Gyrogroups, the grouplike loops
in the service of hyperbolic geometry and
Einstein's special theory of relativity

Abraham A. Ungar

Abstract

In this era of an increased interest in loop theory, the Einstein velocity addition law has fresh resonance. One of the most fascinating aspects of recent work in Einstein's special theory of relativity is the emergence of special grouplike loops. The special grouplike loops, known as *gyrocommutative gyrogroups*, have thrust the Einstein velocity addition law, which previously has operated mostly in the shadows, into the spotlight. We will find that Einstein (Möbius) addition is a gyrocommutative gyrogroup operation that forms the setting for the Beltrami-Klein (Poincaré) ball model of hyperbolic geometry just as the common vector addition is a commutative group operation that forms the setting for the standard model of Euclidean geometry. The resulting analogies to which the grouplike loops give rise lead us to new results in (i) hyperbolic geometry; (ii) relativistic physics; and (iii) quantum information and computation.

1. Introduction

The author's two recent books with the ambitious titles, *"Analytic hyperbolic geometry: Mathematical foundations and applications"* [56], and *"Beyond the Einstein addition law and its gyroscopic Thomas precession: The theory of gyrogroups and gyrovector spaces"* [53, 66], raise expectations for novel applications of special grouplike loops in hyperbolic geometry and in relativistic physics. Indeed, these books lead their readers to see what some

special grouplike loops have to offer, and thereby give them a taste of loops in the service of the hyperbolic geometry of Bolyai and Lobachevsky and the special relativity theory of Einstein.

Seemingly structureless, Einstein's relativistic velocity addition is neither commutative nor associative. Einstein's failure to recognize and advance the rich, grouplike loop structure [52] that regulates his relativistic velocity addition law contributed to the eclipse of his velocity addition law of relativistically admissible 3-velocities, creating a void that could be filled only with the Lorentz transformation of 4-velocities, along with its Minkowski's geometry.

Minkowski characterized his spacetime geometry as evidence that *preestablished harmony* between pure mathematics and applied physics does exist [42]. Subsequently, the study of special relativity followed the lines laid down by Minkowski, in which the role of Einstein velocity addition and its interpretation in the hyperbolic geometry of Bolyai and Lobachevsky are ignored [5]. The tension created by the mathematician Minkowski into the specialized realm of theoretical physics, as well as Minkowski's strategy to overcome disciplinary obstacles to the acceptance of his reformulation of Einstein's special relativity is skillfully described by Scott Walter in [64].

According to Leo Corry [11], Einstein considered Minkowski's reformulation of his theory in terms of four-dimensional spacetime to be no more than "superfluous erudition". Admitting that, unlike his seemingly structureless relativistic velocity addition law, the Lorentz transformation is an elegant group operation, Einstein is quoted as saying:

"If you are out to describe truth, leave elegance to the tailor."

Albert Einstein (1879 – 1955)

One might, therefore, suppose that there is a price to pay in mathematical elegance and regularity when replacing ordinary vector addition approach to Euclidean geometry with Einstein vector addition approach to hyperbolic geometry. But, this is not the case since grouplike loops, called *gyrocommutative gyrogroups*, come to the rescue. It turns out that Einstein addition of vectors with magnitudes $< c$ is a gyrocommutative gyrogroup operation and, as such, it possesses a rich nonassociative algebraic and geometric structure. The best way to introduce the gyrocommutative gyrogroup notion that regulates the algebra of Einstein's relativistic velocity addition law is offered by Möbius transformations of the disc [29]. The subsequent transition from Möbius addition, which regulates the Poincaré ball

model of hyperbolic geometry, Fig. 1, to Einstein addition, which regulates the Beltrami-Klein ball model of hyperbolic geometry, Fig. 6, expressed in gyrolanguage, will then turn out to be remarkably simple and elegant [56, 57].

Evidently, the grouplike loops that we naturally call *gyrocommutative gyrogroups*, along with their extension to gyrovector spaces, form a new tool for the twenty-first century exploration of classical hyperbolic geometry and its use in physics.

2. Möbius transformations of the disc

Möbius transformations of the disc $\mathbb{D}$,

$$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\} \tag{1}$$

of the complex plane $\mathbb{C}$ offer an elegant way to introduce the grouplike loops that we call *gyrogroups*. More than 150 years have passed since August Ferdinand Möbius first studied the transformations that now bear his name [35]. Yet, the rich structure he thereby exposed is still far from being exhausted

Ahlfors' book [1], *Conformal Invariants: Topics in Geometric Function Theory*, begins with a presentation of the Möbius self-transformation of the complex open unit disc $\mathbb{D}$,

$$z \mapsto e^{i\theta} \frac{a+z}{1+\overline{a}z} = e^{i\theta}(a\oplus_{\mathrm{M}} z) \tag{2}$$

$a, z \in \mathbb{D}$, $\theta \in \mathbb{R}$, where $\overline{a}$ is the complex conjugate of a [14, p. 211] [19, p. 185] [36, pp. 177−178]. Suggestively, the *polar decomposition* (2) of Möbius transformation of the disc gives rise to *Möbius addition*, $\oplus_{\mathrm{M}}$,

$$a\oplus_{\mathrm{M}} z = \frac{a+z}{1+\overline{a}z} \, . \tag{3}$$

Naturally, Möbius subtraction, $\ominus_{\mathrm{M}}$, is given by $a\ominus_{\mathrm{M}} z = a\oplus_{\mathrm{M}}(-z)$, so that $z\ominus_{\mathrm{M}} z = 0$ and $\ominus_{\mathrm{M}} z = 0\ominus_{\mathrm{M}} z = 0\oplus_{\mathrm{M}}(-z) = -z$. Remarkably, Möbius addition possesses the *automorphic inverse property*

$$\ominus_{\mathrm{M}}(a\oplus_{\mathrm{M}} b) = \ominus_{\mathrm{M}} a\ominus_{\mathrm{M}} b \tag{4}$$

and the *left cancellation law*

$$\ominus_{\mathrm{M}} a\oplus_{\mathrm{M}}(a\oplus_{\mathrm{M}} z) = z \tag{5}$$

for all $a, b, z \in \mathbb{D}$, [56, 53].

Möbius addition gives rise to the Möbius disc groupoid $(\mathbb{D}, \oplus_{\mathrm{M}})$, recalling that a groupoid $(G, \oplus)$ is a nonempty set, G, with a binary operation, $\oplus$, and that an automorphism of a groupoid $(G, \oplus)$ is a bijective self map f of G that respects its binary operation $\oplus$, that is, $f(a \oplus b) = f(a) \oplus f(b)$. The set of all automorphisms of a groupoid $(G, \oplus)$ forms a group, denoted $Aut(G, \oplus)$.

