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Groups homeomorphisms: topological characteristics, invariant measures and classifications

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Abstract

It is a survey of main results on groups of homeomorphisms of the real line and the circle obtained in the last years.

1. Introduction

One of main problems of the theory of groups is the problem of a classification of abstract groups. Such classification can be based on the Tarski's number connected with the Day's problem. It is known, that the Tarski's number distinguishes among themselves no more than account set of subclasses of the paradoxical groups, but not distinguishes the amenable groups. Nevertheless, the classification is possible on a basis of the scale of the values given by the growth group for finitely generated amenable groups. Unfortunately, there is not any correspondence between good known canonical subclasses of groups and characteristic given by the growth group in a class of finitely generated amenable groups, even though such correspondence takes place for special subclasses of groups. Other important method of investigation of the abstract groups is their realization in the form of subgroups of some selected groups with well investigated properties. The groups of actions on locally compact space and, in particular, the groups of homeo-

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morphisms of the real line and circle belong to such groups. In addition the topological and metric invariants arise for groups of homeomorphisms. From the noted invariants, the invariant measures and topological characteristic connected with them will be considered. The presence of additional invariants generates some natural factorization of such groups. For quotient groups the classification mentioned above appears more informative, as it will be shown in the presented work.

2. The amenability and paradoxical partitions. Tarski's number and Day's problem

The major characteristics of groups are connected with the concept of the amenability and, in particular, the metric invariants. This fact is known since the early works of Krylov and Bogolyubov on invariant measures for groups acting on a compact set.

Definition 1. The discrete group G is called the *amenable group* if it admits a G-invariant probability measure, i.e., the map $\mu : P(G) \longrightarrow [\,0,1\,]$, where $P(G)$ is the collection of all subsets of G, such that

1) μ is finitely additive,
2) $\mu(gA) = \mu(A)$ for all $g \in G$ and $A \subseteq G$,
3) $\mu(G) = 1$.

For a discrete group G by $B(G)$ we will denote the space of all bounded functions on G with the sup-norm.

A linear function m on $B(G)$ is called a *left-invariant mean*, if:

1) $m(\bar{f}) = \overline{m(f)}$,
2) $m(f) \geqslant 0, \quad f \geqslant 0, \quad m(1) = 1$,
3) $m(gf) = m(f)$, where $gf(\bar{g}) = f(g^{-1}\bar{g})$ for all $g, \bar{g} \in G$.

Using this concept we can give the equivalent definition of the amenability.

Definition 1*. The discrete group G is called the *amenable group* if on G there is a left-invariant mean.

Definition 2. A group G is *paradoxical*, if it admits the paradoxical partition, i.e., there are subsets $A_1, \ldots, A_n$, $B_1, \ldots, B_m$ of G and elements

$g_1, \ldots, g_n$, $h_1, \ldots, h_m$ such that

$$G = \begin{cases} A_1 \bigsqcup \cdots \bigsqcup A_n \bigsqcup B_1 \bigsqcup \cdots \bigsqcup B_m \\ g_1 A_1 \bigsqcup \cdots \bigsqcup g_n A_n \\ h_1 B_1 \bigsqcup \cdots \bigsqcup h_m B_m. \end{cases}$$

Theorem 1. (Tarski's alternative [44])
The group G either is amenable or paradoxical. □

The set AG of all amenable groups is closed with respect to the following four operations:

(1) taking of subgroups,

(2) taking of quotient groups,

(3) extensions of groups by elements from AG (G is an extension of a group H by F if $H \trianglelefteq G$ and $G/H \cong F$),

(4) the directed union of amenable subgroups $\{H_\alpha\}$ $(\bigcup_\alpha H_\alpha$ where for any H_{α_1}, H_{α_2} there is $H_\gamma \supset H_{\alpha_1} \cup H_{\alpha_2})$.

Note that any group containing a free subgroup with two generators is paradoxical. Moreover, if a subgroup H of a group G or a quotient group of G/H is paradoxical, then G is paradoxical too.

