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Fuzzy Lie ideals of Lie algebras with interval-valued membership functions

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Abstract

The concept of interval-valued fuzzy sets was first introduced by Zadeh in 1975 as a generalization of fuzzy sets. In this paper we introduce the notion of interval-valued fuzzy Lie ideals in Lie algebras and investigate some of their properties. Using interval-valued fuzzy Lie ideals, characterizations of Noetherian Lie algebras are established. Construction of a quotient Lie algebra via interval-valued fuzzy Lie ideal in a Lie algebra is given. The interval-valued fuzzy isomorphism theorems are also established.

1. Introduction

Lie algebras were discovered by Sophus Lie (1842-1899) when he first attempted to classify certain "smooth" subgroups of general linear groups. The groups he considered are now called Lie groups. By taking the tangent space at the identity element of such a group, he obtained the Lie algebra and hence the problems on groups can be reduced to problems on Lie algebras so that it becomes more tractable. Lie algebra is applied in different domains such as physics, hyperbolic and stochastic differential equations. Lie algebra is also largely used by electrical engineers, mainly in the mobile robot control [5].

The notion of interval-valued fuzzy sets was first introduced by Zadeh [13] in 1975 as a generalization of fuzzy sets. An interval-valued fuzzy set is a fuzzy set whose membership function is many-valued and forms an interval with respect to the membership scale. This idea gives the simplest method

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to capture the imprecision of the membership grades for a fuzzy set. It has been noted by Atansassov [3] that such fuzzy sets have some applications in the technological scheme of the functioning of a silo-farm with pneumatic transportation in a plastic products company and in medicine. The interval valued fuzzy sets provide a more adequate description of uncertainty than the traditional fuzzy sets, it is therefore important to use interval valued fuzzy sets in applications. One of the main applications of fuzzy sets is fuzzy control, and one of the most computationally intensive part of fuzzy control is the defuzzification. Since a transition to interval valued fuzzy sets usually increase the amount of computations, it is vitally important to design faster algorithms for the corresponding defuzzification. Fuzzy and anti fuzzy Lie ideals in Lie algebras have been studied in [1, 6, 7, 10, 11, 12]. In this paper, we apply the concept of interval-valued fuzzy sets to Lie algebras. We introduce the notion of interval-valued fuzzy Lie ideals in Lie algebras and investigate some of their properties. Using interval-valued fuzzy Lie ideals, characterizations of Noetherian Lie algebras are established. Construction of a quotient Lie algebra via interval-valued fuzzy Lie ideal in a Lie algebra is given. The interval-valued fuzzy isomorphism theorems are also established.

2. Preliminaries

In this paper by L will be denoted a *Lie algebra*, i.e., a vector space L over a field F (equal to $\mathbf{R}$ or $\mathbf{C}$) on which the operation $L \times L \to L$ denoted by $(x, y) \to [x, y]$ is defined and satisfies the following axioms:

(L_1) $[x, y]$ is bilinear,

(L_2) $[x, x] = 0$ for all $x \in L$,

(L_3) $[[x, y], z] + [[y, z], x] + [[z, x], y] = 0$ for all $x, y, z \in L$.

A subspace H of L closed under $[\ ,\ ]$ will be called a *Lie subalgebra.* A subspace I of L with the property $[I, L] \subseteq I$ will be called a *Lie ideal* of L. Obviously, any Lie ideal is a subalgebra.

A *fuzzy set* $\mu : L \to [0, 1]$ is called a *fuzzy Lie subalgebra* of L if

(a) $\mu(x + y) \geqslant \min\{\mu(x), \mu(y)\}$,

(b) $\mu(\alpha x) \geqslant \mu(x)$,

(c) $\mu([x, y]) \geqslant \min\{\mu(x), \mu(y)\}$

hold for all $x, y \in L$ and $\alpha \in F$.

According to [1] a fuzzy subset $\mu : L \to [0,1]$ satisfying (a), (b) and

(d) $\mu([x,y]) \geqslant \mu(x)$

is called a *fuzzy Lie ideal* of L. A fuzzy ideal of L is a fuzzy subalgebra [6] such that $\mu(-x) \geqslant \mu(x)$ holds for all $x \in L$.

By an *interval number* D we mean an interval $[a^-, a^+]$, where $0 \leqslant a^- \leqslant a^+ \leqslant 1$. The set of all interval numbers is denoted by $\mathcal{D}[0,1]$. For interval numbers $D_1 = [a_1^-, b_1^+]$, $D_2 = [a_2^-, b_2^+]$, we define

$\min(D_1, D_2) = \min([a_1^-, b_1^+], [a_2^-, b_2^+]) = [\min\{a_1^-, a_2^-\}, \min\{b_1^+, b_2^+\}]$,

$D_1 \leqslant D_2 \longleftrightarrow a_1^- \leqslant a_2^-$ and $b_1^+ \leqslant b_2^+$,

$D_1 = D_2 \longleftrightarrow a_1^- = a_2^-$ and $b_1^+ = b_2^+$.