Möbius addition $\oplus_{\mathrm{M}}$ in the disc is neither commutative nor associative. To measure the extent to which Möbius addition deviates from associativity we define the *gyrator*

$$\mathrm{gyr} : \mathbb{D} \times \mathbb{D} \to Aut(\mathbb{D}, \oplus_{\mathrm{M}}) \tag{6}$$

by the equation

$$\mathrm{gyr}[a, b]z = \ominus_{\mathrm{M}}(a \oplus_{\mathrm{M}} b) \oplus_{\mathrm{M}} \{a \oplus_{\mathrm{M}} (b \oplus_{\mathrm{M}} z)\} \tag{7}$$

for all $a, b, z \in \mathbb{D}$.

The automorphisms

$$\mathrm{gyr}[a, b] \in Aut(\mathbb{D}, \oplus_{\mathrm{M}}) \tag{8}$$

of $\mathbb{D}$, $a, b \in \mathbb{D}$, called gyrations of $\mathbb{D}$, have an important hyperbolic geometric interpretation [63]. Thus, the gyrator in (6) generates the gyrations in (8). In order to emphasize that gyrations of $\mathbb{D}$ are also automorphisms of $(\mathbb{D}, \oplus_{\mathrm{M}})$, as we will see below, they are also called *gyroautomorphisms*.

Clearly, in the special case when the binary operation $\oplus_{\mathrm{M}}$ in (7) is associative, $\mathrm{gyr}[a, b]$ reduces to the trivial automorphism, $\mathrm{gyr}[a, b]z = z$ for all $z \in \mathbb{D}$. Hence, indeed, the self map $\mathrm{gyr}[a, b]$ of the disc $\mathbb{D}$ measures the extent to which Möbius addition $\oplus_{\mathrm{M}}$ in the disc $\mathbb{D}$ deviates from associativity.

One can readily simplify (7) in terms of (3), obtaining

$$\mathrm{gyr}[a, b]z = \frac{1 + a\overline{b}}{1 + \overline{a}b} z \tag{9}$$

$a, b, z \in \mathbb{D}$, so that the gyrations

$$\mathrm{gyr}[a, b] = \frac{1 + a\overline{b}}{1 + \overline{a}b} = \frac{a \oplus_{\mathrm{M}} b}{b \oplus_{\mathrm{M}} a} \tag{10}$$

are unimodular complex numbers. As such, gyrations represent rotations of the disc $\mathbb{D}$ about its center, as shown in (9).

Gyrations are invertible. The inverse, $\mathrm{gyr}^{-1}[a,b] = (\mathrm{gyr}[a,b])^{-1}$, of a gyration $\mathrm{gyr}[a,b]$ is the gyration $\mathrm{gyr}[b,a]$,

$$\mathrm{gyr}^{-1}[a,b] = \mathrm{gyr}[b,a] \tag{11}$$

Moreover, gyrations respect Möbius addition in the disc,

$$\mathrm{gyr}[a,b](c\oplus_{\mathrm{M}} d) = \mathrm{gyr}[a,b]c\oplus_{\mathrm{M}}\mathrm{gyr}[a,b]d \tag{12}$$

for all $a,b,c,d\in\mathbb{D}$, so that gyrations of the disc are automorphisms of the disc, as anticipated in (8).

Identity (10) can be written as

$$a\oplus_{\mathrm{M}} b = \mathrm{gyr}[a,b](b\oplus_{\mathrm{M}} a) \tag{13}$$

thus giving rise to the *gyrocommutative law* of Möbius addition. Furthermore, Identity (7) can be manipulated, by mean of the left cancellation law (5), into the identity

$$a\oplus_{\mathrm{M}}(b\oplus_{\mathrm{M}} z) = (a\oplus_{\mathrm{M}} b)\oplus_{\mathrm{M}}\mathrm{gyr}[a,b]z \tag{14}$$

thus giving rise to the *left gyroassociative law* of Möbius addition.

The gyrocommutative law, (13), and the left gyroassociative law, (14), of Möbius addition in the disc reveal the grouplike structure of Möbius groupoid $(\mathbb{D},\oplus_{\mathrm{M}})$, that we naturally call a *gyrocommutative gyrogroup*. Taking the key features of Möbius groupoid $(\mathbb{D},\oplus_{\mathrm{M}})$ as axioms, and guided by analogies with group theory, we thus obtain the following definitions of gyrogroups and gyrocommutative gyrogroups.

Definition 1. (Gyrogroups). A groupoid $(G,\oplus)$ is a *gyrogroup* if its binary operation satisfies the following axioms. In G there is at least one element, 0, called a *left identity*, satisfying

(G1) $\qquad 0\oplus a = a$

for all $a \in G$. There is an element $0 \in G$ satisfying axiom $(G1)$ such that for each $a \in G$ there is an element $\ominus a \in G$, called a *left inverse* of a, satisfying

(G2) $\qquad \ominus a\oplus a = 0\,.$

Moreover, for any $a,b,c \in G$ there exists a unique element $\mathrm{gyr}[a,b]c \in G$ such that the binary operation obeys the *left gyroassociative law*

(G3) $\qquad a\oplus(b\oplus c) = (a\oplus b)\oplus\mathrm{gyr}[a,b]c\,.$

The map $\mathrm{gyr}[a,b] : G \to G$ given by $c \mapsto \mathrm{gyr}[a,b]c$ is an automorphism of the groupoid $(G, \oplus)$, that is,

(G4) $\qquad \mathrm{gyr}[a,b] \in Aut(G, \oplus)\,,$

and the automorphism $\mathrm{gyr}[a,b]$ of G is called the *gyroautomorphism*, or the *gyration*, of G generated by $a, b \in G$. The operator $\mathrm{gyr} : G \times G \to Aut(G, \oplus)$ is called the gyrator of G. Finally, the gyroautomorphism $\mathrm{gyr}[a,b]$ generated by any $a, b \in G$ possesses the *left loop property*

(G5) $\qquad \mathrm{gyr}[a,b] = \mathrm{gyr}[a \oplus b, b]\,.$

The gyrogroup axioms $(G1)-(G5)$ in Definition 1 are classified into three classes:

(1) The first pair of axioms, $(G1)$ and $(G2)$, is a reminiscent of the group axioms.

(2) The last pair of axioms, $(G4)$ and $(G5)$, presents the gyrator axioms.

(3) The middle axiom, $(G3)$, is a hybrid axiom linking the two pairs of axioms in (1) and (2).