Definition 3. The smallest number $\tau = n+m$ of all paradoxical partitions of a paradoxical group G is called the *Tarski number* and is denoted by $\tau(G)$.

It is easy that $\tau(G) \geqslant 4$.

Fact 1. *If a subgroup H of G or a quotient group G/H is paradoxical, then $\tau(G) \leqslant \tau(H)$.* □

Fact 2. (Johnson, Dekker [46])
For a paradoxical group G we have $\tau(G) = 4$ if and only if G contains a free subgroup with two generators. □

Fact 3. *For a torsion group $\tau(G) \geqslant 6$.* □

Fact 4. *For any paradoxical group G there exists a finitely generated subgroup H such that $\tau(G) = \tau(H)$.* □

More interesting facts about amenable groups and the Tarski number one can find in the surveys [16], [20], [23] and [27].

Problem. *Is it true that for each natural $n \geqslant 4$ there is a paradoxical group G with $\tau(G) = n$?*

The following classes of groups will be considered:

EG – the class of all elementary amenable groups,

FG – the class of groups containing a free subgroup with two generators,

$F_N G$ – the class of groups without free subgroups with two generators.

It is clear that the class EG is the smallest class of groups containing all finite and abelian groups and closed with respect to the operations $(1)-(4)$ defining the class AG. Obviously

$$EG \subseteq AG \subseteq F_N G \subseteq (F_N G \cup FG) \tag{1}$$

and $F_N G \cap FG = \emptyset$. We know that $\tau(G) = 4$ for all groups from the class FG, $\tau(G) \geqslant 5$ for groups from the class $F_N G \backslash AG$ and $\tau(G) = \infty$ for groups from AG. In connection to this, in 1957 Day (cf. [17]) posed the following problem:

Day's problem. *Is it true that*

$$EG \subset AG \subset F_N G\,? \tag{2}$$

The above sequence of is inclusions is called the *dichotomy* or the *extremal property.*

Greenleaf [27] (and others) posed in 1969 the hypothesis that *a discrete group is either amenable or contains a free subgroup with two generators.* This means that

$$AG = F_N G. \tag{3}$$

Then, in 1979, Tits proved [45] the so-called Tits's alternative: *a finitely generated linear group either contains a free subgroup with two generators, or is almost solvable.*

Olshansky [35] (1980), Adyan [1] (19820 and Gromov [25] (1988) had found examples of finitely generated groups from the class $F_N G \backslash AG$. They found *examples of non-amenable finitely generated groups without free subgroups with two generators which are not finitely defined.*

In 1984, Grigorchuk solved [19] the Day's problem by the construction of a finitely generated group from $AG \backslash EG$ (now called the *Grigorchuk's group*). Later, Grigorchuk found [22] the second example of such group but this group is not finitely defined.

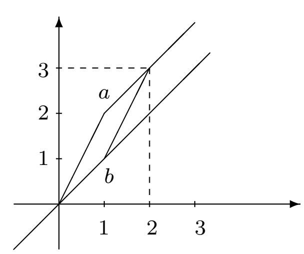
Now it is desirable to know: *is there an example of a finitely generated and finitely defined group from $F_N G \backslash AG$*? In view of Fact 1, such group will be maximally near to $\tau(G) = 5$.

The Richard Thompson's (1965) group seems to be the potential candidate of such group: *F is the set of all piecewise linear homeomorphisms* $[0, 1]$*, having the breaks only in the finite number of binary rational points, and on intervals of differentiability the derivative is equal to a degree two.*

Brin and Squier have shown in 1985 (cf. [14]), that the group F is not elementary and it does not contain a free subgroup with two generators, i.e., $F \in AG \backslash EG$. Such group is isomorphic to the group with two generators and two relations [15]. Namely,

$$F = < A, B : \; [AB^{-1}, A^{-1}BA], \; [AB^{-1}, A^{-2}BA^2] > .$$

It can be realized as a group of homeomorphisms of $\mathbb{R}$ with two generators a, b having the form:



Problem 2. *Which of the statements*

1) $F \in AG \backslash EG$ *(F is amenable),*

2) $F \in F_N G \backslash AG$ *(F is not amenable)*

is true?

