. A fuzzy relation ρ on H is said to be a *fuzzy equivalence relation* if for all $x, y \in H$

$\rho(x, x) = \sup\limits_{(y,z) \in H^2} \rho(y, z)$, (Fuzzy reflexive)

$\rho(y, x) = \rho(x, y)$, (Fuzzy symmetric)

$\rho(x, y) \geqslant \sup_{z \in H} \min(\rho(x, z), \rho(z, y))$, (Fuzzy transitive).

Definition 2.10. Let ρ be a fuzzy equivalence relation on H. Then ρ is said to be a

- *fuzzy left compatible* if for all $u \in a \circ x$ there exists $v \in a \circ y$ and for all $v \in a \circ y$ there exists $u \in a \circ x$ such that $\rho(u, v) \geqslant \rho(x, y)$, for all $a, x, y \in H$.

- *fuzzy right compatible* if for all $z \in x \circ a$ there exists $w \in y \circ a$ and for all $w \in y \circ a$ there exists $z \in x \circ a$ such that $\rho(z, w) \geqslant \rho(x, y)$, for all $a, x, y \in H$.

- *fuzzy congruence relation* if ρ is fuzzy left and fuzzy right compatible.

Theorem 2.11. [5] *Let ρ be a fuzzy relation on H. If ρ is a fuzzy congruence relation on H then for all $t \in [0,1]$, $\rho_t = \{(x,y) \in H \times H : \rho(x,y) \geqslant t\} \neq \emptyset$, is a congruence relation on H. Conversely, if ρ satisfies the sup property and for all $t \in [0,1]$, $\rho_t \neq \emptyset$ is a congruence relation on H, then ρ is a fuzzy congruence relation on H.*

Notation: By $\mathcal{F}_R(H)$, $\mathcal{F}_E(H)$ and $\mathcal{F}_C(H)$ we mean respectively, the set of all fuzzy relations, fuzzy equivalence relations and fuzzy congruence relations on H.

3. Quotient structures

Definition 3.1. $\rho \in \mathcal{F}_R(H)$ is called *fuzzy regular* if for all $x, y \in H$,

$$\rho(x,y) \geqslant \min\left(\sup_{a \in x \circ y} \rho(a,0), \sup_{b \in y \circ x} \rho(b,0)\right).$$

Theorem 3.2. *If $\rho \in \mathcal{F}_R(H)$ is fuzzy regular, then each $\rho_t \neq \emptyset$ is a regular relation on H. Conversely, if ρ satisfies the sup property and each $\rho_t \neq \emptyset$ is a regular relation on H, then ρ is fuzzy regular on H.*

Proof. ($\Leftarrow$) Let for all $s \in [0,1]$, $\rho_s \neq \emptyset$ be a regular relation on H. We first show that ρ is a fuzzy equivalence relation. Let $t = \sup_{(y,z) \in H^2} \rho(y,z)$. Since, ρ satisfies the sup property, then $\rho_t \neq \emptyset$. Now, since ρ_t is a reflexive relation, then $(x,x) \in \rho_t$ and so $\rho(x,x) \geqslant t$ for all $x \in H$. Hence,

$$\rho(x,x) \leqslant \sup_{(y,z) \in H^2} \rho(y,z) = t \leqslant \rho(x,x)$$

and so $\rho(x,x) = \sup_{(y,z) \in H^2} \rho(y,z)$. Thus, ρ is a fuzzy reflexive relation. Moreover, it is easy to check that ρ is a fuzzy symmetric and fuzzy transitive relation. Therefore, ρ is a fuzzy equivalence relationon H. Now, let

$$t = \min\left(\sup_{a \in x \circ y} \rho(a,0), \sup_{b \in y \circ x} \rho(b,0)\right).$$

Since, ρ satisfies the sup property, then there exist $a_0 \in x \circ y$ and $b_0 \in y \circ x$ such that $\rho(a_0, 0) = \sup_{a \in x \circ y} \rho(a,0) \geqslant t$ and $\rho(b_0, 0) = \sup_{b \in y \circ x} \rho(b,0) \geqslant t$

and so $a_0\rho_t 0$ and $b_0\rho_t 0$. This implies that $x \circ y\rho_t\{0\}$ and $y \circ x\rho_t\{0\}$. Since, ρ_t is a regular relation, then $x\rho_t y$ and so

$$\rho(x,y) \geqslant t = \min\left(\sup_{a\in x\circ y}\rho(a,0), \sup_{b\in y\circ x}\rho(b,0)\right).$$

Therefore, ρ is a fuzzy regular relation on H.