An *interval-valued fuzzy set* (briefly, IF set) A on L is defined by

$$A = \{(x, [\mu_A^-, \mu_A^+]) : x \in L\},$$

where μ_A^- and μ_A^+ are fuzzy sets of L such that $\mu_A^-(x) \leqslant \mu_A^+(x)$ for all $x \in L$. Let $\widetilde{\mu}_A(x) = [\mu_A^-(x), \mu_A^+(x)]$, then

$$A = \{(x, \widetilde{\mu}_A(x)) : x \in L\},$$

where $\widetilde{\mu}_A : L \to \mathcal{D}[0,1]$. For $[s,t] \in \mathcal{D}[0,1]$, the set

$$U(\widetilde{\mu}; [s,t]) = \{x \in L : \widetilde{\mu}(x) \geq [s,t]\}$$

is called *upper level* of $\widetilde{\mu}$.

3. Interval-valued fuzzy Lie ideals in Lie algebras

Definition 3.1. An interval-valued fuzzy set $\widetilde{\mu}$ in a Lie algebra L is called an *interval-valued fuzzy Lie subalgebra* of L if

(1) $\widetilde{\mu}(x+y) \geqslant \min\{\widetilde{\mu}(x), \widetilde{\mu}(y)\}$,

(2) $\widetilde{\mu}(\alpha x) \geqslant \widetilde{\mu}(x)$,

(3) $\widetilde{\mu}([x,y]) \geqslant \min\{\widetilde{\mu}(x), \widetilde{\mu}(y)\}$

hold for all $x, y \in L$ and $\alpha \in F$.

Definition 3.2. An interval-valued fuzzy set $\widetilde{\mu}$ satisfying (1), (2) and

(4) $\widetilde{\mu}([x,y]) \geqslant \widetilde{\mu}(x)$

is called an *interval-valued fuzzy Lie ideal* of L.

From (2) it follows that

(5) $\widetilde{\mu}(0) \geqslant \widetilde{\mu}(x)$,

(6) $\widetilde{\mu}(-x) \geqslant \widetilde{\mu}(x)$

for all $x \in L$.

Example 3.3. The set $\Re^3 = \{(x,y,z) : x,y,z \in R\}$ with the operation $[x,y] = x \times y$, is a real Lie algebra. We define an IF set $\widetilde{\mu} : \Re^3 \to \mathcal{D}[0,1]$ by

$$\widetilde{\mu}(x,y,z) = \begin{cases} [s_1,s_2] & \text{if } x=y=z=0, \\ [t_1,t_2] & \text{otherwise,} \end{cases}$$

where $[s_1,s_2] > [t_1,t_2]$ and $[s_1,s_2],[t_1,t_2] \in \mathcal{D}[0,1]$. By routine computations, we can see that it is an IF Lie subalgebra and Lie ideal of $\Re^3$.

Proposition 3.4. *Every interval-valued fuzzy Lie ideal is an interval-valued fuzzy Lie subalgebra.*

The converse of Proposition 3.4 is not true in general.

Example 3.5. Let $\Re^3$ and $[\ ,\ ]$ be as in the previous example. Putting

$$\widetilde{\mu}(x,y,z) = \begin{cases} [1,1] & \text{if } x=y=z=0, \\ [0.5,\ 0.5] & \text{if } x \neq 0, y=z=0, \\ [0,\ 0] & \text{otherwise,} \end{cases}$$

we obtain an example of an interval-valued fuzzy Lie subalgebra which is not an IF Lie ideal. Indeed,

$$\widetilde{\mu}([(1,0,0)\ (1,1,1)]) = \widetilde{\mu}(0,-1,1) = [0,0],$$

$$\widetilde{\mu}(1,0,0) = [0.5,0.5].$$

That is,

$$\widetilde{\mu}([(1,0,0)\ (1,1,1)]) \ngeq \widetilde{\mu}(1,0,0).$$

Theorem 3.6. *An interval-valued fuzzy set* $\widetilde{\mu} = [\mu^-, \mu^+]$ *in* L *is an interval-valued fuzzy Lie ideal if and only if* μ^- *and* μ^+ *are fuzzy Lie ideals of* L.

Proof. Suppose that μ^- and μ^+ are fuzzy Lie ideals of L. Then

$$\begin{aligned}\widetilde{\mu}(x+y) &= [\mu^-(x+y), \mu^+(x+y)]\\ &\geqslant [\min\{\mu^-(x)), \mu^-(y)\}, \min\{\mu^+(x)), \mu^+(y)\}]\\ &= [\min\{\mu^-(x), \mu^+(x)\}, \min\{\mu^-(y), \mu^+(y)\}]\\ &= \min\{\widetilde{\mu}(x)), \widetilde{\mu}(y)\}\end{aligned}$$

for $x, y \in L$. The verification of (2) and (4) is analogous. Hence $\widetilde{\mu}$ is an interval-valued fuzzy Lie ideal of L.