The answer is necessary to determine the way of further systematic investigation of groups of homeomorphisms of the real line, their metric invariants and topological characteristics.

3. The growth of a finitely generated group and a scale of correspondences

For a group $G =< g_1, \dots, g_s >$ the important characteristic is the *growth*

$$\lambda(G) = \lim_{n\to\infty} \sqrt[n]{\gamma_G(n)},$$

where $\gamma_G(n)$ is the number of elements of the set

$$\{g : g = g_{i_1}^{\varepsilon_1} ... g_{i_m}^{\varepsilon_m},\ m \leqslant n,\ i_j \in \{1, \dots, s\},\ \varepsilon_j = \pm 1,\ j = 1, \dots, m\}.$$

We say that the group G has the *exponential growth* if $\lambda(G) > 1$, and the *subexponential growth* if $\lambda(G) = 1$.

If for a given group G the function $\gamma_G(n)$ grows more quickly than a polynomial function, but more slowly than an exponential function, then such group is called the group with the *intermediate growth.*

For a group $G =< g_1, \dots, g_s >$ the growth $\lambda(G)$ is always defined, and its properties do not depend on the choice of generators. For groups from the classes FG and $F_N G \setminus AG$ the growth $\lambda(G)$ is exponential. For groups from the class AG the growth $\lambda(G)$ is no more than exponential.

The growth of groups can be used to the classification of finitely generated amenable groups only.

Finitely generated groups, containing free subsemigroup with two generators, have the exponential growth. On the other hand, *if a group of homeomorphisms of the real line has two generators such that one generator is the shift on unit, the second is an affinity transformation, then this group is solvable of the step two and contains free subsemigroup with two generators, hence it has the exponential growth.*

There are examples of non-amenable finitely generated groups without free subsemigroups with two generators. Hence no connections between the property of the amenability and the existence free subsemigroups with two generators. But there are also "typical" groups with the exponential growth containing free subsemigroups with two generators.

Nevertheless, such one-to-one correspondence takes place for some classes of finitely generated groups. Namely, in 1981 Gromov proved the following theorem [24]:

Theorem 2. *A finitely generated group has the polynomial growth if and only if it is almost nilpotent.* □

Earlier, in 1974, a similar result was proved by Rosenblatt [38] for solvable groups.

Theorem 3. *A finitely generated solvable group without free subsemigroups with two generators is almost nilpotent. This means that it has the polynomial growth.* □

From Theorem 2 it follows that for Grigorchuk's group the growth is more than polynomial. Grigorchuk proved in 1984 (cf. [19]) a stronger result, which can be formulated in the following way:

Theorem 4. *The growth of the Grigorchuk's group is more than polynomial and less than exponential, i.e., this group has the intermediate growth.* □

4. About realizations of abstract groups as groups of action on the real line (circle)

We start with the theorem proved in 1996 by Ghys [26].

Theorem 5. *An account group can be realized as a group of preserving orientation homeomorphisms of the real line if and only if it is rightordered.*

This result was presented for me (independently to [26]), by Grigorchuk who observed that such realization possess one additional important property. Namely, for ordered account groups the graphs of different elements have the form of a cortege, i.e., the graph of one of them is dishosed above the graph of other, though the tangency is possible.

Starting from the construction of the Grigorchuk's group the first nontrivial example of a subgroup of $Homeo_{+}([0,1])$ (the preserving orientation homeomorphisms of an interval $[0,1]$) has been obtained as a result of embedding of $Homeo_+([0,1])$ into some group associated with the Grigorchuk's group and having intermediate growth.

Grigorchuk and Maki have proved in [21] the following theorem.

Theorem 6. *A group $Homeo_+([0,1])$ has a finitely generated subgroup with intermediate growth.* □

Theorem 5 shows that groups of homeomorphisms of an interval, the real line and a circle are the universal object for the abstract theory of groups. By Theorem 6, such groups have a nontrivial structure.

5. Topological characteristics and invariant measures for groups homeomorphisms of the real line and the circle

One of the first results in this direction has been obtained in 1939 by Krylov and Bogolyubov [13], then (in 1961) by Day [18].