The loop property $(G5)$ turns out to be equivalent to the gyration-free identity

$$x \oplus (y \oplus (x \oplus z)) = (x \oplus (y \oplus x)) \oplus z \qquad (15)$$

which loop theorists recognize as the *left Bol identity* [46, 47].

As in group theory, we use the notation $a \ominus b = a \oplus (\ominus b)$ in gyrogroup theory as well.

In full analogy with groups, gyrogroups are classified into gyrocommutative and non-gyrocommutative gyrogroups.

Definition 2. (Gyrocommutative gyrogroups). A gyrogroup $(G, \oplus)$ is *gyrocommutative* if its binary operation obeys the *gyrocommutative law*

(G6) $\qquad a \oplus b = \mathrm{gyr}[a,b](b \oplus a)$

for all $a, b \in G$.

Some first gyrogroup theorems, some of which are analogous to group theorems, are presented in [56, Chap. 2]. Thus, in particular, the gyrogroup left identity and left inverse are identical with their right counterparts, and the resulting identity and inverse are unique, as in group theory. Furthermore, the left gyroassociative law and the left loop property are associated with corresponding right counterparts.

A gyrogroup operation $\oplus$ comes with a dual operation, the *cooperation* (or, *co-operation*, for clarity) $\boxplus$ [56, Def. 2.7], given by the equation

$$a \boxplus b = a \oplus \mathrm{gyr}[a, \ominus b]b \tag{16}$$

so that

$$a \boxminus b = a \ominus \mathrm{gyr}[a, b]b \tag{17}$$

for all $a, b \in G$, where we define $a \boxminus b = a \boxplus (\ominus b)$. The gyrogroup cooperation shares with its associated gyrogroup operation remarkable duality symmetries as, for instance [56, Theorem 2.10],

$$\begin{aligned} a \boxplus b &= a \oplus \mathrm{gyr}[a, \ominus b]b \\ a \oplus b &= a \boxplus \mathrm{gyr}[a, b]b \end{aligned} \tag{18}$$

Interestingly, by [56, Theorem 3.4], a gyrogroup cooperation is commutative if and only if its corresponding gyrogroup is gyrocommutative.

The gyroautomorphisms have their own rich structure as we see, for instance, from the gyroautomorphism inversion property

$$(\mathrm{gyr}[a, b])^{-1} = \mathrm{gyr}[b, a] \tag{19}$$

from the loop property (left and right)

$$\begin{aligned} \mathrm{gyr}[a, b] &= \mathrm{gyr}[a \oplus b, b] \\ \mathrm{gyr}[a, b] &= \mathrm{gyr}[a, b \oplus a] \end{aligned} \tag{20}$$

and from the elegant nested gyroautomorphism identity

$$\mathrm{gyr}[a, b] = \mathrm{gyr}[\ominus \mathrm{gyr}[a, b]b, a] \tag{21}$$

for all $a, b \in G$ in any gyrogroup $G = (G, \oplus)$. More gyroautomorphism identities and important gyrogroup theorems, along with their applications, are found in [53, 56, 62] and in [6, 13, 25, 26, 30, 45, 46, 47, 63].

Thus, without losing the flavor of the group structure we have generalized it into the gyrogroup structure to suit the needs of Möbius addition in the disc and, more generally, in the open ball of any real inner product space [61], as we will show in Sec. 3. Gyrogroups abound in group theory, as shown in [15] and [16], where finite and infinite gyrogroups, both gyrocommutative and non-gyrocommutative, are studied. Plenty of gyrogroup theorems are found in [53, 56, 62]. Furthermore, any gyrogroup can be extended into a group, called a *gyrosemidirect product group* [56, Sec. 2.6] [28].

Hence, the generalization of groups into gyrogroups bears an intriguing resemblance to the generalization of the rational numbers into the real ones. The beginner is initially surprised to discover an irrational number, like $\sqrt{2}$, but soon later he is likely to realize that there are more irrational numbers than rational ones. Similarly, the gyrogroup structure of Möbius addition initially comes as a surprise. But, interested explorers may soon realize that in some sense there are more non-group gyrogroups than groups.

In our "gyrolanguage", as the reader has noticed, we attach the prefix "gyro" to a classical term to mean the analogous term in our study of grouplike loops. The prefix stems from Thomas gyration, which is the mathematical abstraction of the relativistic effect known as Thomas precession, explained in [53]. Indeed, gyrolanguage turns out to be the language we need to articulate novel analogies that the classical and the modern in this paper and in [53, 56, 62] share.

3. Möbius addition in the ball

If we identify complex numbers of the complex plane $\mathbb{C}$ with vectors of the Euclidean plane $\mathbb{R}^2$ in the usual way,

$$\mathbb{C} \ni u = u_1 + iu_2 = (u_1, u_2) = \mathbf{u} \in \mathbb{R}^2 \qquad (22)$$

then the inner product and the norm in $\mathbb{R}^2$ are given by the equations

$$\bar{u}v + u\bar{v} = 2\mathbf{u}\cdot\mathbf{v}$$

$$|u| = \|\mathbf{u}\| \qquad (23)$$

These, in turn, enable us to translate Möbius addition from the complex open unit disc $\mathbb{D}$ into the open unit disc $\mathbb{R}^2_{s=1} = \{\mathbf{v} \in \mathbb{R}^2 : \|\mathbf{v}\| < s = 1\}$ of $\mathbb{R}^2$ [29]:

$$\begin{aligned}
\mathbb{D} \ni u \oplus_{\mathrm{M}} v &= \frac{u + v}{1 + \bar{u}v} \\
&= \frac{(1 + u\bar{v})(u + v)}{(1 + \bar{u}v)(1 + u\bar{v})} \\
&= \frac{(1 + \bar{u}v + u\bar{v} + |v|^2)u + (1 - |u|^2)v}{1 + \bar{u}v + u\bar{v} + |u|^2|v|^2} \\
&= \frac{(1 + 2\mathbf{u}\cdot\mathbf{v} + \|\mathbf{v}\|^2)\mathbf{u} + (1 - \|\mathbf{u}\|^2)\mathbf{v}}{1 + 2\mathbf{u}\cdot\mathbf{v} + \|\mathbf{u}\|^2\|\mathbf{v}\|^2} \\
&= \mathbf{u} \oplus_{\mathrm{M}} \mathbf{v} \in \mathbb{R}^2_{s=1}
\end{aligned} \qquad (24)$$

for all $u, v \in \mathbb{D}$ and all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^2_{s=1}$. The last equation in (24) is a vector equation, so that its restriction to the ball of the Euclidean two-dimensional space $\mathbb{R}^2_{s=1}$ is a mere artifact. As such, it survives unimpaired in higher dimensions, suggesting the following definition of Möbius addition in the ball of any real inner product space.

