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Generalized fuzzy subquasigroups

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Abstract

Different types of (α,β)-fuzzy subquasigroups, for $\alpha,\beta \in \{\in, q, \in\vee q, \in\wedge q\}$, $\alpha \neq \in \wedge q$, are investigated. Various characterizations of $(\in, \in \vee q)$-fuzzy subquasigroups are obtained. Fuzzy subquasigroups with thresholds are studied also.

1. Introduction

During the last decade, there have been many applications of quasigroups in different areas, such as cryptography [13], modern physics [15], coding theory, geometry [14]. In 1965, Zadeh introduced the notion of a fuzzy subset as a method for representing uncertainty. Since than fuzzy set theory has been developed in many directions by many scholars and has evoked great interest among mathematicians working in different fields of mathematics such as topological spaces, functional analysis, loops, groups, rings, semirings, hemirings, nearrings, vector spaces, differential equations, automation. The notion of fuzzy subgroup was made by Rosenfeld [1] in 1971. Das [5] characterized fuzzy subgroups by their level subgroups. The concept of quasi-coincidence of a fuzzy point with a fuzzy subset was introduced by Pu and Liu [17]. Using the idea of quasi-coincidence of a fuzzy point with a fuzzy subset, Bhakat and Das defined in [4] different types of fuzzy subgroups called, (α,β)-fuzzy subgroups. In particular, they introduced the concept of $(\in, \in \vee q)$-fuzzy subgroups which was an important and useful generalization of Rosenfeld's fuzzy subgroups. Dudek [7] introduced the notion of fuzzy subquasigroups and studied some their properties.

In this paper we introduce the notion of (α,β)- fuzzy subquasigroups where α, $\beta \in \{\in, q, \in\vee q, \in\wedge q\}$ and $\alpha \neq \in\wedge q$, and investigate some related properties. We characterize $(\in, \in\vee q)$- fuzzy subquasigroups by their levels subquasigroups. Finally we study fuzzy subquasigroups with thresholds.

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2. Preliminaries

In this section we review some facts which are necessary for this paper.

A groupoid $(G, \cdot)$ is called a *quasigroup* if for any $a, b \in G$ each of the equations $a \cdot x = b$, $x \cdot a = b$ have a unique solution in G. A quasigroup may be also defined as an *equasigroup*, i.e., an algebra $(G, \cdot, \backslash, /)$ with three binary operations $\cdot$, $\backslash$, $/$ satisfying the following identities:

$$(x \cdot y)/y = x, \qquad x \backslash (x \cdot y) = y,$$

$$(x/y) \cdot y = x, \qquad x \cdot (x \backslash y) = y.$$

A nonempty subset S of a quasigroup $\mathcal{G} = (G, \cdot, \backslash, /)$ is called a *subquasigroup* if it is closed with respect to these three operations, that is, if $x * y \in S$ for all $x, y \in S$ and $* \in \{\cdot, \backslash, /\}$.

A homomorphic image of an equasigroup is an equasigroup. Also every subset of an equasigroup closed with respect to these three operations is an equasigroup. In theory of quasigroups an important role play *unipotent quasigroups*, i.e., quasigroups with the identity $x \cdot x = y \cdot y$. These quasigroups are connected with Latin squares which have one fixed element on the diagonal [6]. Such quasigroups may be defined as quasigroups G with the special fixed element θ satisfying the identity $x \cdot x = \theta$. Obviously, θ is uniquely determined and it is an idempotent, but, in general, it is not the (left, right) neutral element.

To avoid repetitions we use the following convection: a *quasigroup* $\mathcal{G}$ always denotes an equasigroup $(G, \cdot, \backslash, /)$, G always denotes the nonempty set.

A mapping $\mu : G \to [0, 1]$ is called a *fuzzy set* on G. For any fuzzy set μ on G and any $t \in [0, 1]$, we define the set

$$U(\mu; t) = \{x \in G \mid \mu(x) \geqslant t\},$$

which is called the *upper t-level cut* of μ. The set $\underline{\mu} = \{x \in G \,|\, \mu(x) > 0\}$ is called the *support* of μ.

Definition 2.1. (cf. [7]) A fuzzy set μ on G is called a *fuzzy subquasigroup* of $\mathcal{G}$ if

$$\mu(x * y) \geqslant \min\{\mu(x), \mu(y)\}$$

for all $x, y \in G$ and $* \in \{\cdot, \backslash, /\}$.

The following two results are proved in [7].

Proposition 2.2. *A fuzzy set μ on a quasigroup $\mathcal{G}$ is a fuzzy subquasigroup if and only if every its nonempty upper level cut is a subquasigroup of $\mathcal{G}$.* □

Proposition 2.3. *If μ is a fuzzy subquasigroup of a unipotent quasigroup $(G,\cdot,\backslash,/,\theta)$, then $\mu(\theta)\geqslant\mu(x)$ for all $x\in G$.* □

Definition 2.4. A fuzzy set μ of the form

$$\mu(y)=\begin{cases} t\in(0,1] & \text{for} \quad y=x,\\ 0 & \text{for} \quad y\neq x,\end{cases}$$

is called a *fuzzy point* with the support x and the value t and is denoted by x_t.