Theorem 7. *For discrete amenable groups G, acting continuously on a compact space, there is a G–invariant Borel measure.* □

Various aspects of proofs of this theorem are analyzed in the review [3].

Note that the existence of an invariant Borel measure is equivalent to the existence of his topologically characteristic support. Therefore, it is difficult to expect the presence of a criterion of the existence of an invariant Borel measure by the terms of the amenability, or the algebraic characteristics of the initial group.

Nevertheless, Plante [36] has formulated such type criterion for some finitely generated groups. This criterion is formulated in the term of the subexponential growth of orbits of points (Theorem 8 below).

Definition 4. We say that the orbit $G(t)$ of the point $t \in \mathbb{R}$ has the *subexponential growth*, if

$$\lim_{n\to\infty} \inf \frac{1}{n} \log |G^n(t)| = 0,$$

where G^n is the set of all words of length no more than n, and $|G^n|$ is the cardinality of G^n.

The form of the condition of the subexponential growth has the asymptotic character, but, in fact, it is a topological characteristic.

Theorem 8. *Let $G \subseteq Homeo_+(\mathbb{R})$ be a finitely generated group. The existence of a Borel measure finite on compact subsets and invariant with respect to the group G is equivalent to the existence of a point $t \in \mathbb{R}$ with the orbit having the subexponential growth.* □

Unfortunately, this theorem does not admit a generalization to groups which are not finitely generated.

Nevertheless Ghys (1998) posed a hypothesis that Theorem 7 can be proved also for groups acting on the circle. This hypothesis was verified by Margulis (2000). Namely, he proved in [33]:

Theorem 9. *For any group of homeomorphisms of the circle there exists a free subgroup with two generators or a probability Borel measure invariant with respect to this group.* □

This alternative is not strong. There are groups of homeomorphisms of the circle, for which there are both a free subgroup with two generators and a probability invariant Borel measure simultaneously.

Note, that result analogous to the above theorem was obtained earlier (1984) by Solodov. His result was formulated in other terms (cf. [42]). The equivalence of these two results was proved by Beklaryan in 2002 (cf. [11]).

Denote by $Homeo(X)$ the group of homeomorphisms of $X = \mathbb{R}$, S^1 and by $Homeo_+X$ – the group of orientation-preserving homeomorphisms of $X = \mathbb{R}$, S^1.

Since for a group $G \subseteq Homeo(X)$ the set G_+ of all preserving orientation homeomorphisms defines the normal subgroup of an index no more than two, the study of such groups can be reduced to the study of groups of preserving orientation homeomorphisms of X.

For a group $G \subseteq Homeo_+(X)$ we additionally define the set

$$G^s = \{g \in G : \exists t \in \mathbb{R},\ g(t) = t\},$$

which is the union of stabilizers.

For $X = \mathbb{R}$ we also define the set

$$G^s_\infty = \{g \in G^s : \sup\{t : g(t) = t\} = +\infty,\ \inf\{t : g(t) = t\} = -\infty\}.$$

Note that G^s is not a group, in general, but

$$G^s \subseteq < G^s > \subseteq G\,. \tag{4}$$

Moreover, the following lemma is true (cf. [4]).

Lemma 1. *For $G \subseteq Homeo_+(\mathbb{R})$ we have $G^s = < G^s >$ or $< G^s > = G$.* $\square$

This alternative is not strong. There are groups for which $G^s = G$.

Theorem 10. *For $G \subseteq Homeo_+(X)$ the quotient group $G/{<}G^s{>}$ is commutative and isomorphic to some subgroup of the additive group of X.* $\square$

In the proof of this theorem Lemma 1 and the Hölder's theorem about archimedean groups (cf. [4]) are used.

For many special cases (for finitely generated groups, for finitely generated groups without free subsemigroups with two generators, ...) this theorem has been easier proved by Novikov [34], Imanishi [29] and Salhi [39], [41].