($\Rightarrow$) Let ρ be a fuzzy regular relation on H, $t \in [0,1]$ and $\rho_t \neq \emptyset$. We first show that ρ_t is an equivalence relation on H. Since, ρ_t is a non-empty subset of H, there exists $y, z \in H$ such that $(y,z) \in \rho_t$ and so $\rho(y,z) \geqslant t$. Since, for all $x \in H$, $\rho(x,x) = \sup_{(y,z)\in H^2}\rho(y,z) \geqslant t$, then $(x,x) \in \rho_t$ and so ρ_t is a reflexive relation. It is easy to check that ρ_t is a symmetric and transitive relation on H. Therefore, ρ_t is an equivalence relation on H. Now, let $x \circ y\rho_t\{0\}$ and $y \circ x\rho_t\{0\}$, for $x, y \in H$. Then, there exist $a \in x \circ y$ and $b \in y \circ x$ such that $a\rho_t 0$ and $b\rho_t 0$. Since, ρ is a fuzzy regular relation on H, then

$$\rho(x,y) \geqslant \min\left(\sup_{u\in x\circ y}\rho(u,0), \sup_{v\in y\circ x}\rho(v,0)\right) \geqslant \min(\rho(a,0), \rho(b,0)) \geqslant t$$

and so $x\rho_t y$. Hence, ρ_t is a regular relation on H. □

Definition 3.3. Let $\rho \in \mathcal{F}_R(H)$. Then for all $x \in H$, the fuzzy subset $\mu_x : H \longrightarrow [0,1]$ is defined as follows: for all $y \in H$,

$$\mu_x(y) = \rho(y,x).$$

Notation. From now on, in this paper, for all $y \in H$ we let

$$\mu(y) = \mu_0(y)(= \rho(y,0)).$$

Lemma 3.4. *Let $\rho \in \mathcal{F}_E(H)$. Then,*

(*i*) *for all $x, y \in H$, $\mu_x = \mu_y$ if and only if $\rho(x,y) = \sup_{(w,z)\in H^2}\rho(w,z)$,*

(*ii*) *if $t \in [0,1]$ and $\rho_t \neq \emptyset$, then $[0]_{\rho_t} = \mu_t$.*

Proof. (*i*) Let $\mu_x = \mu_y$, for $x, y \in H$. Since, ρ is a fuzzy reflexive relation, then

$$\rho(x,y) = \mu_y(x) = \mu_x(x) = \rho(x,x) = \sup_{(w,z)\in H^2}\rho(w,z).$$

Conversely, let $\rho(x,y)=\sup_{(u,v)\in H^2}\rho(u,v)$, for $x,y\in H$ and $w\in H$. Since, ρ is a fuzzy symmetric and fuzzy transitive relation, then

$$\begin{aligned}\mu_x(w)&=\rho(w,x)=\rho(x,w)\geqslant\min(\rho(x,y),\rho(y,w))\\&=\min\left(\sup_{(u,v)\in H^2}\rho(u,v),\rho(y,w)\right)=\rho(y,w)=\rho(w,y)=\mu_y(w).\end{aligned}$$

Similarly, we can show that $\mu_y(w)\geqslant\mu_x(w)$. Hence, for all $w\in H$, $\mu_x(w)=\mu_y(w)$ and so $\mu_x=\mu_y$.

(ii) Let $x\in[0]_{\rho_t}$. Then, $x\rho_t 0$ and so $\mu(x)=\rho(x,0)\geqslant t$. Hence, $x\in\mu_t$. Conversely, if $x\in\mu_t$ then $\rho(x,0)=\mu(x)\geqslant t$ and so $x\rho_t 0$. Hence, $x\in[0]_{\rho_t}$. Therefore, $[0]_{\rho_t}=\mu_t$. □

Theorem 3.5. *Let ρ be a fuzzy regular congruence relation on H and*

$$H/\mu=\{\mu_x : x\in H\}.$$

If a hyperoperation "$\circ$" and a hyperorder "$\ll$" on H/μ are defined as follows:

$$\mu_x\circ\mu_y=\mu_{x\circ y}=\{\mu_z : z\in x\circ y\},$$

$$\mu_x\ll\mu_y\longleftrightarrow\ \mu\in\mu_x\circ\mu_y,$$

then $(H/\mu,\circ,\mu)$ is a hyper BCK-algebra.