Conversely, assume that $\widetilde{\mu}$ is an interval-valued fuzzy Lie ideal of L. Then

$$\begin{aligned}[\mu^-(x+y), \mu^+(x+y)] &= \widetilde{\mu}(x+y) \geqslant \min\{\widetilde{\mu}(x), \widetilde{\mu}(y)\}\\ &= \min\{[\mu^-(x)), \mu^+(x)], [\mu^-(y), \mu^+(y)]\}\\ &= [\min\{\mu^-(x), \mu^-(y)\}, \min\{\mu^+(x), \mu^+(y)\}]\end{aligned}$$

for $x, y \in L$. So,

$\mu^-(x+y) \geqslant \min\{\mu^-(x), \mu^-(y)\}$ and $\mu^+(x+y) \geqslant \min\{\mu^+(x), \mu^+(y)\}$.

In a similar way we can verify (2) and (4). This means that μ^- and μ^+ are fuzzy Lie ideals of L. □

Theorem 3.7. *All nonempty upper levels of interval-valued Lie ideals of a Lie algebra* L *are Lie ideals of* L.

Proof. Assume that $\widetilde{\mu}$ is an interval-valued fuzzy Lie ideal of L and let $[t_1, t_2] \in \mathcal{D}[0,1]$ be such that $U(\widetilde{\mu}; [t_1, t_2]) \neq \emptyset$. If $x \in U(\widetilde{\mu}; [t_1, t_2])$, and $y \in U(\widetilde{\mu}; [t_1, t_2])$, then $\widetilde{\mu}(x) \geqslant [t_1, t_2]$ and $\widetilde{\mu}(y) \geqslant [t_1, t_2]$. Hence

$$\widetilde{\mu}(x+y) \geqslant \min(\widetilde{\mu}(x), \widetilde{\mu}(y)) \geqslant [t_1, t_2],$$

$$\widetilde{\mu}(\alpha x) \geqslant \widetilde{\mu}(x) \geqslant [t_1, t_2],$$

$$\widetilde{\mu}([x, y]) \geqslant \widetilde{\mu}(x) \geqslant [t_1, t_2].$$

So, $x + y \in U(\widetilde{\mu}; [t_1, t_2])$, $\alpha x \in U(\widetilde{\mu}; [t_1, t_2])$ and $[x, y] \in U(\widetilde{\mu}; [t_1, t_2])$. This proves that $U(\widetilde{\mu}; [t_1, t_2])$ is a Lie ideal of L. □

Definition 3.8. Let $f : L_1 \to L_2$ be a homomorphism of Lie algebras. For any interval-valued fuzzy set $\widetilde{\mu}$ in a Lie algebra L_2, we define an interval-valued fuzzy set $\widetilde{\mu}^f$ in L by $\widetilde{\mu}^f(x) = \widetilde{\mu}(f(x))$ for all $x \in L_1$.

Lemma 3.9. *Let $f : L_1 \to L_2$ be a homomorphism of Lie algebras. If $\widetilde{\mu}$ is an interval-valued fuzzy Lie ideal of L_2, then $\widetilde{\mu}^f$ is an interval-valued fuzzy Lie ideal of L_1.*

Proof. Let $x, y \in L_1$ and $\alpha \in F$. Then

$$\widetilde{\mu}^f(x+y) = \widetilde{\mu}(f(x+y)) = \widetilde{\mu}(f(x)+f(y)) \geqslant \min\{\widetilde{\mu}(f(x)), \widetilde{\mu}(f(y))\}$$
$$= \min\{\widetilde{\mu}^f(x), \widetilde{\mu}^f(y)\},$$

$$\widetilde{\mu}^f(\alpha x) = \widetilde{\mu}(f(\alpha x)) = \widetilde{\mu}(\alpha f(x)) \geqslant \widetilde{\mu}(f(x)) = \mu^f(x),$$

$$\widetilde{\mu}^f([x,y]) = \widetilde{\mu}(f([x,y])) = \widetilde{\mu}([f(x), f(y)]) \geqslant \widetilde{\mu}(f(x)) = \widetilde{\mu}^f(x),$$

which proves that $\widetilde{\mu}^f$ is an interval-valued fuzzy Lie ideal of L_1. □

Theorem 3.10. *Let $f : L_1 \to L_2$ be an epimorphism of Lie algebras. Then $\widetilde{\mu}^f$ is an interval-valued fuzzy Lie ideal of L_1 if and only if $\widetilde{\mu}$ is an interval-valued fuzzy Lie ideal of L_2.*