Definition 3. (Möbius addition in the ball). Let $\mathbb{V}$ be a real inner product space [33], and let $\mathbb{V}_s$ be the s-ball of $\mathbb{V}$,

$$\mathbb{V}_s = \{\mathbb{V}_s \in \mathbb{V} : \|\mathbf{v}\| < s\} \tag{25}$$

for any fixed $s > 0$. *Möbius addition $\oplus_M$ in the ball $\mathbb{V}_s$ is a binary operation in $\mathbb{V}_s$ given by the equation*

$$\mathbf{u}\oplus_{\mathrm{M}}\mathbf{v} = \frac{(1 + \frac{2}{s^2}\mathbf{u}\cdot\mathbf{v} + \frac{1}{s^2}\|\mathbf{v}\|^2)\mathbf{u} + (1 - \frac{1}{s^2}\|\mathbf{u}\|^2)\mathbf{v}}{1 + \frac{2}{s^2}\mathbf{u}\cdot\mathbf{v} + \frac{1}{s^4}\|\mathbf{u}\|^2\|\mathbf{v}\|^2} \tag{26}$$

$\mathbf{u}, \mathbf{v} \in \mathbb{V}_s$, *where $\cdot$ and $\|\cdot\|$ are the inner product and norm that the ball $\mathbb{V}_s$ inherits from its space $\mathbb{V}$.*

Without loss of generality, one may select $s = 1$ in Definition 3. We, however, prefer to keep s as a free positive parameter in order to exhibit the result that in the limit as $s \to \infty$, the ball $\mathbb{V}_s$ expands to the whole of its real inner product space $\mathbb{V}$, and Möbius addition $\oplus_{\mathrm{M}}$ in the ball reduces to vector addition in the space. Remarkably, like the Möbius disc groupoid $(\mathbb{D}, \oplus_{\mathrm{M}})$, also the Möbius ball groupoid $(\mathbb{V}_s, \oplus_{\mathrm{M}})$ forms a gyrocommutative gyrogroup, called a Möbius gyrogroup.

Möbius addition in the ball $\mathbb{V}_s$ is known in the literature as a *hyperbolic translation* [2, 43]. Following the discovery of the gyrocommutative gyrogroup structure in 1988 [50], Möbius hyperbolic translation in the ball $\mathbb{V}_s$ now deserves the title *"Möbius addition"* in the ball $\mathbb{V}_s$, in full analogy with the standard vector addition in the space $\mathbb{V}$ that contains the ball.

Möbius addition in the ball $\mathbb{V}_s$ satisfies the *gamma identity*

$$\gamma_{\mathbf{u}\oplus_{\mathrm{M}}\mathbf{v}} = \gamma_{\mathbf{u}}\gamma_{\mathbf{v}}\sqrt{1 + \frac{2}{s^2}\mathbf{u}\cdot\mathbf{v} + \frac{1}{s^4}\|\mathbf{u}\|^2\|\mathbf{v}\|^2} \tag{27}$$

for all $\mathbf{u}, \mathbf{v} \in \mathbb{V}_s$, where $\gamma_{\mathbf{u}}$ is the gamma factor

$$\gamma_{\mathbf{v}} = \frac{1}{\sqrt{1 - \frac{\|\mathbf{v}\|^2}{s^2}}} \tag{28}$$

in the s-ball $\mathbb{V}_s$.

Following (16), Möbius cooperation, also called Möbius coaddition, in the ball is commutative, given by the equation

$$\mathbf{u} \boxplus_{\mathrm{M}} \mathbf{v} = \frac{\gamma_{\mathbf{u}}^2 \mathbf{u} + \gamma_{\mathbf{v}}^2 \mathbf{v}}{\gamma_{\mathbf{u}}^2 + \gamma_{\mathbf{v}}^2 - 1} \tag{29}$$

for all $\mathbf{u}, \mathbf{v} \in \mathbb{V}_s$. Note that $\mathbf{v} \boxplus_{\mathrm{M}} \mathbf{0} = \mathbf{v}$ and $\mathbf{v} \boxminus_{\mathrm{M}} \mathbf{v} = \mathbf{0}$, as expected.

4. Gyrogroups are loops

A *loop* is a groupoid $(G, \oplus)$ with an identity element, 0, such that each of its two *loop equations* for the unknowns x and y,

$$\begin{aligned} a \oplus x &= b \\ y \oplus a &= b \end{aligned} \tag{30}$$

possesses a unique solution in G for any $a, b \in G$ [39, 40]. Any gyrogroup is a loop. Indeed, if $(G, \oplus)$ is a gyrogroup then the respective unique solutions of the gyrogroup *loop equations* (30) are [56, Sec. 2.4]

$$\begin{aligned} x &= \ominus a \oplus b \\ y &= b \boxminus a \end{aligned} \tag{31}$$

The *cogyrogroup* $(G, \boxplus)$, associated with any gyrogroup $(G, \oplus)$, is also a loop. The unique solutions of its two loop equations

$$\begin{aligned} a \boxplus x &= b \\ y \boxplus a &= b \end{aligned} \tag{32}$$

are [56, Theorem 2.38]

$$\begin{aligned} x &= \ominus(\ominus b \oplus a) \\ y &= b \ominus a \end{aligned} \tag{33}$$

Note that, in general, the two loop equations in (32) are identically the same equation if and only if the gyrogroup cooperation $\boxplus$ is commutative. Hence, their solutions must be, in general, identical if and only if the gyrogroup cooperation $\boxplus$ is commutative. Indeed, a gyrogroup $(G, \oplus)$ possesses the *gyroautomorphic inverse property*, $\ominus(a \oplus b) = \ominus a \ominus b$, if and

only if it is gyrocommutative [56, Theorem 3.2]. Hence, the two solutions, x and y, in (33) are, in general, equal if and only if the gyrogroup $(G, \oplus)$ is gyrocommutative. This result is compatible with the result that a gyrogroup is gyrocommutative if and only if its cooperation $\boxplus$ is commutative [56, Theorem 3.4].

The cogyrogroup is an important and interesting loop. Its algebraic structure is not grouplike, but it plays a crucial role in the study of the gyroparallelogram law of Einstein's special relativity theory and its underlying hyperbolic geometry, Figs. 4, 5 and 8.