Proof. First we show that a hyperoperation "$\circ$" is well-defined. Let $\mu_x=\mu_{x'}$ and $\mu_y=\mu_{y'}$. Then, by Lemma 3.4(i),

$$\rho(x,x')=\sup_{(u,z)\in H^2}\rho(u,z)=\rho(y,y').$$

Let $t=\sup_{(u,z)\in H^2}\rho(u,z)$. Hence, $x\rho_t x'$ and $y\rho_t y'$. Since, by Theorem 2.11, ρ_t is a congruence relation on H, then $x\circ y\overline{\rho_t}x'\circ y$ and $x'\circ y\overline{\rho_t}x'\circ y'$ and so $x\circ y\overline{\rho_t}x'\circ y'$. Now, let $\mu_z\in\mu_x\circ\mu_y$. Then, there exists $z'\in x\circ y$ such that $\mu_z=\mu_{z'}$. Since, $z'\in x\circ y$ and $x\circ y\overline{\rho_t}x'\circ y'$, there exists $w\in x'\circ y'$ such that $z'\rho_t w$ and so $\rho(z',w)\geqslant t=\sup_{(u,z)\in H^2}\rho(u,z)\geqslant\rho(z',w)$. Hence $\rho(z',w)=t$. Since ρ is a fuzzy equivalence relation, then for all $u\in H$,

$$\begin{aligned}\mu_z(u)&=\mu_{z'}(u)=\rho(u,z')=\rho(z',u)\geqslant\min(\rho(z',w),\rho(w,u))\\&=\min(t,\rho(w,u))=\rho(w,u)=\rho(u,w)=\mu_w(u).\end{aligned}$$

Conversely, for all $u \in H$,

$$\begin{aligned}\mu_w(u) = \rho(u,w) = \rho(w,u) \geqslant \min(\rho(w,z'),\rho(z',u)) = \min(\rho(z',w),\rho(z',u))\\ = \min(t,\rho(z',u)) = \rho(z',u) = \rho(u,z') = \mu_{z'}(u) = \mu_z(u).\end{aligned}$$

Hence, $\mu_z(u) = \mu_w(u)$, for all $u \in H$ and so $\mu_z = \mu_w$. Since, $w \in x' \circ y'$, then $\mu_z = \mu_w \in \mu'_x \circ \mu'_y$ and so $\mu_x \circ \mu_y \subseteq \mu'_x \circ \mu'_y$. Similarly, we can show that $\mu'_x \circ \mu'_y \subseteq \mu_x \circ \mu_y$. Therefore, $\mu_x \circ \mu_y = \mu_{x'} \circ \mu_{y'}$.

Now we establish the axioms of a hyper BCK-algebra.

(HK1) Let $\mu_v \in (\mu_x \circ \mu_z) \circ (\mu_y \circ \mu_z)$. Then, there exist $\mu_u \in \mu_x \circ \mu_z$ and $\mu_w \in \mu_y \circ \mu_z$ such that $\mu_v \in \mu_u \circ \mu_w$ and so there exists $a \in u \circ w$ such that $\mu_v = \mu_a$. Since, $a \in u \circ w \subseteq (x \circ z) \circ (y \circ z) \ll x \circ y$, then there exists $b \in x \circ y$ such that $a \ll b$ and so $0 \in a \circ b$. This implies that $\mu \in \mu_{a \circ b} = \mu_a \circ \mu_b = \mu_v \circ \mu_b \subseteq ((\mu_u \circ \mu_w) \circ (\mu_x \circ \mu_y)) \subseteq ((\mu_x \circ \mu_z) \circ (\mu_y \circ \mu_z)) \circ (\mu_x \circ \mu_y)$. Thus, $(\mu_x \circ \mu_z) \circ (\mu_y \circ \mu_z) \ll \mu_x \circ \mu_y$.

(HK2) Let $\mu_u \in (\mu_x \circ \mu_y) \circ \mu_z$. Then, there exists $v \in (x \circ y) \circ z$ such that $\mu_u = \mu_v$. Since, $v \in (x \circ y) \circ z = (x \circ z) \circ y$ then $\mu_u = \mu_v \in (\mu_x \circ \mu_z) \circ \mu_y$. This implies that $(\mu_x \circ \mu_y) \circ \mu_z \subseteq (\mu_x \circ \mu_z) \circ \mu_y$. Similarly, we can show that $(\mu_x \circ \mu_z) \circ \mu_y \subseteq (\mu_x \circ \mu_y) \circ \mu_z$. Thus, $(\mu_x \circ \mu_y) \circ \mu_z = (\mu_x \circ \mu_z) \circ \mu_y$.