For any fuzzy set μ the symbol $x_t\in\mu$ means that $\mu(x)\geqslant t$. In the case $\mu(x)+t>1$ we say that a fuzzy point x_t is *quasicoincident* with a fuzzy set μ and write $x_tq\mu$. The symbol $x_t\in\vee q\mu$ means that $x_t\in\mu$ or $x_tq\mu$. Similarly, $x_t\in\wedge q\mu$ denotes that $x_t\in\mu$ and $x_tq\mu$. $x_t\overline{\in}\mu$, $x_t\overline{q}\mu$ and $x_t\overline{\in\vee q}\mu$ mean that $x_t\in\mu$, $x_tq\mu$ and $x_t\in\vee q\mu$ do not hold, respectively.

3. (α,β)-fuzzy subquasigroups

Let α and β denote one of the symbols $\in$, q, $\in\vee q$ or $\in\wedge q$ unless otherwise specified.

Definition 3.1. *A fuzzy set μ in $\mathcal{G}$ is called a (α,β)-fuzzy subquasigroup of $\mathcal{G}$, if it satisfies the following condition:*

$$x_{t_1}\alpha\mu,\ y_{t_2}\alpha\mu\Longrightarrow(x*y)_{\min\{t_1,t_2\}}\beta\mu$$

for all $x,y\in G$, $t_1,t_2\in(0,1]$, $\alpha\neq\in\wedge q$ and $*\in\{\cdot,\backslash,/\}$.

Remark 3.2. (1) It is easy to construct 12 different types of fuzzy subquasigroups by the replacement of $\alpha(\neq\in\wedge q)$ and β in the Definition 3.1 by any two of $\{\in,q,\in\vee q,\in\wedge q\}$.

(2) Why $\alpha\neq\in\wedge q$? Since for a fuzzy set μ such that $\mu(x)\leqslant 0.5$ for all $x\in G$ and $x_t\in\wedge q\mu$ for some $t\in(0,1]$, we have $\mu(x)\geqslant t$ and $\mu(x)+t>1$. Thus

$$1<\mu(x)+t\leqslant\mu(x)+\mu(x)=2\mu(x),$$

so, $\mu(x)>0.5$. Hence $\{x_t\,|\,x_t\in\wedge q\mu\}=\emptyset$. This explains why $\alpha=\in\wedge q$ can be omitted in the above definition.

(3) $(\in,\in)$-fuzzy subquasigroups are in fact fuzzy subquasigroups.

(4) (α,β)-fuzzy subquasigroups are a generalization of fuzzy subquasigroups described in [7].

It is not difficult to see that the following proposition is true.

Proposition 3.3. *Every $(\in,\in)$-fuzzy subquasigroup is an $(\in,\in\vee q)$-fuzzy subquasigroup.* □

Corollary 3.4. *For any subset S of $\mathcal{G}$, the characteristic function χ_S of S is an $(\in,\in\vee q)$-fuzzy subquasigroup of $\mathcal{G}$ if and only if S is a subquasigroup of $\mathcal{G}$.*

Proof. Suppose that characteristic function χ_S is an $(\in,\in\vee q)$-fuzzy subquasigroup of $\mathcal{G}$. Let $x,y\in S$. Then $\chi_S(x)=1=\chi_S(y)$, and so $x_1\in\chi_S$ and $y_1\in\chi_S$. It follows that $(x*y)_1=(x*y)_{\min\{1,1\}}\in\vee q\chi_S$, which implies $\chi_S(x*y)>0$. Thus $x*y\in S$, and hence χ_S is a fuzzy subquasigroup of $\mathcal{G}$.

Conversely, if S is a fuzzy subquasigroup of $\mathcal{G}$, then χ_S is an $(\in,\in)$-fuzzy subquasigroup of $\mathcal{G}$ and, by Proposition 3.3, it is an $(\in,\in\vee q)$-fuzzy subquasigroup of $\mathcal{G}$. □

Proposition 3.5. *Every $(\in\vee q,\in\vee q)$-fuzzy subquasigroup is an $(\in,\in\vee q)$-fuzzy subquasigroup.*

Proof. Let μ be an $(\in\vee q,\in\vee q)$-fuzzy subquasigroup of $\mathcal{G}$. Let $x,y\in\mathcal{G}$ and $t_1,t_2\in(0,1]$ be such that $x_{t_1}\in\mu$ and $y_{t_2}\in\mu$. Then $x_{t_1}\in\vee q\mu$ and $y_{t_2}\in\vee q\mu$. Thus $(x*y)_{\min\{t_1,t_2\}}\in\vee q\mu$, which proves that μ is an $(\in,\in\vee q)$-fuzzy subquasigroup of $\mathcal{G}$. □

The converse statement of Proposition 3.5 is not true as we can see in the following example.

Example 3.6. The set $G=\{0,a,b,c\}$ with the multiplication:

$\cdot$	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

is a commutative quasigroup (Klein's group) in which the operations $\backslash$ and $/$ coincide with the group inverse operation.