It follows from the solutions of the loop equations in (30) and (32) that any gyrogroup $(G, \oplus)$ possesses the following cancellation laws [56, Table 2.1]:

$$
\begin{aligned}
a \oplus (\ominus a \oplus b) &= b \\
(b \boxminus a) \oplus a &= b \\
a \boxminus (\ominus b \oplus a) &= b \\
(b \ominus a) \boxplus a &= b
\end{aligned}
\tag{34}
$$

The first (second) cancellation law in (34) is called the *left* (*right*) *cancellation law*. The last cancellation law in (34) is called the *second right cancellation law*. The two right cancellation laws in (34) form one of the duality symmetries that the gyrogroup operation and cooperation share, mentioned in the paragraph of (18). It is thus clear that in order to maintain analogies between gyrogroups and groups, we need both the gyrogroup operation and its associated gyrogroup cooperation.

In the special case when a gyrogroup is gyrocommutative, it is also known as (i) a *K-loop* (a term coined by Ungar in [51]; see also [27, pp. 1, 169-170]); and (ii) a *Bruck loop* [27, pp. 168]. A new term, (iii) "dyadic symset", which emerges from an interesting work of Lawson and Lim in [31], turns out, according to [31, Theorem 8.8], to be identical with a two-divisible, torsion-free, gyrocommutative gyrogroup [56, p. 71].

5. Möbius scalar multiplication in the ball

Having developed the Möbius gyrogroup as a grouplike loop, we do not stop at the loop level. Encouraged by analogies gyrogroups share with groups, we now seek analogies with vector spaces as well. Accordingly, we uncover the scalar multiplication, $\otimes_{\mathrm{M}}$, between a real number $r \in \mathbb{R}$ and a vector

$\mathbf{v} \in \mathbb{V}_s$, that a Möbius gyrogroup $(\mathbb{V}_s, \oplus_{\mathrm{M}})$ admits, so that we can turn the Möbius gyrogroup into a Möbius gyrovector space $(\mathbb{V}_s, \oplus_{\mathrm{M}}, \otimes_{\mathrm{M}})$. For any natural number $n \in \mathbb{N}$ we define and calculate $n \otimes_{\mathrm{M}} \mathbf{v} := \mathbf{v} \oplus_{\mathrm{M}} \ldots \oplus_{\mathrm{M}} \mathbf{v}$ (n-terms), obtaining a result in which we formally replace n by a real number r, suggesting the following definition of the Möbius scalar multiplication.

Definition 4. (Möbius scalar multiplication). Let $(\mathbb{V}_s, \oplus_{\mathrm{M}})$ be a Möbius gyrogroup. Then its corresponding Möbius gyrovector space $(\mathbb{V}_s, \oplus_{\mathrm{M}}, \otimes_{\mathrm{M}})$ involves the *Möbius scalar multiplication* $r \otimes_{\mathrm{M}} \mathbf{v} = \mathbf{v} \otimes_{\mathrm{M}} r$ in $\mathbb{V}_s$, given by the equation

$$
r \otimes_{\mathrm{M}} \mathbf{v} = s \frac{\left(1 + \dfrac{\|\mathbf{v}\|}{s}\right)^r - \left(1 - \dfrac{\|\mathbf{v}\|}{s}\right)^r}{\left(1 + \dfrac{\|\mathbf{v}\|}{s}\right)^r + \left(1 - \dfrac{\|\mathbf{v}\|}{s}\right)^r} \frac{\mathbf{v}}{\|\mathbf{v}\|}
\tag{35}
$$

$$
= s \tanh(r \tanh^{-1} \frac{\|\mathbf{v}\|}{s}) \frac{\mathbf{v}}{\|\mathbf{v}\|}
$$

where $r \in \mathbb{R}$, $\mathbf{v} \in \mathbb{V}_s$, $\mathbf{v} \neq \mathbf{0}$; and $r \otimes_{\mathrm{M}} \mathbf{0} = \mathbf{0}$.

Extending Definition 4 by abstraction, we obtain the abstract gyrovector space, studied in [56, Chap. 6]. As we go through the study of gyrovector spaces, we see remarkable analogies with classical results unfolding. In particular, armed with the gyrovector space structure, we offer a gyrovector space approach to the study of hyperbolic geometry [56], which is fully analogous to the common vector space approach to the study of Euclidean geometry [24]. Our basic examples are presented in the sequel and shown in several figures.

6. Möbius gyroline and more

In full analogy with straight lines in the standard vector space approach to Euclidean geometry, let us consider the *gyroline* equation in the ball $\mathbb{V}_s$,

$$
L_{AB} := A \oplus (\ominus A \oplus B) \otimes t
\tag{36}
$$

$t \in \mathbb{R}$, $A, B \in \mathbb{V}_s$, in a Möbius gyrovector space $(\mathbb{V}_s, \oplus, \otimes)$. For simplicity, we use in this section the notation $\oplus_{\mathrm{M}} = \oplus$ and $\otimes_{\mathrm{M}} = \otimes$. The gyrosegment AB is the part of the gyroline (36) that links the points A and B. Hence, it is given by (36) with $0 \leqslant t \leqslant 1$, Fig. 1.
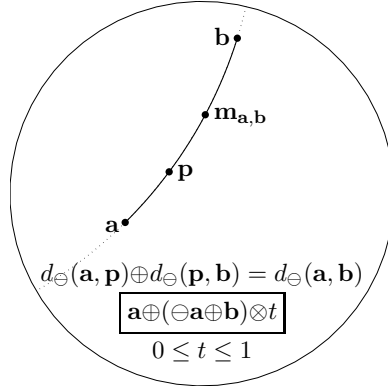
$d_\ominus(\mathbf{a}, \mathbf{p}) \oplus d_\ominus(\mathbf{p}, \mathbf{b}) = d_\ominus(\mathbf{a}, \mathbf{b})$

$\boxed{\mathbf{a} \oplus (\ominus \mathbf{a} \oplus \mathbf{b}) \otimes t}$

$0 \leq t \leq 1$

$d_\boxminus(\mathbf{a}, \mathbf{p}) \boxplus d_\boxminus(\mathbf{p}, \mathbf{b}) = d_\boxminus(\mathbf{a}, \mathbf{b})$

$\boxed{(\mathbf{b} \boxminus \mathbf{a}) \otimes t \oplus \mathbf{a}}$

$0 \leq t \leq 1$

Figure 1. The gyrosegment that links the two points **a** and **b** in the Möbius gyrovector plane $(\mathbb{R}^2_s, \oplus, \otimes)$. **p** is a generic point between **a** and **b**, and $m_{\mathbf{a},\mathbf{b}}$ is the midpoint of the points **a** and **b**.
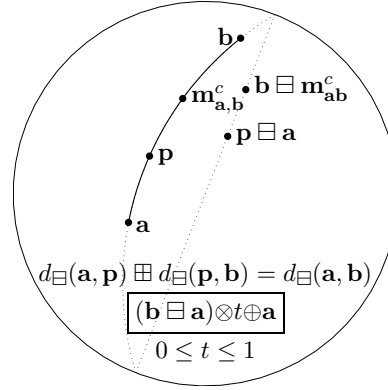
Figure 2. The cogyrosegment that links the two points **a** and **b** in the Möbius gyrovector plane $(\mathbb{R}^2_s, \oplus, \otimes)$. **p** is a generic point cobetween **a** and **b** and $m_{\mathbf{a},\mathbf{b}}$ is the comidpoint of the points **a** and **b**.