(HK3) Let $\mu_z \in \mu_x \circ H/\mu$. Then, there exists $\mu_y \in H/\mu$ such that $\mu_z \in \mu_x \circ \mu_y$ and so there exists $w \in x \circ y$ such that $\mu_z = \mu_w$. Since, $x \circ y \ll x$ then $w \ll x$ and so $0 \in w \circ x$. Thus, $\mu \in \mu_{w \circ x} = \mu_w \circ \mu_x = \mu_z \circ \mu_x$. This implies that $\mu_z \ll \mu_x$ and so $\mu_x \circ H/\mu \ll \mu_x$.

(HK4) Let $\mu_x \ll \mu_y$ and $\mu_y \ll \mu_x$. Then, $\mu \in \mu_x \circ \mu_y$ and $\mu \in \mu_y \circ \mu_x$. Hence, there exist $z \in x \circ y$ and $w \in y \circ x$ such that $\mu_z = \mu_0 = \mu_w$ and so by Lemma 3.4(i), $\rho(z,0) = \sup_{(a,b)\in H^2} \rho(a,b) = \rho(w,0)$. Let $t = \sup_{(a,b)\in H^2} \rho(a,b)$. Then, $z\rho_t 0$ and $w\rho_t 0$. Since, $z \in x \circ y$ and $w \in y \circ x$, then $x \circ y \rho_t \{0\}$ and $y \circ x \rho_t \{0\}$. Since, by Theorem 3.2, ρ_t is a regular relation, hence $x\rho_t y$ and so $\rho(x,y) \geqslant t = \sup_{(a,b)\in H^2} \rho(a,b) \geqslant \rho(x,y)$. Thus, $\rho(x,y) = \sup_{(y,u)\in H^2} \rho(y,u)$ and so by Lemma 3.4(i), $\mu_x = \mu_y$. □

Theorem 3.6. *If ρ is a fuzzy congruence relation on H, then μ is a fuzzy hyper BCK-ideal of H.*

Proof. Let $x \ll y$, for $x, y \in H$. Then $0 \in x \circ y$. Since, $x \in x \circ 0$ and ρ is fuzzy left compatible, then $\mu(x) = \rho(x,0) \geqslant \rho(y,0) = \mu(y)$. Now, let $x, y \in H$ and $a \in x \circ y$. Since, $x \in x \circ 0$, then $\rho(x,a) \geqslant \rho(y,0)$ and since ρ is fuzzy transitive, then

$$\mu(x) = \rho(x,0) \geqslant \min(\rho(x,a),\rho(a,0)) \geqslant \min(\rho(y,0),\rho(a,0))$$

$$\geqslant \min\left(\inf_{a\in x\circ y}\rho(a,0),\rho(y,0)\right)=\min\left(\inf_{a\in x\circ y}\mu(a),\mu(y)\right).$$

This implies that μ is a fuzzy hyper BCK-ideal of H. □

4. Isomorphism theorems

Theorem 4.1. [6] *Let $f : H \longrightarrow H'$ be a homomorphism of hyper BCK-algebras. Then,*

(i) *$kerf$ is a hyper BCK-ideal of H,*

(ii) *if Θ is a regular congruence on H and $kerf = I$, then $H/I \simeq f(H)$.*

Theorem 4.2. *Let ρ be a fuzzy regular congruence relation on H and $t = \sup_{(z,w)\in H^2}\rho(z,w)$. Then there is a hyper BCK-ideal J of H/μ such that*