5.1. Topological characterizations

For any group $G \subseteq Homeo_+(X)$ we define the set:

$$Fix\, G^s = \{t \in X : \forall g \in G^s,\ g(t) = t\}.$$

Definition 5. By a *minimal set* of a group $G \subseteq Homeo(X)$ we mean such closed G-invariant subset of X which do not contains any proper closed G-invariant subsets. If there is no such set, then we say that minimal set is empty. If for a group G there exists only one minimal set, then it is denoted by $E(G)$.

A very important characterization of minimal sets was given in 1996 by Beklaryan [7]. Namely, he proved that

Theorem 11. *For a group $G \subseteq Homeo_+(\mathbb{R})$ the following four cases are possible:*

a) any minimal set is discrete and is contained in $Fix\, G^s$ (in this case $Fix\, G^s$ is the union of minimal sets), *b*) *the minimal set is a perfect anywhere dense subset of* $\mathbb{R}$ (*in this case it is a unique minimal set and it is contained in the closure of the orbit* $G(t)$ *of an arbitrary point* $t \in \mathbb{R}$),

c) *the minimal set coincides with* $\mathbb{R}$,

d) *the minimal set is empty.* □

Earlier, the minimal sets of cyclic groups of homeomorphisms of the circle were investigated in [2] and [31]. Note that in the compact case for any group $G \subseteq Homeo_+(S^1)$ the non-empty minimal set always exists.

Account groups of homeomorphisms and groups of diffeomorphisms of the real line were investigated by Salhi. The minimal sets of groups $G \subseteq Homeo_+(S^1)$ were studied by many authors.

The problem of existence of non-empty minimal sets was partially solved in [7], where the following is proved:

Proposition 1. *If a group* $G \subseteq Homeo_+(\mathbb{R})$

a) *is finitely generated, or*

b) $Fix\, G^s \neq \emptyset$, *or*

c) $G \neq G^s_\infty$,

then it has a non-empty minimal set. □

The proof of this proposition is based on the axiom of choice, so the minimal set cannot be described constructively. But in the case $Fix\, G^s \neq \emptyset$ we have a stronger result [6]:

Theorem 12. *Let* $G \subseteq Homeo_+(X)$. *If* $Fix\, G^s \neq \emptyset$, *then:*

1) *for every* $t \in Fix\, G^s$ *the set* $\mathbb{P}(G)$ *of all limit points of* $G(t)$ *does not depends on the point* t,

2) $\mathbb{P}(G) \subseteq Fix\, G^s$,

3) *either* $\mathbb{P}(G) = X$, *or* $\mathbb{P}(G)$ *is the perfect anywhere dense subset of* $\mathbb{X}$, *or* $\mathbb{P}(G) = \emptyset$,

4) *if* $\mathbb{P}(G) \neq \emptyset$, *then* G *has the unique non-discrete minimal set* $E(G)$ *and* $\mathbb{P}(G)$ *coincides with* $E(G)$,

5) *in the case* $\mathbb{P}(G) = \emptyset$, *all minimal sets are discrete, belong to* $Fix\, G^s$ *and* $Fix\, G^s$ *is the union of these minimal sets.* □

In the study of some problems, for example, in the study of tracks of groups of quasiconformal maps of the upper half plane [10], a very important role plays the possibility of replacement of the initial group of homeomorphisms by its subgroup with the same topological complexity (i.e., with the same minimal set). Therefore, the investigation of connections between subgroups of initial groups and their minimal sets represents the big interest. We present two lemmas proved in [9] as examples of such results.

Lemma 2. *If the minimal set* $E(\Gamma)$ *of a subgroup* $\Gamma \subseteq G \subseteq Homeo_+(X)$ *is non-empty and non-discrete, then the minimal set* $E(G)$ *of* G *is also non-empty and non-discrete and* $E(\Gamma) \subseteq E(G)$. □

Lemma 3. *If the minimal set* $E(\Gamma)$ *of a normal subgroup* $\Gamma \subseteq G \subseteq Homeo_+(X)$ *is non-empty and non-discrete, then it coincides with the minimal set of the initial groups* G, *i.e.,* $E(\Gamma) = E(G)$. □

The latter lemma gives the possibility to reduce the study of the initial groups of homeomorphisms and its minimal sets to the study of smallest and simplest groups and their minimal sets.