For any $t \in \mathbb{R}$ the point $P(t) = A \oplus (\ominus A \oplus B) \otimes t$ lies on the gyroline L_{AB}. Thinking of t as time, at time $t = 0$ the point P lies at $P(0) = A$ and, owing to the left cancellation law in (34), at time $t = 1$ the point P lies at $P(1) = B$. Furthermore, the point P reaches the *gyromidpoint* M_{AB} of the points A and B at time $t = 1/2$,

$$M_{AB} = A \oplus (\ominus A \oplus B) \otimes \tfrac{1}{2} = \tfrac{1}{2} \otimes (A \boxplus B) \tag{37}$$

[56, Sec. 6.5]. Here M_{AB} is the unique gyromidpoint of the points A and B in the gyrodistance sense, $d(A, M_{AB}) = d(B, M_{AB})$, the gyrodistance function being $d(A, B) = \|\ominus A \oplus B\| = \|B \ominus A\|$.

In the special case when $\mathbb{V}_s = \mathbb{R}^2_s$, the gyroline L_{AB}, shown in Fig. 1, is a circular arc that intersects the boundary of the s-disc $\mathbb{R}^2_s$ orthogonally. A study of the connection between gyrovector spaces and differential geometry [56, Chap. 7] [57] reveals that this gyroline is the unique geodesic that passes through the points A and B in the Poincaré disc model of hyperbolic geometry.

The *cogyroline* equation in the ball $\mathbb{V}_s$, similar to (36), is

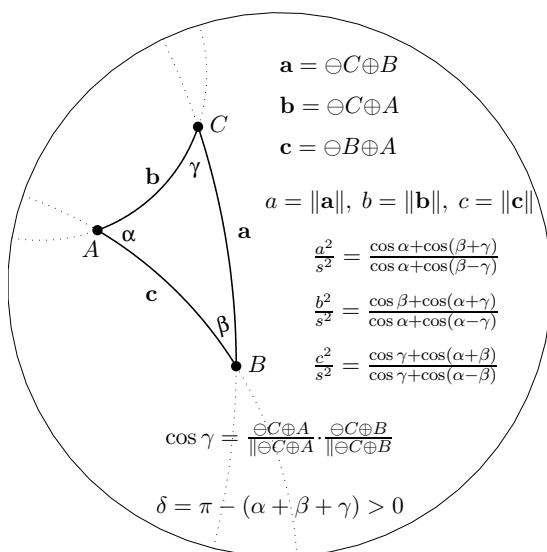$$L^c_{AB} := (B \boxminus A) \otimes t \oplus A \tag{38}$$

Figure 3. Möbius gyrotriangle and its standard notation and identities in a Möbius gyrovector space $(\mathbb{V}_s, \oplus, \otimes)$. Remarkably, in the limit as $s \to \infty$ the equations in the figure reduce to their Euclidean counterparts. Thus, for instance, in that limit we have $\cos \alpha + \cos(\beta + \gamma) = 0$ implying the Euclidean theorem according to which the triangle angle sum is π, $\alpha + \beta + \gamma = \pi$.

$t \in \mathbb{R}$, $A, B \in \mathbb{V}_s$, in a Möbius gyrovector space $(\mathbb{V}_s, \oplus, \otimes)$. The cogyrosegment AB is the part of the cogyroline (38) that links the points A and B. Hence, it is given by (38) with $0 \leqslant t \leqslant 1$, Fig. 2.

For any $t \in \mathbb{R}$ the point $P(t) = (B \boxminus A) \otimes t \oplus A$ lies on the cogyroline L_{AB}^c in (38). Thinking of t as time, at time $t = 0$ the point P lies at $P(0) = A$ and, owing to the right cancellation law in (34), at time $t = 1$ the point P lies at $P(1) = B$. Furthermore, the point P reaches the *cogyromidpoint* M_{AB}^c of the points A and B at time $t = 1/2$,

$$M_{AB}^c = (B \boxminus A) \otimes \tfrac{1}{2} \oplus A = \tfrac{1}{2} \otimes (A \oplus B) \qquad (39)$$

[56, Theorem 6.34]. Here M_{AB}^c is the unique cogyromidpoint of the points A and B in the cogyrodistance sense, $d^c(A, M_{AB}^c) = d^c(B, M_{AB}^c)$, the cogyrodistance function being $d^c(A, B) = \|\ominus A \boxplus B\| = \|B \boxminus A\|$.

In the special case when $\mathbb{V}_s = \mathbb{R}_s^2$, the cogyroline L_{AB}^c, shown in Fig. 2, is a circular arc that intersects the boundary of the s-disc $\mathbb{R}_s^2$ diametrically.

Let $A, B, C \in G$ be any three non-gyrocollinear points of a Möbius gyrovector space $G = (G, \oplus, \otimes)$. In Fig. 3 we see a gyrotriangle ABC whose vertices, A, B, and C, are linked by the gyrovectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$; and whose
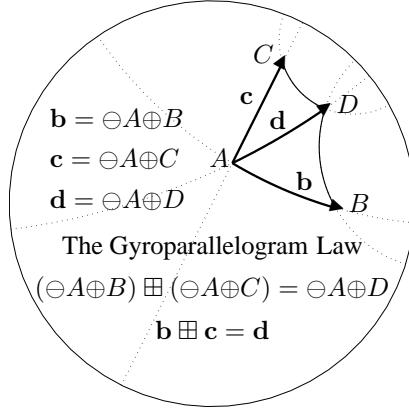
Figure 4. The Möbius gyroparallelogram *ABDC* and its associated gyroparallelogram addition law of gyrovectors in a Möbius gyrovector space $(\mathbb{V}_s, \oplus, \otimes)$ is shown.
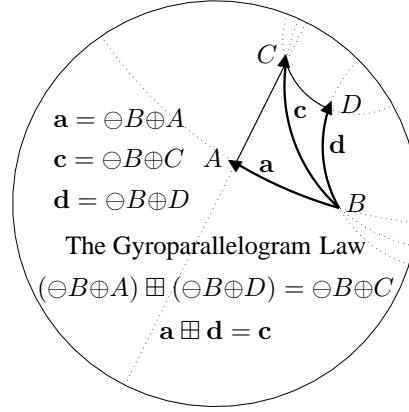
Figure 5. As a second example, the same Möbius gyroparallelogram *ABDC* in Fig. 4 gives rise to a second gyroparallelogram addition of gyrovectors.