$$(H/\mu)/J \simeq H/\mu_t.$$

Proof. Since ρ is a fuzzy reflexive relation on H, then

$$\rho(0,0)=\sup_{(y,z)\in H^2}\rho(y,z)=t$$

and so $(0,0)\in\rho_t$ and ρ satisfies the sup property. Hence, by Theorems 2.11 and 3.2, ρ_t is a regular congruence relation on H. By Lemma 3.4(ii), $[0]_{\rho_t}=\mu_t$ and so by Theorem 2.6, H/μ_t is a hyper BCK-algebra. Let $I = \mu_t$. Then, by Theorem 2.6, $H/\mu_t = H/I = \{I_x : x\in H\}$, where $I_x = [x]_{\rho_t}$. Now, let $\psi : H/\mu \to H/I$ be defined by $\psi(\mu_x) = I_x$. Let $\mu_x = \mu_y$, for $\mu_x,\mu_y \in H/\mu$. Then, by Lemma 3.4(i), $\rho(x,y) = \sup_{(z,w)\in H^2}\rho(z,w) = t$. Thus, $x\rho_t y$ and so $I_x = I_y$; i.e. $\psi(\mu_x) = \psi(\mu_y)$. This implies that ψ is well-defined. Moreover, for all $\mu_x,\mu_y\in H/\mu$,

$$\begin{aligned}\psi(\mu_x\circ\mu_y)&=\psi(\mu_{x\circ y})=\psi(\{\mu_z : z\in x\circ y\})=\{\psi(\mu_z) : z\in x\circ y\}\\&=\{I_z : z\in x\circ y\}=I_x\circ I_y=\psi(\mu_x)\circ\psi(\mu_y)\end{aligned}$$

and so ψ is a homomorphism. Now, let

$$\mu_x\Theta\mu_y \;\longleftrightarrow\; x\rho_t y$$

for all $x,y\in H$. Since, ρ_t is a regular congruence relation on H, then Θ is a regular congruence relation on H/μ, too. Now, we show that $ker\psi = [\mu]_\Theta$.

$$\begin{aligned}\mu_x\in ker\psi &\longleftrightarrow \psi(\mu_x)=I\longleftrightarrow I_x=I\longleftrightarrow x\in I=[0]_{\rho_t}\\&\longleftrightarrow x\rho_t 0\longleftrightarrow \mu_x\Theta\mu\longleftrightarrow \mu_x\in[\mu]_\Theta.\end{aligned}$$

It is clear that ψ is onto. Therefore, by Theorem 4.1(ii), $(H/\mu)/ker\psi \simeq H/\mu_t$. Now, let $J = ker\psi$. Then $(H/\mu)/J \simeq H/\mu_t$ and by Theorem 4.1(i), J is a hyper BCK-ideal of H/μ. □

Theorem 4.3. *Let ρ be a fuzzy regular congruence relation on H and $\mu^* = \{x \in H : \mu(x) = \mu(0)\}$. Then $H/\mu \simeq H/\mu^*$.*

Proof. Let $t = \mu(0)$. Since, ρ is fuzzy reflexive, then $t = \mu(0) = \rho(0,0) = \sup_{(y,z)\in H^2} \rho(y,z)$. Now, we must show that $\mu^* = [0]_{\rho_t}$. Let $x \in \mu^*$. Then, $\mu(x) = \mu(0)$ and so $\rho(x,0) = \rho(0,0) = t$. Hence, $x\rho_t 0$ and so $x \in [0]_{\rho_t}$. Hence, $\mu^* \subseteq [0]_{\rho_t}$. Moreover, if $x \in [0]_{\rho_t}$, then $x\rho_t 0$; i.e, $(x,0) \in \rho_t$ and so $\rho(x,0) \geqslant t = \sup_{(y,z)\in H^2} \rho(y,z) \geqslant \rho(x,0)$. Thus, $\rho(x,0) = t = \rho(0,0)$ and so $\mu(x) = \mu(0)$; i.e, $x \in \mu^*$. Hence, $[0]_{\rho_t} \subseteq \mu^*$. Therefore, $[0]_{\rho_t} = \mu^*$ and so by Theorem 2.6, H/μ^* is well-defined.

Now, let $\psi : H/\mu \longrightarrow H/\mu^*$ be defined by $\psi(\mu_x) = I_x$, for all $x \in H$, where $I = \mu^*$. By the proof of Theorem 4.2, ψ is an epimorphism. Now, we show that ψ is one-to-one. For this, let $\mu_x \in ker\psi$. Then, $\psi(\mu_x) = I$ and so $I_x = I$. Hence, $x \in I = \mu^*$ and so $\mu(x) = \mu(0)$. Since, ρ is fuzzy reflexive, then $\rho(x,0) = \rho(0,0) = \sup_{(y,z)\in H^2} \rho(y,z)$ and so by Lemma 3.4(i), $\mu_x = \mu_0 = \mu$. Hence, $ker\psi = \{\mu\} = 0_{H/\mu^*}$ and so $H/\mu \simeq H/\mu^*$. □

Theorem 4.4. (First Isomorphism Theorem)
Let ρ be a fuzzy regular congruence relation on H and $f : H \to H'$ be an epimorphism of hyper BCK-algebras such that $\mu^* = kerf$. Then $H/\mu \simeq H'$.