Proof. The sufficiency follows from Lemma 3.9. To prove the necessity observe that f is surjective, so for any $x, y \in L_2$ there are $x_1, y_1 \in L_1$ such that $x = f(x_1)$, $y = f(y_1)$. Thus $\widetilde{\mu}(x) = \widetilde{\mu}^f(x_1)$, $\widetilde{\mu}(y) = \widetilde{\mu}^f(y_1)$, whence

$$\widetilde{\mu}(x+y) = \widetilde{\mu}(f(x_1)+f(y_1)) = \widetilde{\mu}(f(x_1+y_1)) = \widetilde{\mu}^f(x_1+y_1)$$
$$\geqslant \min\{\widetilde{\mu}^f(x_1), \widetilde{\mu}^f(y_1)\} = \min\{\widetilde{\mu}(x), \widetilde{\mu}(y)\},$$

$$\widetilde{\mu}(\alpha x) = \widetilde{\mu}(\alpha f(x_1)) = \widetilde{\mu}(f(\alpha x_1)) = \widetilde{\mu}^f(\alpha x_1) \geqslant \widetilde{\mu}^f(x_1) = \widetilde{\mu}(x),$$

$$\widetilde{\mu}([x,y]) = \widetilde{\mu}([f(x_1), f(y_1)]) = \widetilde{\mu}(f([x_1,y_1])) = \widetilde{\mu}^f([x_1,y_1]) \geqslant \widetilde{\mu}^f(x_1) = \widetilde{\mu}(x).$$

This proves that $\widetilde{\mu}$ is an interval-valued fuzzy Lie ideal of L_2. □

Definition 3.11. Two interval-valued fuzzy ideals $\widetilde{\mu}$ and $\widetilde{\lambda}$ of L have the *same type* if there exists $f \in Aut(L)$ such that $\widetilde{\mu}(x) = \widetilde{\lambda}(f(x))$ for all $x \in L$.

Theorem 3.12. *Let $\widetilde{\mu}$ and $\widetilde{\lambda}$ be interval-valued fuzzy Lie ideals of L. Then the following are equivalent:*

(*i*) *$\widetilde{\mu}$ and $\widetilde{\lambda}$ have the same type,*

(*ii*) *$\widetilde{\mu} \circ f = \widetilde{\lambda}$ for some $f \in Aut(L)$,*

(*iii*) *$g(\widetilde{\mu}) = \widetilde{\lambda}$ for some $g \in Aut(L)$,*

(*iv*) *$h(\widetilde{\lambda}) = \widetilde{\mu}$ for some $h \in Aut(L)$,*

(v) *there exist* $h \in Aut(L)$ *such that* $U(\widetilde{\mu}; [t_1, t_2]) = h(U(\widetilde{\lambda}; [t_1, t_2]))$ *for all* $[t_1, t_2] \in \mathcal{D}[0, 1]$.

Proof. $(i) \to (ii)$: Proof follows immediately from the definition.

$(ii) \to (iii)$: Suppose that $\widetilde{\mu} \circ f = \widetilde{\lambda}$ for some $f \in Aut(L)$. Then $\widetilde{\mu}(f(x)) = \widetilde{\lambda}(x)$ and $f^{-1}(\widetilde{\mu})(x) = \sup_{y \in f(x)} \widetilde{\mu}(y) = \widetilde{\mu}(f(x)) = \widetilde{\lambda}(x)$ for all $x \in L$. If $g = f^{-1}$, then $g \in Aut(L)$ and $g(\widetilde{\mu}) = \widetilde{\lambda}$.

$(iii) \to (iv)$: Suppose that $g(\widetilde{\mu}) = \widetilde{\lambda}$ for some $g \in Aut(L)$. Then $\widetilde{\lambda}(x) = g(\widetilde{\mu}) = \sup_{y \in g^{-1}(x)} \widetilde{\mu}(y) = \widetilde{\mu}(g^{-1}(x))$. Hence $g^{-1}(x) = \sup_{y \in g(x)} \widetilde{\lambda}(y) = \widetilde{\lambda}(g(y)) = \widetilde{\mu}(g^{-1}(g(x))) = \widetilde{\mu}(x)$ for all $x \in L$. If $h = g^{-1}$, then $h \in Aut(L)$ and $h(\widetilde{\lambda}) = \widetilde{\mu}$.

$(iv) \to (v)$: If $h(\widetilde{\lambda}) = \widetilde{\mu}$ for some $h \in \mathrm{Aut}(L)$, then $\widetilde{\mu}(x) = h(\widetilde{\lambda})(x) = \sup_{y \in h^{-1}(x)} \widetilde{\lambda}(y) = \widetilde{\lambda}(h^{-1}(x))$ for all $x \in L$.