Consider on this quasigroup the fuzzy set μ such that $\mu(0) = 0.5$, $\mu(a) = 0.6$ and $\mu(b) = \mu(c) = 0.3$. By routine computations, it is easy to verify that:

(1) μ is an $(\in, \in\vee q)$-fuzzy subquasigroup,

(2) μ is not an $(\in, \in)$-fuzzy subquasigroup because $a_{0.65} \in \mu$ and $a_{0.67} \in \mu$, but $(a * a)_{\min\{0.65,0.67\}} = 0_{0.65}\overline{\in}\mu$,

(3) μ is not an $(q, \in\vee q)$-fuzzy subquasigroup because $a_{0.51}q\mu$ and $b_{0.81}q\mu$, but $(a * b)_{\min\{0.51,0.81\}} = c_{0.51}\overline{\in\vee q}\mu$,

(4) μ is not an $(\in\vee q, \in\vee q)$-fuzzy subquasigroup because $a_{0.63} \in q\mu$ and $c_{0.77} \in q\mu$, but $(a * c)_{\min\{0.63,0.77\}} = c_{0.63}\overline{\in\vee q}\mu$. □

Now we prove some basic properties of (α, β)-fuzzy quasigroups.

Lemma 3.7. *If μ is a nonzero $(\in, \in)$-fuzzy subquasigroup of $\mathcal{G}$, then $\underline{\mu}$ is a subquasigroup of $\mathcal{G}$.*

Proof. If $\underline{\mu}$ is not a subquasigroup, then $\mu(x) > 0$, $\mu(y) > 0$ and $\mu(x*y) = 0$ for some $x, y \in \underline{\mu}$. But in this case $x_{\mu(x)}, y_{\mu(y)} \in \mu$ and $(x*y)_{\min\{\mu(x),\mu(y)\}}\overline{\in}\mu$, which is a contradiction. Hence $\mu(x * y) > 0$, i.e., $x * y \in \underline{\mu}$. So, $\underline{\mu}$ is a subquasigroup. □

Lemma 3.8. *If μ is a nonzero $(\in, q)$-fuzzy subquasigroup of $\mathcal{G}$, then $\underline{\mu}$ is a subquasigroup of $\mathcal{G}$.*

Proof. Similarly as in the previous proof suppose that $x, y \in \underline{\mu}$ and $x*y \notin \underline{\mu}$. Then $\mu(x) > 0$, $\mu(y) > 0$ and $\mu(x * y) = 0$. Consequently,

$$\mu(x * y) + \min\{\mu(x), \mu(y)\} = \min\{\mu(x), \mu(y)\} \leqslant 1.$$

Hence $(x * y)_{\min\{\mu(x),\mu(y)\}}\overline{q}\mu$, which is impossible. Thus $\mu(x * y) > 0$, so $x * y \in \underline{\mu}$. □

Lemma 3.9. *If μ is a nonzero $(q, \in)$-fuzzy subquasigroup of $\mathcal{G}$, then $\underline{\mu}$ is a subquasigroup of $\mathcal{G}$.*

Proof. Let $x, y \in \underline{\mu}$. Then $\mu(x) > 0$ and $\mu(y) > 0$. Thus $\mu(x) + 1 > 1$ and $\mu(y) + 1 > 1$, which imply that $x_1q\mu$ and $y_1q\mu$. If $\mu(x * y) = 0$, then $\mu(x * y) < 1 = \min\{1, 1\}$. Therefore, $(x * y)_{\min\{1,1\}}\overline{\in}\mu$, which is a contradiction. Therefore $\mu(x * y) > 0$, i.e., $x * y \in \underline{\mu}$. □

Lemma 3.10. *If μ is a nonzero (q,q)-fuzzy subquasigroup of $\mathcal{G}$, then $\underline{\mu}$ is a subquasigroup of $\mathcal{G}$.*

Proof. Let $x,y \in \underline{\mu}$. Then $\mu(x) > 0$ and $\mu(y) > 0$. Thus $\mu(x)+1 > 1$ and $\mu(y)+1 > 1$. This implies that $x_1 q\mu$ and $y_1 q\mu$. If $\mu(x*y)=0$, then $\mu(x*y)+\min\{1,1\} = 0+1 = 1$, and so $(x*y)_{\min\{1,1\}}\overline{q}\mu$. This is impossible, and hence $\mu(x*y)>0$, i.e., $x*y \in \underline{\mu}$. □

By using a very similar argumentation as in the proof of the above four lemmas we can prove the following theorem.

Theorem 3.11. *If μ is a nonzero (α,β)-fuzzy subquasigroup of $\mathcal{G}$, then $\underline{\mu}$ is a subquasigroup of $\mathcal{G}$.* □

Theorem 3.12. *Let S be a subquasigroup of $\mathcal{G}$. Then any fuzzy set μ of $\mathcal{G}$ such that $\mu(x) \geqslant 0.5$ for all $x \in S$ and $\mu(x)=0$ otherwise is a $(\alpha, \in\vee q)$-fuzzy subquasigroup.*

Proof. (i) Let $x,y \in G$ and $t_1,t_2 \in (0,1]$ be such that $x_{t_1} \in \mu$ and $y_{t_2} \in \mu$. Then $\mu(x) \geqslant t_1$ and $\mu(y) \geqslant t_2$. Thus $x,y \in S$, and so $x*y \in S$, i.e., $\mu(x*y) \geqslant 0.5$. If $\min\{t_1,t_2\} \leqslant 0.5$, then $\mu(x*y) \geqslant 0.5 \geqslant \min\{t_1,t_2\}$. Hence $(x*y)_{\min\{t_1,t_2\}} \in \mu$. If $\min\{t_1,t_2\} > 0.5$, then $\mu(x*y)+\min\{t_1,t_2\} > 0.5+0.5 = 1$ and so $(x*y)_{\min\{t_1,t_2\}} q\mu$. Therefore $(x*y)_{\min\{t_1,t_2\}} \in\vee q\mu$.