Proof. Let $\varphi : H/\mu \to H'$ be defined by $\varphi(\mu_x) = f(x)$, for all $x \in H$. First we show that φ is well-defined. For this, let $\mu_x = \mu_y$, for $x, y \in H$. Then by Lemma 3.4(i), $\rho(x,y) = \sup_{(z,w)\in H^2} \rho(z,w)$. Let $t = \sup_{(z,w)\in H^2} \rho(z,w)$. Hence, $x\rho_t y$. Since, by Theorem 2.11, ρ_t is a congruence relation on H, then $x \circ y\overline{\rho_t}y \circ y$ and $x \circ x\overline{\rho_t}y \circ x$. Since, $0 \in y \circ y$, then there exists $a \in x \circ y$ such that $a\rho_t 0$ and so $\rho(a,0) \geqslant t = \sup_{(z,w)\in H^2} \rho(z,w) \geqslant \rho(a,0)$. Hence, $\rho(a,0) = \sup_{(z,w)\in H^2} \rho(z,w)$. Since, ρ is fuzzy reflexive, then $\mu(0) = \rho(0,0) = \sup_{(z,w)\in H^2} \rho(z,w) = \rho(a,0) = \mu(a)$ and so $a \in \mu^* = kerf$. Hence, $0' = f(a) \in f(x \circ y) = f(x) \circ f(y)$ and so $f(x) \ll f(y)$. Similarly, since $0 \in x \circ x$, then there exists $b \in y \circ x$ such that $b \in kerf$. Hence, $0' = f(b) \in f(y \circ x) = f(y) \circ f(x)$ and so $f(y) \ll f(x)$. Thus, $f(x) = f(y)$ and so φ is well-defined. Let $\mu_x, \mu_y \in H/\mu$. Then, $\varphi(\mu_x \circ \mu_y) = \varphi(\mu_{x\circ y}) = f(x \circ y) = f(x) \circ f(y) = \varphi(\mu_x) \circ \varphi(\mu_y)$, and so φ is a homomorphism. Now, let $\mu_x \in ker\varphi$. Then $f(x) = \varphi(\mu_x) = 0'$ and so $x \in kerf = \mu^*$; i.e., $\mu(x) = \mu(0)$. Thus, $\rho(x,0) = \mu(x) = \mu(0) = \rho(0,0) = \sup_{(z,w)\in H^2} \rho(z,w)$.

Hence, by Lemma 3.4(i), $\mu_x = \mu$ and so $ker\varphi = \{\mu\}$. Hence, φ is one to one. Since, f is onto, then φ is onto, too. Therefore, φ is an isomorphism and so $H/\mu \simeq H'$. □

Theorem 4.5. *Let $f : H \to H'$ be an epimorphism of hyper BCK-algebras, ρ and σ (resp.) be fuzzy regular congruence relations on H and H', μ and μ' (resp.) be fuzzy subsets on H and H' such that $\mu_y = f^{-1}(\mu'_{f(y)})$, for all $y \in H$. Then $H/\mu \simeq H'/\mu'$.*

Proof. Let $\varphi : H/\mu \to H'/\mu'$ be defined by $\varphi(\mu_x) = \mu'_{f(x)}$, for all $x \in H$. Now, let $\mu_x = \mu_y$, for $\mu_x, \mu_y \in H/\mu$ and $z' \in H'$. Since, f is onto, then there exists $z \in H$ such that $f(z) = z'$. Hence,

$$\begin{aligned}\mu'_{f(x)}(z') &= \mu'_{f(x)}(f(z)) = f^{-1}(\mu'_{f(x)})(z) = \mu_x(z) = \mu_y(z)\\ &= f^{-1}(\mu'_{f(y)})(z) = \mu'_{f(y)}(f(z)) = \mu'_{f(y)}(z').\end{aligned}$$