Let $[t_1, t_2] \in \mathcal{D}[0, 1]$. We need to show $U(\widetilde{\mu}; [t_1, t_2]) = h(U(\widetilde{\lambda}; [t_1, t_2]))$. If $x \in U(\widetilde{\mu}; [t_1, t_2])$, then $\widetilde{\lambda}(h^{-1}(x)) = \widetilde{\mu} \geqslant [t_1, t_2]]$ which implies that $h^{-1}(x) \in U(\widetilde{\lambda}; [t_1, t_2])$, i.e., $x \in h(U(\widetilde{\lambda}; [t_1, t_2]))$. Thus we obtain $U(\widetilde{\mu}; [t_1, t_2]) \subseteq h(U(\widetilde{\lambda}; [t_1, t_2]))$. On the other hand, let $x \in h(U(\widetilde{\lambda}; [t_1, t_2]))$. Then $h^{-1}(x) \in U(\widetilde{\lambda}; [t_1, t_2])$ and so $\widetilde{\mu}(x) = \widetilde{\lambda}(h^{-1}(x)) \geqslant [t_1, t_2]$. It follows that $x \in U(\widetilde{\mu}; [t_1, t_2])$. Hence $h(U(\widetilde{\lambda}; [t_1, t_2])) \subseteq U(\widetilde{\mu}; [t_1, t_2])$ and (v) holds.

$(v) \to (i)$: Suppose that there exists $h \in \mathrm{Aut}(L)$ such that $U(\widetilde{\mu}; [t_1, t_2]) = h(U(\widetilde{\lambda}; [t_1, t_2]))$ for all $[t_1, t_2] \in \mathcal{D}[0, 1]$. Let $\widetilde{\lambda}(h^{-1}(x)) = [s_1, s_2]$. Then $h^{-1}(x) \in U(\widetilde{\lambda}; [s_1, s_2])$, hence $x \in h(U(\widetilde{\lambda}; [s_1, s_2])) = U(\widetilde{\mu}; [s_1, s_2])$. Thus $\widetilde{\mu}(x) \geqslant [s_1, s_2] = \widetilde{\lambda}(h^{-1}(x))$. Hence $\widetilde{\mu}(x) = \widetilde{\lambda}(h^{-1}(x))$ for all $x \in L$, which proves that $\widetilde{\mu}$ and $\widetilde{\lambda}$ have the same type. □

4. Characterizations of Noetherian Lie algebras

Definition 4.1. A Lie algebra L is said to be *Noetherian* if for every ascending sequence $I_1 \subseteq I_2 \subseteq I_3 \subseteq \ldots$ of Lie ideals of L there exists a natural number n such that $I_n = I_k$ for all $n \geqslant k$.

Theorem 4.2. *A Lie algebra L is Noetherian if and only if the set of values of any its interval-valued fuzzy Lie ideal is well-ordered.*

Proof. Suppose that $\widetilde{\mu}$ is an interval-valued fuzzy Lie ideal whose set of values is not well-ordered subset of $\mathcal{D}[0, 1]$. Then there exists a strictly decreasing sequence $\{[\alpha_n, \beta_n]\}$ such that $[\alpha_n, \beta_n] = \widetilde{\mu}(x_n)$ for some $x_n \in L$. Let $B_n := \{x \in L \,|\, \widetilde{\mu}(x) \geqslant [\alpha_n, \beta_n]\}$. Then $B_1 \subset B_2 \subset B_3 \subset \ldots$ form a strictly ascending chain of Lie ideals of L, contradicting the assumption that L is Noetherian.

Conversely, suppose that the set of values of any interval-valued fuzzy Lie ideal of L but L is not Noetherian. Then there exists a strictly ascending chain $A_1 \subset A_2 \subset A_3 \subset \ldots$ of Lie ideals of L. Suppose that $A = \bigcup_{k=1}^{\infty} A_k$ is a Lie ideal of L. Define an interval-valued fuzzy set $\widetilde{\mu}$ in L by putting

$$\widetilde{\mu}(x) := \begin{cases} [\frac{1}{k+1}, \frac{1}{k}] & \text{for } x \in A_k \backslash A_{k-1}, \\ [0,\, 0] & \text{for } x \notin A. \end{cases}$$

We claim that $\widetilde{\mu}$ is an interval-valued fuzzy Lie ideals of L. Let $x, y \in L$.