(ii) Let $x,y \in G$ and $t_1,t_2 \in (0,1]$ be such that $x_{t_1} q\mu$ and $y_{t_2} q\mu$. Then $x,y \in S$, $\mu(x)+t_1 > 1$ and $\mu(y)+t_2 > 1$. Since $x*y \in S$, we have $\mu(x*y) \geqslant 0.5$. If $\min\{t_1,t_2\} \leqslant 0.5$, then $\mu(x*y) \geqslant 0.5 \geqslant \min\{t_1,t_2\}$. Hence $(x*y)_{\min\{t_1,t_2\}} \in \mu$. If $\min\{t_1,t_2\} > 0.5$, then $\mu(x*y)+\min\{t_1,t_2\} > 0.5+0.5 = 1$ and so $(x*y)_{\min\{t_1,t_2\}} q\mu$. Therefore $(x*y)_{\min\{t_1,t_2\}} \in\vee q\mu$.

(iii) Let $x,y \in G$ and $t_1,t_2 \in (0,1]$ be such that $x_{t_1} \in \mu$ and $y_{t_2} q\mu$. Then $\mu(x) \geqslant t_1$ and $\mu(y)+t_2 > 1$. Since $x,y \in S$, also $x*y \in S$, i.e., $\mu(x*y) \geqslant 0.5$. Analogously as in (i) and (ii) we obtain $(x*y)_{\min\{t_1,t_2\}} \in \mu$ for $\min\{t_1,t_2\} \leqslant 0.5$ and $(x*y)_{\min\{t_1,t_2\}} q\mu$ for $\min\{t_1,t_2\} > 0.5$. Thus $(x*y)_{\min\{t_1,t_2\}} \in\vee q\mu$.

(iv) The case $x_{t_1} q\mu$ and $y_{t_2} \in \mu$ is analogous to (iii). □

Theorem 3.13. *A fuzzy set μ of $\mathcal{G}$ is an $(\in, \in\vee q)$-fuzzy subquasigroup if and only if it satisfies the inequality*

$$\mu(x*y) \geqslant \min\{\mu(x),\mu(y),0.5\}. \tag{1}$$

Proof. Let μ be an $(\in, \in\vee q)$-fuzzy subquasigroup of $\mathcal{G}$. Suppose that for $x, y \in G$ we have $\min\{\mu(x), \mu(y)\} < 0.5$. If $\mu(x*y) < \min\{\mu(x), \mu(y)\}$, then $x_t \in \mu$ and $y_t \in \mu$ for any t such that $\mu(x*y) < t < \min\{\mu(x), \mu(y)\}$. but in this case $(x*y)_{\min\{t,t\}} = (x*y)_t \overline{\in\vee q}\mu$, a contradiction. This means that in the case $\min\{\mu(x), \mu(y)\} < 0.5$ must be $\mu(x*y) \geqslant \min\{\mu(x), \mu(y)\}$.

If $\min\{\mu(x), \mu(y)\} \geqslant 0.5$, then $x_{0.5} \in \mu$ and $y_{0.5} \in \mu$, which imply

$$(x*y)_{\min\{0.5,0.5\}} = (x*y)_{0.5} \in\vee q\mu.$$

Hence $\mu(x*y) \geqslant 0.5$. Otherwise, $\mu(x*y) + 0.5 < 0.5 + 0.5 = 1$, a contradiction. Consequently, $\mu(x*y) \geqslant 0.5 = \min\{\mu(x), \mu(y), 0.5\}$ for all $x, y \in G$.

Conversely, assume that the inequality mentioned in the above theorem is valid. Let $x, y \in G$ and $t_1, t_2 \in (0, 1]$ be such that $x_{t_1} \in \mu$ and $y_{t_2} \in \mu$. Then $\mu(x) \geqslant t_1$ and $\mu(y) \geqslant t_2$. In the case $\mu(x*y) \geqslant \min\{t_1, t_2\}$ we obtain $(x*y)_{\min\{t_1,t_2\}} \in \mu$. In the case $\mu(x*y) < \min\{t_1, t_2\}$ we have $\min\{\mu(x), \mu(y)\} \geqslant 0.5$. If not, then

$$\mu(x*y) \geqslant \min\{\mu(x), \mu(y), 0.5\} \geqslant \min\{\mu(x), \mu(y)\} \geqslant \min\{t_1, t_2\},$$

which is a contradiction. So, in this case

$$\mu(x*y) + \min\{t_1, t_2\} > 2\mu(x*y) \geqslant 2\min\{\mu(x), \mu(y), 0.5\} = 1,$$

i.e., $(x*y)_{\min\{t_1,t_2\}} q\mu$. Hence μ is an $(\in, \in\vee q)$-fuzzy subquasigroup of $\mathcal{G}$. □