Thus, $\mu'_{f(x)} = \mu'_{f(y)}$ and so φ is well-defined. Now, let $\mu_x, \mu_y \in H/\mu$. Then,

$$\varphi(\mu_x \circ \mu_y) = \varphi(\mu_{x\circ y}) = \mu'_{f(x\circ y)} = \mu'_{f(x)\circ f(y)} = \mu'_{f(x)} \circ \mu'_{f(y)} = \varphi(\mu_x) \circ \varphi(\mu_y),$$

and this implies that φ is a homomorphism. Moreover, since f is onto then φ is onto, too. Now, let $\varphi(\mu_x) = \varphi(\mu_y)$, for $\mu_x, \mu_y \in H/\mu$. Then $\mu'_{f(x)} = \mu'_{f(y)}$ and so for all $z \in H$,

$$\mu_x(z) = f^{-1}(\mu'_{f(x)})(z) = \mu'_{f(x)}(f(z)) = \mu'_{f(y)}(f(z)) = f^{-1}(\mu'_{f(y)})(z) = \mu_y(z).$$

This implies that φ is one-to-one. Therefore, φ is an isomorphism and so $H/\mu \simeq H'/\mu'$. □

Lemma 4.6. *Let ρ and σ be two fuzzy regular congruence relations on H such that $\mu_y(x) = \sigma(x, y)$, $\mu(x) = \sigma(x, 0)$, for all $x, y \in H$ and ρ satisfies the sup property. Then ρ/μ is a fuzzy regular congruence relation on H/μ, where fuzzy relation ρ/μ on H/μ is defined by $\rho/\mu(\mu_x, \mu_y) = \rho(x, y)$.*

Proof. Since, σ is a fuzzy regular congruence relation on H, then by Theorem 3.5, H/μ is well-defined. Moreover, since ρ is a fuzzy regular congruence relation on H, then by some modifications we can show that ρ/μ is a fuzzy congruence relation on H, too. Now, let

$$s = \min\left(\sup_{\mu_a \in \mu_x \circ \mu_y} \rho/\mu(\mu_a, \mu), \sup_{\mu_b \in \mu_y \circ \mu_x} \rho/\mu(\mu_b, \mu)\right).$$

Then, $\sup_{\mu_a \in \mu_x \circ \mu_y} \rho/\mu(\mu_a, \mu) \geqslant s$ and $\sup_{\mu_b \in \mu_y \circ \mu_x} \rho/\mu(\mu_b, \mu) \geqslant s$. Since, ρ satisfies the sup property, then ρ/μ so is. Thus, there exist $a_0 \in x \circ y$ and $b_0 \in y \circ x$ such that

$$\rho(a_0, 0) = \rho/\mu(\mu_{a_0}, \mu) = \sup_{\mu_a \in \mu_x \circ \mu_y} \rho/\mu(\mu_a, \mu) \geqslant s$$

and
$$\rho(b_0, 0) = \rho/\mu(\mu_{b_0}, \mu) = \sup_{\mu_b \in \mu_y \circ \mu_x} \rho/\mu(\mu_b, \mu)) \geqslant s.$$

Since, ρ is a fuzzy regular relation on H, then

$$\rho/\mu(\mu_x, \mu_y) = \rho(x, y) \geqslant \min(\sup_{a \in x \circ y} \rho(a, 0), \sup_{b \in y \circ x} \rho(b, 0))$$

$$\geqslant \min(\rho(a_0, 0), \rho(b_0, 0)) \geqslant s = \min(\sup_{\mu_a \in \mu_x \circ \mu_y} \rho/\mu(\mu_a, \mu), \sup_{\mu_b \in \mu_y \circ \mu_x} \rho/\mu(\mu_b, \mu)).$$

Hence, ρ/μ is a fuzzy regular relation on H/μ and so it is a fuzzy regular congruence relation on H/μ. □

Theorem 4.7. (Second Isomorphism Theorem)
Let ρ and σ be two fuzzy regular congruence relations on H such that $\sigma \subseteq \rho$ and there exists $a \in H$ such that $\sigma(a, a) = 1$. Let fuzzy subsets η_y and μ_y on H are defined by $\eta_y(x) = \rho(x, y)$ and $\mu_y(x) = \sigma(x, y)$, for all $x, y \in H$. Then

$$(H/\mu)/(\eta/\mu) \simeq H/\eta,$$

where $(\eta/\mu)(\mu_x) = \rho/\mu(\mu_x, \mu)$ *and* $(\rho/\mu)(\mu_x, \mu_y) = \rho(x, y)$.