If $x, y \in A$ then there are m, n such that $x \in A_n \backslash A_{n-1}$, $y \in A_m \backslash A_{m-1}$. Obviously $x + y \in A_k \backslash A_{k-1} \subset A_p$, where $k \leqslant p = \max\{m, n\}$. So, $\widetilde{\mu}(x) = \left[\frac{1}{n+1}, \frac{1}{n}\right]$, $\widetilde{\mu}(y) = [\frac{1}{m+1}, \frac{1}{m}]$ and

$$\widetilde{\mu}(x+y) = [\frac{1}{k+1}, \frac{1}{k}] \geqslant [\frac{1}{p+1}, \frac{1}{p}] = \min\{\widetilde{\mu}(x), \widetilde{\mu}(y)\}.$$

In the case $x \notin A$, $y \in A$ we have $y \in A_m \backslash A_{m-1}$ for some natural m. Hence $\widetilde{\mu}(x) = [0, 0]$, $\widetilde{\mu}(y) = [\frac{1}{m+1}, \frac{1}{m}]$, consequently

$$\widetilde{\mu}(x+y) \geqslant [0, 0] = \min\{\widetilde{\mu}(x), \widetilde{\mu}(y)\}.$$

The case $x \in A$, $y \notin A$ is analogous. The case $x \notin A$, $y \notin A$ is obvious. The verification of (2) and (4) is analogous. Thus $\widetilde{\mu}$ is an interval-valued fuzzy Lie ideal of L. Consequently, $\widetilde{\mu}$ is an interval-valued fuzzy Lie ideal. Since the chain $A_1 \subset A_2 \subset A_3 \subset \ldots$ is not terminating, $\widetilde{\mu}$ has a strictly descending sequence of values. This contradicts that the value set of any interval-valued fuzzy Lie ideal is well-ordered. This completes the proof. □

We note that a set is well ordered if and only if it does not contain any infinite decreasing sequence.

Theorem 4.3. *Let $S = \{[s_1, t_1], [s_2, t_2], \ldots\} \cup \{[0, 0]\}$, where $\{[s_n, t_n]\}$ is a strictly decreasing sequence in $\mathcal{D}[0, 1]$. Then a Lie algebra L is Noetherian if and only if for each interval-valued fuzzy Lie ideal $\widetilde{\mu}$ of L, $Im(\widetilde{\mu}) \subseteq S$ implies that there exists a positive integer m such that $Im(\widetilde{\mu}) \subseteq \{[s_1, t_1], [s_2, t_2], \ldots, [s_m, t_m]\} \cup \{[0, 0]\}$.*

Proof. If L is a Noetherian Lie algebra, then $Im(\widetilde{\mu})$ is a well ordered subset of $\mathcal{D}[0,1]$.

Conversely, if the above condition is satisfied and L is not Noetherian, then there exists a strictly ascending chain $A_1 \subset A_2 \subset A_3 \subset \dots$ of Lie ideals of L. Define an interval-valued fuzzy set $\widetilde{\mu}$ by

$$\widetilde{\mu}(x) := \begin{cases} [s_1, t_1] & \text{if } x \in A_1, \\ [s_n, t_n] & \text{if } x \in A_n \backslash A_{n-1}, n = 2, 3, 4, \dots \\ [0,\ 0] & \text{if } x \in G \backslash \bigcup_{n=1}^{\infty} A_n. \end{cases}$$

Let $x, y \in L$. If either x or y belongs to $G \backslash \bigcup_{n=1}^{\infty} A_n$, then either $\widetilde{\mu}(x) = [0,0]$ or $\widetilde{\mu}(y) = [0,0]$. Thus $\widetilde{\mu}(x+y) \geqslant \min\{\widetilde{\mu}(x), \widetilde{\mu}(y)\}$.

If $x, y \in A_1$, then $x \in A_1$ and so $\widetilde{\mu}(x+y) = [s_1, t_1] \geqslant \min\{\widetilde{\mu}(x), \widetilde{\mu}(y)\}$.

If $x, y \in A_n \backslash A_{n-1}$, then $x \in A_n$ and $\widetilde{\mu}(x+y) \geq [s_n, t_n] = \min\{\widetilde{\mu}(x), \widetilde{\mu}(y)\}$. Assume that $x \in A_1$ and $y \in A_n \backslash A_{n-1}$ for $n = 2, 3, 4, \dots$, then $x + y \in A_n$ and hence

$$\widetilde{\mu}(x+y) \geq [s_n, t_n] = \min\{[s_1, t_1], [s_n, t_n]\} = \min\{\widetilde{\mu}(x), \widetilde{\mu}(y)\}.$$

Similarly for $x \in A_n \backslash A_{n-1}$ and $y \in A_1$ for $n = 2, 3, 4, \dots$, we have

$$\widetilde{\mu}(x+y) \geq [s_n, t_n] = \min\{\widetilde{\mu}(x), \widetilde{\mu}(y)\}.$$

Hence $\widetilde{\mu}$ is an interval-valued fuzzy Lie ideal of Lie algebra. This contradicts our assumption. The verification of (2) and (4) is analogous and we omit the details. This completes the proof. □

5. Quotient Lie algebra via IF Lie ideals

Theorem 5.1. *Let I be a Lie ideal of a Lie algebra L. If $\widetilde{\mu}$ is an interval-valued Lie ideal of L, then an interval-valued fuzzy set $\overline{\widetilde{\mu}}$ defined by*

$$\overline{\widetilde{\mu}}(a+I) = \sup_{x \in I} \widetilde{\mu}(a+x)$$

is an interval-valued Lie ideal of the quotient Lie algebra L/I.