Corollary 3.14. *Any $(\in, \in\vee q)$-fuzzy subquasigroup μ of $\mathcal{G}$ satisfying the inequality $\mu(x) < 0.5$ is an ordinary fuzzy subquasigroup of $\mathcal{G}$.* □

Theorem 3.15. *A fuzzy set μ of $\mathcal{G}$ is its $(\in, \in\vee q)$-fuzzy subquasigroup if and only if for every $t \in (0, 0.5]$ each nonempty level $U(\mu; t)$ is a subquasigroup of $\mathcal{G}$.*

Proof. Assume that μ is an $(\in, \in\vee q)$-fuzzy subquasigroup of $\mathcal{G}$ and let $t \in (0, 0.5]$ be such that $U(\mu; t) \neq \emptyset$. If $x, y \in U(\mu; t)$, then $\mu(x) \geqslant t$ and $\mu(y) \geqslant t$. Thus

$$\mu(x*y) \geqslant \min\{\mu(x), \mu(y), 0.5\} \geqslant \min\{t, 0.5\} = t.$$

So, $x*y \in U(\mu; t)$. Hence $U(\mu; t)$ is a subquasigroup of $\mathcal{G}$.

Conversely, suppose that each nonempty level $U(\mu; t)$, $t \in (0, 0.5]$, is a subquasigroup of $\mathcal{G}$. If there are $x, y \in G$ such that

$$\mu(x*y) < \min\{\mu(x), \mu(y), 0.5\},$$

then also

$$\mu(x*y) < t_1 < \min\{\mu(x), \mu(y), 0.5\}$$

for some t_1. This means that $x, y \in U(\mu; t_1)$ and $x*y \notin U(\mu; t_1)$, which contradicts to the assumption that all $U(\mu; t)$ are subquasigroups. Therefore

$$\mu(x*y) < \min\{\mu(x), \mu(y), 0.5\}.$$

So, μ is an $(\in, \in\vee q)$-fuzzy subquasigroup of $\mathcal{G}$. □

Theorem 3.16. *The nonempty intersection of any family of $(\in, \in\vee q)$-fuzzy subquasigroups of $\mathcal{G}$ is an $(\in, \in\vee q)$-fuzzy subquasigroups of $\mathcal{G}$.*

Proof. Let $\{\lambda_i : i \in \Lambda\}$ be a fixed family of $(\in, \in\vee q)$-fuzzy subquasigroups of $\mathcal{G}$ and let λ be the nonempty intersection of this family. If $x_{t_1}, y_{t_2} \in \lambda$ and $(x*y)_{\min\{t_1,t_2\}}\overline{\in\vee q}\lambda$ for some $x, y \in G$ and $t_1, t_2 \in (0, 1]$, then

$$\lambda(x*y) < \min\{t_1, t_2\} \quad\text{and}\quad \lambda(x*y) + \min\{t_1, t_2\} \leqslant 1.$$

Thus $\lambda(x*y) < 0.5$.

Since each λ_i is an $(\in, \in\vee q)$-fuzzy subquasigroup, the family $\{\lambda_i : i \in \Lambda\}$ can be divided into two disjoint parts:

$$\Lambda' = \{\lambda_i \,|\, \lambda_i(x*y) \geqslant \min\{t_1, t_2\}\}$$

and

$$\Lambda'' = \{\lambda_i \,|\, \lambda_i(x*y) < \min\{t_1, t_2\} \quad\text{and}\quad \lambda_i(x*y) + \min\{t_1, t_2\} > 1\}.$$

If $\lambda_i(x*y) \geqslant \min\{t_1, t_2\}$ for all λ_i, then also $\lambda(x*y) \geqslant \min\{t_1, t_2\}$, which is a contradiction. So, for some λ_i we have $\lambda_i(x*y) < \min\{t_1, t_2\}$ and $\lambda_i(x*y) + \min\{t_1, t_2\} > 1$. Thus $\min\{t_1, t_2\} > 0.5$, whence $\lambda_i(x) \geqslant \lambda(x) \geqslant t_1 \geqslant \min\{t_1, t_2\} > 0.5$ for all $\lambda_i \in \Lambda''$. Similarly $\lambda_i(y) > 0.5$ for all $\lambda_i \in \Lambda''$. Now suppose that $t = \lambda_i(x*y) < 0.5$ for some λ_i. Let $t' \in (0, 0.5)$ be such that $t < t'$, then $\lambda_i(x) > 0.5 > t'$ and $\lambda_i(y) > 0.5 > t'$, that is $x_{t'} \in \lambda_i$ and $y_{t'} \in \lambda_i$ but $\lambda_i(x*y) = t < t'$ and $\lambda_i(x*y) + t' < 1$. So, $(x*y)_{t'}\overline{\in\vee q}\lambda_i$. This contradicts that λ_i is a $(\in, \in\vee q)$ fuzzy subquasigroup of $\mathcal{G}$. Hence $\lambda_i(x*y) \geqslant 0.5$ for all λ_i, and thus $\lambda(x*y) \geqslant 0.5$. This is impossible because for all $x, y \in G$ we have $\lambda(x*y) < 0.5$. Therefore $(x*y)_{\min\{t_1,t_2\}} \in\vee q\lambda$. □

For any fuzzy subset μ of $\mathcal{G}$ and any $t \in (0, 1]$ we consider two subsets:

$$Q(\mu; t) = \{x \in G \,|\, x_t q \mu\} \quad\text{and}\quad [\mu]_t = \{x \in G \,|\, x_t \in\vee q \mu\}.$$

It is clear that $[\mu]_t = U(\mu;t) \cup Q(\mu;t)$.