Proof. Since, by Lemma 4.6, ρ/μ is a fuzzy regular congruence relation on H/μ and $(\eta/\mu)(\mu_x) = (\rho/\mu)(\mu_x, \mu)$, then $(H/\mu)/(\eta/\mu)$ is a hyper BCK-algebra. Also, it is easy to see that

$$\sup_{(z,w) \in H^2} \rho(z, w) = \sup_{(\mu_z, \mu_w) \in (H/\mu)^2} \rho/\mu(\mu_z, \mu_w).$$

Now, let $\psi : H/\mu \to H/\eta$ be defined by $\psi(\mu_x) = \eta_x$. We have to show that ψ is well-defined. Let $\mu_x = \mu_y$, for $\mu_x, \mu_y \in H/\mu$. Then, by Lemma 3.4(i), $\sigma(x, y) = \sup_{(z,w) \in H^2} \sigma(z, w) = 1$. Since, $\sigma \subseteq \rho$, then $\rho(x, y) \geqslant \sigma(x, y) = 1$ and so $\rho(x, y) = \sup_{(z,w) \in H^2} \rho(z, w)$. Hence, by Lemma 3.4($i$), $\eta_x = \eta_y$, which this shows that ψ is well-defined. It is easy to check that ψ is an epimorphism. Now,

$$\begin{aligned} ker\psi &= \{\mu_x \in H/\mu : \eta_x = \psi(\mu_x) = \eta\} \\ &= \{\mu_x \in H/\mu : \rho(x, 0) = \sup\nolimits_{(z,w) \in H^2} \rho(z, w) = \rho(0, 0)\} \\ &= \{\mu_x \in H/\mu : \rho/\mu(\mu_x, \mu) = \rho/\mu(\mu, \mu)\} \end{aligned}$$

$$= \{\mu_x \in H/\mu : (\eta/\mu)(\mu_x) = (\eta/\mu)(\mu)\} = (\eta/\mu)^*.$$

So, $(H/\mu)/(\eta/\mu) \simeq H/\eta$. □

References

[1] **R. Ameri, M. Bakhshi, S. A. Nematollahzadeh, R. A. Borzooei**: *Fuzzy (strong) congruence relations on a hypergroupoid and hyper BCK-algebra*, Quasigroups Related Systems **15** (2007), 11 − 24.

[2] **M. Bakhshi, R. A. Borzooei**: *Lattice structure on fuzzy congruence relations of a hypergroupoid*, Information Sci. **177** (2007), 3305 − 3313.

[3] **R. A. Borzooei, M. Bakhshi**: *Some results on hyper BCK-algebras*, Quasigroups Related Systems **11** (2004), 9 − 27.

[4] **R. A. Borzooei, M. Bakhshi**: *Weak implicative hyper BCK-ideals*, Quasigroups Related Systems **12** (2004), 13 − 28.

[5] **R. A. Borzooei, M. Bakhshi, Y. B. Jun**: *Fuzzy congruence relations on hyper BCK-algebras*, J. Fuzzy Math. **13** (2005), 627 − 636.

[6] **R. A. Borzooei, H. Harizavi**: *Regular congruence relations on hyper BCK-algebras*, Sci. Math. Jpn. **61** (2005), 83 − 97.

[7] **Y. Imai, K. Iséki**: *On axiom systems of propositional calculi XIV*, Proc. Japan Academy **42** (1966), 19 − 22.

[8] **Y. B. Jun, X. L. Xin**: *Fuzzy hyper BCK-ideals of hyper BCK-algebras*, Sci. Math. Jpn. **53** (2001), 353 − 360.

[9] **Y. B. Jun, X. L. Xin, E. H. Roh, M. M. Zahedi**: *Strong hyper BCK-ideals of hyper BCK-algebras*, Math. Jpn. **51** (2000), 493 − 498.

[10] **Y. B. Jun, M. M. Zahedi, X. L. Xin, R. A. Borzooei**: *On hyper BCK-algebras*, Ital. J. Pure Appl. Math. **10** (2000), 127 − 136.

[11] **F. Marty**: *Sur une generalization de la notion de groups*, 8th congress Math. Scandinaves, Stockholm (1934), 45 − 49.

[12] **V. Murali**: *Fuzzy congruence relations*, Fuzzy Sets and Systems **41** (1991), 359 − 369.

[13] **H. T. Nguyen, E. A. Walker**: *A First Course In Fuzzy Logic*, 3rd ed. (2006), Chapman and Hall/CRC, USA, Florida.

[14] **L. A. Zadeh**: *Fuzzy sets*, Inform. and control **8** (1965), 338 − 353.

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