Proof. Clearly, $\overline{\widetilde{\mu}}$ is well-defined. Let $x+I$, $y+I \in L/I$, then

$$\begin{aligned} \overline{\widetilde{\mu}}(x+I)+(y+I)) &= \overline{\widetilde{\mu}}_A((x+y)+I) = \sup\nolimits_{z \in I} \widetilde{\mu}((x+y)+z) \\ &= \sup\nolimits_{z=s+t \in I} \widetilde{\mu}((x+y)+(s+t)) \\ &\geqslant \sup\nolimits_{s,\ t \in I} \min\{\widetilde{\mu}(x+s), \widetilde{\mu}(y+t)\} \\ &= \min\{\sup\nolimits_{s \in I} \widetilde{\mu}(x+s), \sup\nolimits_{t \in I} \widetilde{\mu}(y+t)\} \\ &= \min\{\overline{\widetilde{\mu}}(x+I), \overline{\widetilde{\mu}}(y+I)\}, \end{aligned}$$

$$\widetilde{\overline{\mu}}(\alpha(x+I)) = \widetilde{\overline{\mu}}(\alpha x+I) = \sup_{z\in I}\widetilde{\mu}(\alpha x+z) \geqslant \sup_{z\in I}\widetilde{\mu}(x+z) = \widetilde{\overline{\mu}}(x+I),$$

$$\begin{aligned}\widetilde{\overline{\mu}}([x+I, y+I]) &= \widetilde{\overline{\mu}}([x,y]+I) = \sup_{z\in I}\widetilde{\mu}([x,y]+z)\\ &\geqslant \sup_{z\in I}\widetilde{\mu}(x+z) = \widetilde{\overline{\mu}}(x+I).\end{aligned}$$

Hence $\widetilde{\overline{\mu}}$ is an interval-valued fuzzy Lie ideal of L/I. □

Theorem 5.2. *Let $f : L_1 \to L_2$ be a homomorphism of a Lie algebra L_1 onto a Lie algebra L_2.*

(*i*) *If $\widetilde{\mu}$ is an interval-valued fuzzy Lie ideal of L_1, then $f(\widetilde{\mu})$ is an interval-valued fuzzy Lie ideal of L_2,*

(*ii*) *If $\widetilde{\lambda}$ is an interval-valued fuzzy Lie ideal of L_2, then $f^{-1}(\widetilde{\lambda})$ is an interval-valued fuzzy Lie ideal of L_1.*

Proof. Straightforward. □

For an interval-valued fuzzy Lie ideal $\widetilde{\mu}$ of a Lie algebra L we define a binary relation $\sim$ by putting

$$x \sim y \longleftrightarrow \widetilde{\mu}(x-y) = \widetilde{\mu}(0).$$

This relation is a congruence. The set of all its equivalence classes $\widetilde{\mu}[x]$ is denoted by $L/\widetilde{\mu}$. It is a Lie algebra under the following operations:

$$\widetilde{\mu}[x] + \widetilde{\mu}[y] = \widetilde{\mu}[x+y], \quad \alpha\widetilde{\mu}[x] = \widetilde{\mu}[\alpha x], \quad [\widetilde{\mu}[x], \widetilde{\mu}[y]] = \widetilde{\mu}[[x,y]],$$

where $x, y \in L$, $\alpha \in F$.

Theorem 5.3. (First IF isomorphism theorem)
Let $f : L_1 \to L_2$ be an epimorphism of Lie algebras and let $\widetilde{\mu}$ be an interval-valued fuzzy Lie ideal of L_2. Then $L_1/f^{-1}(\widetilde{\mu}) \cong L_2/\widetilde{\mu}$.

Proof. Define a map $\theta : L_1/f^{-1}(\widetilde{\mu}) \to L_2/\widetilde{\mu}$ by $\theta(f^{-1}(\widetilde{\mu})[x]) = \widetilde{\mu}[f(x)]$.

θ is well-defined since $f^{-1}(\widetilde{\mu})[x] = f^{-1}(\widetilde{\mu})[y]$ implies $f^{-1}(\widetilde{\mu})(x-y) = f^{-1}(\widetilde{\mu})(0)$. Whence $\widetilde{\mu}(f(x) - f(y)) = \widetilde{\mu}(f(0)) = \widetilde{\mu}(0)$. Thus $\widetilde{\mu}[f(x)] = \widetilde{\mu}[f(y)]$.

θ is one to one because $\widetilde{\mu}[f(x)] = \widetilde{\mu}[f(y)]$ gives $\widetilde{\mu}(f(x) - f(y)) = \widetilde{\mu}(0)$, i.e., $\widetilde{\mu}(f(x) - f(y)) = \widetilde{\mu}(f(0))$, which proves $f^{-1}(\widetilde{\mu})(x-y) = f^{-1}(\widetilde{\mu})(0)$. Thus $f^{-1}(\widetilde{\mu})[x] = f^{-1}(\widetilde{\mu})[y]$.