In Theorem 3.15 we have shown that a fuzzy subset μ of a quasigroup $\mathcal{G}$ is an $(\in, \in\vee q)$-fuzzy subquasigroup of $\mathcal{G}$ if and only if $U(\mu;t) \neq \emptyset$ is a subquasigroup of $\mathcal{G}$ for all $0 < t \leqslant 0.5$. Now we show a similar result for $[\mu]_t$.

Theorem 3.17. *A fuzzy subset μ of $\mathcal{G}$ is an $(\in, \in\vee q)$-fuzzy subquasigroup of $\mathcal{G}$ if and only if $[\mu]_t$ is a subquasigroup of $\mathcal{G}$ for all $t \in (0, 0.5]$.*

Proof. Let μ be an $(\in, \in\vee q)$-fuzzy subquasigroup of $\mathcal{G}$. Let $x, y \in [\mu]_t$ for some $t \in (0, 0.5]$. Then $\mu(x) \geqslant t$ or $\mu(x) + t > 1$ and $\mu(y) \geqslant t$ or $\mu(y)+t > 1$. Since μ is an $(\in, \in\vee q)$-fuzzy subquasigroup, we have $\mu(x*y) \geqslant \min\{\mu(x), \mu(y), 0.5\}$ (Theorem 3.13). This implies $\mu(x*y) \geqslant \min\{t, 0.5\} = t$. So, $x * y \in [\mu]_t$.

Conversely, let μ be a fuzzy subset of $\mathcal{G}$ and let $[\mu]_t$ be a subquasigroup of $\mathcal{G}$ for all $t \in (0, 0.5]$. If $\mu(x * y) < t < \min\{\mu(x), \mu(y), 0.5\}$ for some $t \in (0, 0.5]$, then $x, y \in [\mu]_t$ and $x*y \in [\mu]_t$. Hence $\mu(x*y) \geqslant t$ or $\mu(x*y)+t > 1$, a contradiction. Therefore $\mu(x*y) \geqslant \min\{\mu(x), \mu(y), 0.5\}$ for all $x, y \in G$. □

Lemma 3.18. *Let μ be an arbitrary fuzzy set defined on $\mathcal{G}$ and let $x \in G$. Then $\mu(x) = t$ if and only if $x \in U(\mu;t)$, $x \notin U(\mu;s)$ for all $s > t$.* □

Theorem 3.19. *Let $\{A_t\}_{t\in\Gamma}$, where $\Gamma \subseteq (0, 0.5]$ be a collection of subquasigroups of $\mathcal{G}$ such that $G = \bigcup\limits_{t\in\Gamma} A_t$, and for $s, t \in \Gamma$, $s < t$ if and only if $A_t \subset A_s$. Then a fuzzy set μ defined by*

$$\mu(x) = \sup\{t \in \Gamma \,|\, x \in A_t\},$$

is an $(\in, \in\vee q)$-fuzzy subquasigroup of $\mathcal{G}$.

Proof. According to Theorem 3.15, it is sufficient to show that for every $t \in (0, 0.5]$, each nonempty $U(\mu;t)$ is a subquasigroup of $\mathcal{G}$. We consider two cases:

$$(i) \quad t = \sup\{s \in \Gamma \,|\, s < t\}$$

$$(ii) \quad t \neq \sup\{s \in \Gamma \,|\, s < t\}.$$

In the first case

$$x \in U(\mu;t) \longleftrightarrow (x \in A_s \ \forall s < t) \longleftrightarrow x \in \bigcap_{s<t} A_s.$$

So, $U(\mu;t) = \bigcap\limits_{s<t} A_s$, which is a subquasigroup of $\mathcal{G}$. In the second case, we have $U(\mu;t) = \bigcup\limits_{s\geq t} A_s$. Indeed, if $x \in \bigcup\limits_{s\geqslant t} A_s$, then $x \in A_s$ for some $s \geqslant t$. Thus $\mu(x) \geqslant s \geqslant t$, i.e., $x \in U(\mu;t)$. This proves $\bigcup\limits_{s\geqslant t} A_s \subset U(\mu;t)$. To prove the converse inclusion consider $x \notin \bigcup\limits_{s\geqslant t} A_s$. Then $x \notin A_s$ for all $s \geqslant t$. Since $t \neq \sup\{s \in \Gamma \,|\, s < t\}$, there exists $\varepsilon > 0$ such that $(t-\varepsilon, t) \cap \Gamma = \emptyset$. Hence $x \notin A_s$ for all $s > t-\varepsilon$, which means that if $x \in A_s$, then $s \leqslant t-\varepsilon$. Thus $\mu(x) \leq t-\varepsilon < t$, and so $x \notin U(\mu;t)$. Therefore $U(\mu;t) = \bigcup\limits_{s\geqslant t} A_s$. Since, as it is not difficult to verify, $\bigcup\limits_{s\geqslant t} A_s$ is a subquasigroup of $\mathcal{G}$, we see that $U(\mu;t)$ is a subquasigroup in any case. □