side gyrolengths are a, b, and c, given by the equations

$$
\begin{aligned}
\mathbf{a} &= \ominus C \oplus B, & a &= \|\mathbf{a}\| \\
\mathbf{b} &= \ominus C \oplus A, & b &= \|\mathbf{b}\| \\
\mathbf{c} &= \ominus B \oplus A, & c &= \|\mathbf{c}\|
\end{aligned}
\tag{40}
$$

With the gyrodistance function $d(A, B) = \|\ominus A \oplus B\| = \|B \ominus A\|$, we have the gyrotriangle inequality [56, Theorem 6.9] $d(A, C) \leqslant d(A, B) \oplus d(B, C)$, in full analogy with the Euclidean triangle inequality.

A gyrovector $\mathbf{v} = \ominus A \oplus B$ in a Möbius gyrovector plane $(\mathbb{R}_s^2, \oplus, \otimes)$ and in a Möbius three-dimensional gyrovector space $(\mathbb{R}_s^3, \oplus, \otimes)$ is represented graphically by the directed gyrosegment AB from A to B as, for instance, in Figs. 4 – 5 and 8.

Two gyrovectors, (i) $\ominus A \oplus B$, from A to B, and (ii) $\ominus A' \oplus B'$, from A' to B', in a gyrovector space $G = (G, \oplus, \otimes)$ are equivalent if

$$
\ominus A \oplus B = \ominus A' \oplus B'
\tag{41}
$$

In the same way that vectors in Euclidean geometry are equivalence classes of directed segments that add according to the parallelogram law,

gyrovectors in hyperbolic geometry are equivalence classes of directed gyrosegments that add according to the gyroparallelogram law. A gyroparallelogram, the hyperbolic parallelogram, sounds like a contradiction in terms since parallelism in hyperbolic geometry is denied. However, in full analogy with Euclidean geometry, but with no reference to parallelism, the gyroparallelogram is defined as a hyperbolic quadrilateral whose gyrodiagonals intersect at their gyromidpoints, as in Figs. $4-5$. Indeed, any three non-gyrocollinear points A, B, C in a gyrovector space $(G, \oplus, \otimes)$ form a gyroparallelogram $ABDC$ if and only if D satisfied the *gyroparallelogram condition* $D = (B \boxplus C) \ominus A$ [56, Sec. 6.7].

An interesting contrast between Euclidean and hyperbolic geometry is observed here. In Euclidean geometry vector addition coincides with the parallelogram addition law. In contrast, in hyperbolic geometry gyrovector addition, given by Möbius addition, and the Möbius gyroparallelogram addition law are distinct.

7. Einstein operations in the ball

Definition 5. (Einstein addition in the ball). Let $\mathbb{V}$ be a real inner product space and let $\mathbb{V}_s$ be the s-ball of $\mathbb{V}$,

$$\mathbb{V}_s = \{\mathbf{v} \in \mathbb{V} : \|\mathbf{v}\| < s\} \tag{42}$$

where $s > 0$ is an arbitrarily fixed constant (that represents in physics the vacuum speed of light c). *Einstein addition* $\oplus_{\mathrm{E}}$ is a binary operation in $\mathbb{V}_s$ given by the equation

$$\mathbf{u} \oplus_{\mathrm{E}} \mathbf{v} = \frac{1}{1 + \frac{\mathbf{u} \cdot \mathbf{v}}{s^2}} \left\{ \mathbf{u} + \frac{1}{\gamma_{\mathbf{u}}} \mathbf{v} + \frac{1}{s^2} \frac{\gamma_{\mathbf{u}}}{1 + \gamma_{\mathbf{u}}} (\mathbf{u} \cdot \mathbf{v}) \mathbf{u} \right\} \tag{43}$$

where $\gamma_{\mathbf{u}}$ is the gamma factor, (28), in $\mathbb{V}_s$, and where $\cdot$ and $\|\cdot\|$ are the inner product and norm that the ball $\mathbb{V}_s$ inherits from its space $\mathbb{V}$.

We may note that the Euclidean 3-vector algebra was not so widely known in 1905 and, consequently, was not used by Einstein. Einstein calculated in his founding paper [12] the behavior of the velocity components parallel and orthogonal to the relative velocity between inertial systems, which is as close as one can get without vectors to the vectorial version (43).
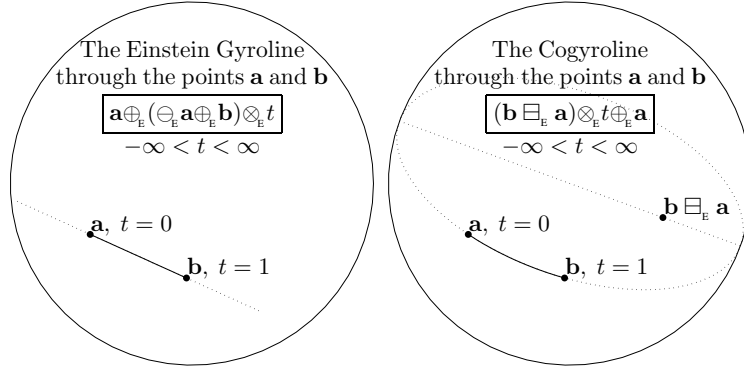
Figure 6. The unique gyroline in an Einstein gyrovector space $(\mathbb{V}_s, \oplus_{\mathrm{E}}, \otimes_{\mathrm{E}})$ through two given points **a** and **b**. The case of the Einstein gyrovector plane, when $\mathbb{V}_s = \mathbb{R}^2_s$ is the real open unit disc, is shown graphically.

Figure 7. The unique cogyroline in an Einstein gyrovector space $(\mathbb{V}_s, \oplus_{\mathrm{E}}, \otimes_{\mathrm{E}})$ through two given points **a** and **b**. The case of the Einstein gyrovector plane, when $\mathbb{V}_s = \mathbb{R}^2_s$ is the real open unit disc, is shown graphically.