Since f is an onto, θ is an onto. Finally, θ is a homomorphism because

$$\begin{aligned}\theta(f^{-1}(\widetilde{\mu})[x]+f^{-1}(\widetilde{\mu})[y]) &= \theta(f^{-1}(\widetilde{\mu})[x+y]) = \widetilde{\mu}[f(x+y)] = \widetilde{\mu}[f(x)+f(y)]\\ &= \widetilde{\mu}[f(x)] + \widetilde{\mu}[f(y)] = \theta(f^{-1}(\widetilde{\mu})[x]) + \theta(f^{-1}(\widetilde{\mu})[y]),\end{aligned}$$

$$\theta(\alpha f^{-1}(\widetilde{\mu})[x]) = \theta(f^{-1}(\widetilde{\mu})[\alpha x]) = \widetilde{\mu}[f(\alpha x)] = \alpha\widetilde{\mu}[f(x)] = \alpha\theta(f^{-1}(\widetilde{\mu})[x]),$$

$$\begin{aligned}\theta([f^{-1}(\widetilde{\mu})[x], f^{-1}(\widetilde{\mu})[y]]) &= \theta([f^{-1}(\widetilde{\mu})[x,y]]) = \widetilde{\mu}[f([x,y])]\\ &= \widetilde{\mu}[[f(x), f(y)]] = [\widetilde{\mu}[f(x)], \widetilde{\mu}[f(y)]]\\ &= [\theta(f^{-1}(\widetilde{\mu})[x]), \theta(f^{-1}(\widetilde{\mu})[y])].\end{aligned}$$

Hence $L_1/f^{-1}(\widetilde{\mu}) \cong L_2/\widetilde{\mu}$. □

We state the following IF isomorphism Theorems without proofs.

Theorem 5.4. (Second IF isomorphism theorem)
Let $\widetilde{\mu}$ and $\widetilde{\lambda}$ be two interval-valued fuzzy subsets of the same Lie algebra. If $\widetilde{\mu}$ is a subalgebra and $\widetilde{\lambda}$ is a Lie ideal, then

(i) $\widetilde{\lambda}$ *is an interval-valued fuzzy Lie ideal of* $\widetilde{\mu}+\widetilde{\lambda}$,

(ii) $\widetilde{\mu}\cap\widetilde{\lambda}$ *is an interval-valued fuzzy ideal of* $\widetilde{\mu}$,

(iii) $(\widetilde{\mu}+\widetilde{\lambda})/\lambda \cong \widetilde{\mu}/(\widetilde{\mu}\cap\widetilde{\lambda})$.

Theorem 5.5. (Third IF isomorphism theorem)
Let $\widetilde{\mu}$ and $\widetilde{\lambda}$ be interval-valued fuzzy Lie ideals of the same Lie algebra such that $\widetilde{\mu}\leqslant\widetilde{\lambda}$. Then

(i) $\widetilde{\lambda}/\widetilde{\mu}$ *is an interval-valued fuzzy Lie ideal of* $L/\widetilde{\mu}$,

(ii) $(L/\widetilde{\mu})/(\widetilde{\lambda}/\widetilde{\mu}) \cong L/\widetilde{\lambda}$.

Theorem 5.6. (IF Zassenhaus lemma)
Let $\widetilde{\mu}$ and $\widetilde{\lambda}$ be interval-valued fuzzy subalgebras of a Lie algebra L and let $\widetilde{\mu}_1$ and $\widetilde{\lambda}_1$ be an interval-valued fuzzy Lie ideals of $\widetilde{\mu}$ and $\widetilde{\lambda}$ respectively. Then

(a) $\widetilde{\mu}_1+(\widetilde{\mu}\cap\widetilde{\lambda}_1)$ *is an interval-valued fuzzy Lie ideal of* $\widetilde{\mu}_1+(\widetilde{\mu}\cap\widetilde{\lambda})$,

(b) $\widetilde{\lambda}_1+(\widetilde{\mu}_1\cap\widetilde{\lambda})$ *is an interval-valued fuzzy ideal of* $\widetilde{\lambda}_1+(\widetilde{\mu}\cap\widetilde{\lambda})$,

(c) $\frac{\widetilde{\mu}_1+(\widetilde{\mu}\cap\widetilde{\lambda})}{\widetilde{\mu}_1+(\widetilde{\mu}\cap\widetilde{\lambda}_1)} \simeq \frac{\widetilde{\lambda}_1+(\widetilde{\mu}\cap\widetilde{\lambda})}{\widetilde{\lambda}_1+(\widetilde{\mu}_1\cap\widetilde{\lambda})}$.

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