Theorem 3.20. *For any chain $A_0 \subset A_1 \subset A_2 \subset \ldots \subset A_n = G$ of subquasigroups of $\mathcal{G}$ there exists an $(\in, \in\!\vee q)$-fuzzy subquasigroup of $\mathcal{G}$ for which level sets coincide with this chain.*

Proof. Let $t_0, t_1, \ldots, t_n$ be a finite decreasing sequence in $[0,1]$. Consider the fuzzy set μ on $\mathcal{G}$ defined by $\mu(A_0) = t_0$ and $\mu(A_k \setminus A_{k-1}) = t_k$ for $0 < k \leqslant n$. Let $x, y \in G$. If $x, y \in A_k \setminus A_{k-1}$, then $x * y \in A_k$ and

$$\mu(x*y) \geqslant t_k = \min\{\mu(x), \mu(y)\}.$$

Now let $x \in A_i \setminus A_{i-1}$ and $y \in A_j \setminus A_{j-1}$, where $i \neq j$. If $i > j$, then $A_j \subset A_i$, $\mu(x) = t_i < t_j = \mu(y)$, $x * y \in A_i$. Thus

$$\mu(x*y) \geqslant t_i = \min\{\mu(x), \mu(y)\}.$$

Analogously for $i < j$. So, μ is a fuzzy subquasigroup. It is not difficult to see that it is an $(\in, \in\!\vee q)$-fuzzy subquasigroup.

Such defined μ has only the values $t_0, t_1, \ldots, t_n$. Their level subsets are subquasigroups and form the chain

$$U(\mu;t_0) \subset U(\mu;t_1) \subset \ldots \subset U(\mu;t_n) = G.$$

We now prove that $U(\mu;t_k) = A_k$ for $0 \leqslant k \leqslant n$. Indeed,

$$U(\mu;t_0) = \{x \in G \,|\, \mu(x) \geqslant t_0\} = A_0.$$

Moreover, $A_k \subseteq U(\mu;t_k)$ for $0 < k \geqslant n$. If $x \in U(\mu;t_k)$, then $\mu(x) \geqslant t_k$ and so $x \notin A_i$ for $i > k$. Hence $\mu(x) \in \{t_0, t_1, \ldots, t_k\}$, which implies $x \in A_i$ for some $i \leqslant k$. Since $A_i \subseteq A_k$, it follows that $x \in A_k$. Consequently, $U(\mu;t_k) = A_k$ for every $0 < k \leqslant n$. This completes the proof. □

4. Fuzzy subquasigroups with thresholds

Definition 4.1. Let $0 \leqslant \lambda_1 < \lambda_2 \leqslant 1$ be fixed. A fuzzy set μ of a quasigroup $\mathcal{G}$ is called a *fuzzy subquasigroup with thresholds* (λ_1, λ_2), if

$$\max\{\mu(x*y), \lambda_1\} \geqslant \min\{\mu(x), \mu(y), \lambda_2\}$$

for all $x, y \in G$.

It is not difficult to see that:

- for $\lambda_1 = 0$ and $\lambda_2 = 1$ we have an ordinary fuzzy subquasigroup,
- for $\lambda_1 = 0$ and $\lambda_2 = 0.5$ we have an $(\in, \in\vee q)$-fuzzy subquasigroup,
- a fuzzy subquasigroup is a fuzzy subquasigroup with some thresholds,
- also any $(\in, \in\vee q)$-fuzzy subquasigroup is a fuzzy subquasigroup with some thresholds.

Example 4.2. Let $\mathcal{G}$ be a commutative quasigroup defined in Example 3.6 and let $\mu(0) = 0.5$, $\mu(a) = 0.7$, $\mu(b) = 0.4$, $\mu(c) = 0.3$. Then:

1. μ is a fuzzy subquasigroup with thresholds $\lambda_1 = 0.4$ and $\lambda_2 = 0.65$, but it is not a fuzzy subquasigroup with thresholds $\lambda_1 = 0.6$ and $\lambda_2 = 0.8$ since $\max\{\mu(a*a), 0.6\} = 0.6 < 0.7 = \min\{\mu(a), \mu(a), 0.8\}$,
2. μ is a fuzzy subquasigroup with thresholds $\lambda_1 = 0.77$ and $\lambda_2 = 0.88$, but it is not an ordinary fuzzy subquasigroup because $\mu(a*b) = \mu(c) = 0.3 < 0.4 = \min\{\mu(a), \mu(b)\}$. □

Theorem 4.3. *A fuzzy set μ of a quasigroup $\mathcal{G}$ is a fuzzy subquasigroup with thresholds (λ_1, λ_2) if and only if for all $t \in (\lambda_1, \lambda_2]$ each nonempty $U(\mu; t)$ is a subquasigroup of $\mathcal{G}$.*

Proof. The proof is similar to the proof of Theorem 3.15. □

Note that in the above theorem the restriction $t \in (\lambda_1, \lambda_2]$ is essential. $U(\mu; t)$ for $t \in (0, \lambda_1]$ may not be a subquasigroup of $\mathcal{G}$.