5.2. Invariant measures

Since the existence of invariant Borel measures is equivalent to the existence of their closed supports, the criterion of the existence of such measure can be formulated in terms of topological characteristics.

Theorem 13. *For* $G \subseteq Homeo_+(X)$ *the set* $Fix\, G^s$ *is either empty, or it is a Borel (probabilistic, in the case* $X = S^1$*) measure* μ, *finite on compact sets and invariant with respect to the group* G. □

The proof of this theorem [5] is based on our Theorems 10, 11 and 12.

If in the Margulis theorem (Theorem 9) the existence of an invariant measure will be guaranteed by $Fix\, G^s \neq \emptyset$, then we obtain the result proved earlier (1984) by Solodov [42]. In terms of homomorphisms (characters) this result was proved (1983) by Hector and Hirsch [28] for finitely generated groups of homeomorphisms of the circle. For arbitrary groups of homeomorphisms of the circle it has been obtained in 1996 by Beklaryan [7]. More interesting facts about various criterions of the existence of an invariant measure for groups of homeomorphisms of the real line (circle) one can find in the review [12].

Now we focus our attention on four theorems proved in [6] and their consequences.

Theorem 14. *If for $G \subseteq Homeo_+(X)$ there exists a Borel (probabilistic, in the case $X = S^1$) measure μ, finite on compact sets and invariant with respect to the group G, then $supp\, \mu \subseteq Fix\, G^s$ and $supp\, \mu = \mathbb{P}(G) = E(G)$, if $\mathbb{P}(G) \neq \emptyset$ (in this case μ is continuous). In the case $\mathbb{P}(G) = \emptyset$ the support of μ is the union of some discrete minimal sets.* □

Definition 6. A group $G \subseteq Homeo_+(X)$ is *strictly ergodic*, if there is a Borel measure, finite on compact sets and invariant with respect to the group G, and for any two invariant measures μ_1, μ_2 there is a constant $c > 0$ such that $\mu_1 = c\mu_2$.

Theorem 15. *If for the group $G \subseteq Homeo_+(X)$ there is a Borel (probabilistic, in the case $X = S^1$) measure, finite on compact sets and invariant with respect to the group G, then G is strictly ergodic if and only if*

1) $\mathbb{P}(G) \neq \emptyset$, *or*
2) $\mathbb{P}(G) = \emptyset$ *and $Fix\, G^s$ coincides with the unique non-empty minimal set.* □

Now, using the above results, especially Theorem 13, we can present the criterion of the existence of invariant measures in another form.

Theorem 16. *Let $G \subseteq Homeo_+(\mathbb{R})$. For the existence of Borel measures, finite on compact sets and invariant with respect to the group G, it is necessary and sufficient, that:*

1) *for any finitely generated subgroup $\Gamma \subseteq G$ there is a Borel measure, finite on compact sets and invariant with respect to the subgroup Γ,*

2) *there is a natural number n such that $[-n, n] \cap Fix\, \Gamma^s \neq \emptyset$ for any finitely generated subgroup Γ of G.* □

Theorem 17. *Let $G \subseteq Homeo_+(\mathbb{R})$. If the quotient group $G/< G^s >$ is non-trivial, i.e., $G/< G^s > \neq < e >$, then there is a Borel measure finite on compact sets and invariant with respect to the group G. Moreover, if the quotient group $G/< G^s >$ is non-cyclic, then the group G is strictly ergodic.* □

5.3. Combinatorial aspects

In view of Theorem 14 the support of an invariant measure is the union of minimal sets.

The natural problem is: *What are the combinatorial obstacles for a group with the non-empty minimal set to have an invariant measure*?

Various aspects of this problem were studied by many authors. Below we present some results obtained in [11] by Beklaryan.

In the formulation of these results a normal subgroup H_G of G plays an important role.

Definition 7. For a group $G \subseteq Homeo(X)$ we define the *normal subgroup* H_G in the following way:

1) if a minimal set is non-empty and non-discrete, then

$$H_G = \{h \in G_+ : E(G_+) \subseteq Fix < h >\},$$

2) if a minimal set is non-empty and discrete, then $H_G = G_+^s$,

   (since a minimal set is discrete, $Fix\, G_+^s$ is non-empty, consequently G_+^s is a normal subgroup),

3) if a minimal set is empty, then we put $H_G =< e >$.

Note that $H_G = < e >$ also in the case when a minimal set coincides with the real line.

Theorem 18. *Let $G \subseteq Homeo\,(S^1)$. Then either the quotient group G/H_G contains a free subgroup with two generators, or there is a probabilistic Borel measure invariant with respect to the group G.* □

Theorem 19. *Let $G \subseteq Homeo\,(\mathbb{R})$ be a group with a non-empty minimal set. Then either the quotient group G/H_G contains a free subsemigroup with two generators, or there is a Borel measure finite on compact sets and invariant with respect to the group G.* □

5.4. About analogs of the Tits's alternative

For groups $G \subseteq Homeo\,(X)$ with an invariant measure we have $H_G = G_+^s$, where G_+ is the maximal normal subgroup of all orientation-preserving homeomorphisms with the index no more than two. Thus, the quotient group G_+/H_G is commutative.

Theorem 18 about existence of an invariant measure on the circle can be reformulated to the form analogous to the Tits's alternative (cf. [11]).

Theorem 20. *For groups $G \subseteq Homeo\,(S^1)$ either the quotient group G/H_G contains a free subgroup with two generators, or contains a commutative normal subgroup G_+/H_G of index no more than two.* □

For any group $G \subseteq Homeo_+\,(S^1)$ the action of an element $\tilde{g} \in G/H_G$ can be realized as action on the circle. If $\tilde{g}$ corresponds to $g \in G$, then the action of $\tilde{g}$ coincides with the action of g on the minimal set.

If we denote by KG the class of almost commutative groups, then the chain (1) can be expanded to the chain

$$KG \subseteq EG \subseteq AG \subseteq F_N G \subseteq (F_N G \cup FG)\,. \tag{5}$$

Note that for groups of homeomorphisms of the circle Theorem 18 is equivalent to the condition

$$KG = F_N G$$

for corresponding quotient groups. This means that groups of homeomorphisms of the circle satisfy the extremal property for quotient groups.

Moreover, for such finitely generated groups, the growth of the quotient group satisfies the extremal property as well: either it is polynomial (for quotient groups from $F_N G$), or it is exponential (for quotient groups from FG) and there are no quotient groups having the intermediate growth.

Theorem 19 about existence of an invariant measure on the real line can be reformulated to the form analogous to the Tits's alternative (cf. [11]).

Theorem 21. *If $G \subseteq Homeo(\mathbb{R})$ is a group with the non-empty minimal set, then it either has the quotient group G/H_G containing a free subsemigroup with two generators, or it contains the commutative normal subgroup G_+/H_G with the index no more than two.* □

Let FPG be a class of groups containing the free subsemigroup with two generators, $F_N PG$ – a class of groups without free subsemigroups with two generators. Then

$$KG \subseteq F_N PG \subseteq (F_N PG \cup FPG) \,. \tag{6}$$

For groups of homeomorphisms of the real line Theorem 19 is equivalent to the condition

$$KG = F_N PG$$

for corresponding quotient groups and means that groups of homeomorphisms of the real line satisfy the extremal property for quotient groups. Moreover, for such finitely generated groups, the growth of the quotient group satisfies the extremal property also: either it is polynomial (for quotient groups from $F_N PG$), or it is exponential (for quotient groups from FPG) and there are no quotient groups of intermediate growth.

Thus, an investigation of groups of homeomorphisms G of the circle (the real line) can be reduced to the study of the canonical subgroups H_G in which all algebraic properties of the initial group are concentrated. For this purpose the additional metric invariants in the form of a projectively-invariant measure [5], [7], [8], [37] and a ω-projectively-invariant measure [9] can be applied.

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