Example 4.4. The set $\mathbb{Z}$ of all integers with three operations $\circ$, $\backslash$, $/$ defined as follows: $x \circ y = x - y$, $x \backslash y = x - y$, $x/y = x + y$, is a quasigroup. Consider the following fuzzy set

$$\mu(x) = \begin{cases} 0 & \text{if} \quad x < 0 \quad \text{and} \quad x \neq 2k, \\ 0.3 & \text{if} \quad x > 0 \quad \text{and} \quad x \neq 2k, \\ 0.5 & \text{if} \quad x = 2n \quad \text{and} \quad x \neq 4k, \\ 0.8 & \text{if} \quad x = 4n \quad \text{and} \quad x \neq 8k, \\ 0.9 & \text{if} \quad x = 8n \quad \text{and} \quad x < 0, \\ 1 & \text{if} \quad x = 8n \quad \text{and} \quad x > 0, \end{cases}$$

where k and n are arbitrary integers. Then

$$U(\mu;t)=\begin{cases}\mathbb{Z} & \text{for} \quad t=0,\\ 2\mathbb{Z}\cup\mathbb{Z}^{+} & \text{for} \quad t\in(0,0.3],\\ 2\mathbb{Z} & \text{for} \quad t\in(0.3,0.5],\\ 4\mathbb{Z} & \text{for} \quad t\in(0.5,0.8],\\ 8\mathbb{Z} & \text{for} \quad t\in(0.8,0.9],\\ 8\mathbb{Z}^{+} & \text{for} \quad t\in(0.9,1],\end{cases}$$

where $p\mathbb{Z}$ denotes the set of all integers divided by p, $\mathbb{Z}^{+}$ – the set of all positive integers. It is clear that for $t\in(0.3,0.9]$ each $U(\mu;t)$ is a subguasigroup of this quasigroup. For $t\in(0,0.3]$ and $t\in(0.9,1]$ $U(\mu;t)$ are not subquasigroups. So, in view of Theorem 4.3, μ is a fuzzy subquasigroup with thresholds $\lambda_1=0.3$ and $\lambda_2=0.9$. But μ is not a fuzzy subquasigroup since

$$\mu(3\circ 8)=\mu(-5)=0\not\geqslant 0.3=\min\{\mu(3),\mu(8)\}.$$

It is not an $(\in,\in\vee q)$-fuzzy subquasigroup too because $3_{0.2}\in\mu$ and $8_{0.5}\in\mu$ but $(3\circ 8)_{\min\{0.2,0.5\}}\overline{\in\vee q}\mu$. □

Theorem 4.5. *Let $f:\mathcal{G}_1\to\mathcal{G}_2$ be an epimorphism of quasigroups and let μ and ν be fuzzy subquasigroups of $\mathcal{G}_1$ and $\mathcal{G}_2$, respectively. Then $f(\mu)$ defined by*

$$f(\mu)(y)=\sup\{\mu(x)\,|\,f(x)=y \quad for\ all\ y\in\mathcal{G}_2\}$$

and $f^{-1}(\nu)$ defined by

$$f^{-1}(\nu)(x)=\nu(f(x)) \quad for\ all \quad x\in\mathcal{G}_1$$

are fuzzy subquasigroups of $\mathcal{G}_2$ and $\mathcal{G}_1$, respectively. Moreover, if μ and ν are with thresholds (λ_1,λ_2), then also $f(\mu)$ and $f^{-1}(\nu)$ are with thresholds (λ_1,λ_2).

Proof. Let $y_1,y_2\in G_2$. Then

$$\begin{aligned}\max\{f(\mu)(y_1*y_2),\lambda_1\}&=\max\{\sup\{\mu(x_1*x_2),|f(x_1*x_2)=y_1*y_2\},\lambda_1\}\\&=\sup\{\max\{\mu(x_1*x_2),\lambda_1\}\mid f(x_1*x_2)=y_1*y_2\}\\&\geqslant\sup\{\min\{\mu(x_1),\mu(x_2),\lambda_1\}|f(x_1)=y_1,f(x_2)=y_2\}\\&=\min\{\sup\{\mu(x_1)\mid f(x_1)=y_1\},\\&\qquad\qquad\sup\{\mu(x_2)\mid f(x_2)=y_2\},\lambda_2\}\\&=\min\{f(\mu)(y_1),f(\mu)(y_2),\lambda_2\}.\end{aligned}$$

Similarly, for $x, y \in G_1$ we obtain

$$\begin{aligned}\max\{f^{-1}(\nu)(x*y),\lambda_1\} &= \max\{\nu(f(x*y)),\lambda_1\} = \max\{\mu(f(x)*f(y)),\lambda_1\} \\ &\geqslant \min\{\nu(f(x)),\nu(f(y)),\lambda_2\} = \min\{f^{-1}(\nu)(x), f^{-1}(\nu)(y),\lambda_2\},\end{aligned}$$

which completes the proof. □

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