Seemingly structureless, Einstein velocity addition could not play in Einstein's special theory of relativity a central role. Indeed, Borel's attempt to "repair" the seemingly "defective" Einstein velocity addition in the years following 1912 is described in [65, p. 117]. Fortunately, however, there is no need to "repair" the Einstein velocity addition law since, like Möbius addition in the ball, Einstein addition in the ball is a gyrocommutative gyrogroup operation, which gives rise to the Einstein ball gyrogroups $(\mathbb{V}_s, \oplus_{\mathrm{E}})$ and gyrovector spaces $(\mathbb{V}_s, \oplus_{\mathrm{E}}, \otimes_{\mathrm{E}})$, Figs. $6-7$ [53, 8]. Furthermore, Einstein's gyration turns out to be the Thomas precession of relativity physics [52], so that Thomas precession is a kinematic effect rather than a dynamic effect as it is usually portrayed [58]. A brief history of the discovery of Thomas precession is presented in [53, Sec. 1.1].

The gamma factor is related to Einstein addition by the *gamma identity*

$$\gamma_{\mathbf{u} \oplus_{\mathrm{E}} \mathbf{v}} = \gamma_{\mathbf{u}} \gamma_{\mathbf{v}} \left(1 + \frac{\mathbf{u} \cdot \mathbf{v}}{s^2} \right) \tag{44}$$

This gamma identity provided the historic link between Einstein's special theory of relativity and the hyperbolic geometry of Bolyai and Lobachevsky, as explained in [60].

Einstein scalar multiplication in the ball $\mathbb{V}_s$ is identical with Möbius scalar multiplication, (35), in the ball $\mathbb{V}_s$, $r \otimes_{\mathrm{E}} \mathbf{v} = r \otimes_{\mathrm{M}} \mathbf{v}$ for all $r \in \mathbb{R}$ and $\mathbf{v} \in \mathbb{V}_s$. Hence Einstein and Möbius scalar multiplication are denoted here, collectively, by $\otimes$.

The isomorphism between Einstein addition $\oplus_{\mathrm{E}}$ and Möbius addition $\oplus_{\mathrm{M}}$ in the ball $\mathbb{V}_s$ is surprisingly simple when expressed in gyrolanguage,

the language of gyrovector spaces. As we see from [56, Table 6.1], the gyrovector space isomorphism between $(\mathbb{V}_s, \oplus_{\mathrm{E}}, \otimes)$ and $(\mathbb{V}_s, \oplus_{\mathrm{M}}, \otimes)$ is given by the equations

$$\mathbf{u} \oplus_{\mathrm{E}} \mathbf{v} = 2 \otimes (\tfrac{1}{2} \otimes \mathbf{u} \oplus_{\mathrm{M}} \tfrac{1}{2} \otimes \mathbf{v})$$
$$\mathbf{u} \oplus_{\mathrm{M}} \mathbf{v} = \tfrac{1}{2} \otimes (2 \otimes \mathbf{u} \oplus_{\mathrm{E}} 2 \otimes \mathbf{v}) \tag{45}$$

Following (16), Einstein cooperation, also called Einstein coaddition, in the ball is commutative, given by the equation

$$\mathbf{u} \boxplus_{\mathrm{E}} \mathbf{v} = 2 \otimes \frac{\gamma_{\mathbf{u}} \mathbf{u} + \gamma_{\mathbf{v}} \mathbf{v}}{\gamma_{\mathbf{u}} + \gamma_{\mathbf{v}}} \tag{46}$$

for all $\mathbf{u}, \mathbf{v} \in \mathbb{V}_s$. Clearly, $\mathbf{v} \boxminus_{\mathrm{E}} \mathbf{v} = \mathbf{0}$. Noting the *Einstein half*,

$$\tfrac{1}{2} \otimes \mathbf{v} = \frac{\gamma_{\mathbf{v}}}{1 + \gamma_{\mathbf{v}}} \mathbf{v} \tag{47}$$

and the *scalar associative law* of gyrovector spaces [56, p. 138], it is clear from (46) – (47) that $\mathbf{v} \boxplus_{\mathrm{E}} \mathbf{0} = \mathbf{v}$, as expected.

Einstein noted in 1905 that

> "Das Gesetz vom Parallelogramm der Geschwindigkeiten gilt
> also nach unserer Theorie nur in erster Annäherung."
>
> A. Einstein [12], 1905

[Thus the law of velocity parallelogram is valid according to our theory only to a first approximation.]

We now see that with our gyrovector space approach to hyperbolic geometry, Einstein's noncommutative addition $\oplus_{\mathrm{E}}$ gives rise to an *exact* hyperbolic parallelogram addition $\boxplus_{\mathrm{E}}$, Fig. 8, which is commutative. The cogyrogroup $(\mathbb{V}_s, \boxplus)$ is thus an important commutative loop that regulates algebraically the hyperbolic parallelogram [59].

An interesting contrast between Euclidean and hyperbolic geometry is thus observed here. In Euclidean geometry and in classical mechanics vector addition coincides with the parallelogram addition law. In contrast, in hyperbolic geometry and in relativistic mechanics gyrovector addition, given by Einstein addition, $\mathbf{u} \oplus_{\mathrm{E}} \mathbf{v}$, and the gyroparallelogram addition, $\mathbf{u} \boxplus_{\mathrm{E}} \mathbf{v}$ in $\mathbb{V}_s$, are distinct. We thus face the problem of whether the ultimate relativistic velocity addition is given by the (i) non-commutative Einstein velocity addition law in (43), or by the (ii) commutative Einstein gyroparallelogram
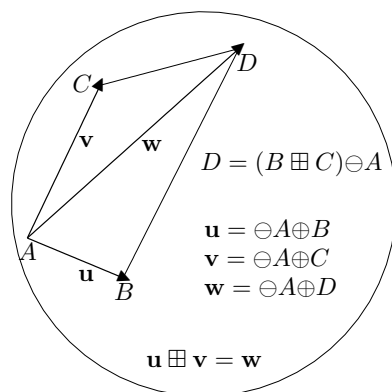
Figure 8.   The Einstein gyroparallelogram addition law of relativistically admissible veloci-
ties. Let $A, B, C \in \mathbb{R}_s^3$ be any three nongyrocollinear points of an Einstein gyrovector space
$(\mathbb{R}_s^3, \oplus, \otimes)$, giving rise to the two gyrovectors $\mathbf{u} = \ominus A \oplus B$ and $\mathbf{v} = \ominus A \oplus C$. Furthermore,
let D be a point of the gyrovector space such that $ABDC$ is a gyroparallelogram, that is,
$D = (B \boxplus C) \ominus A$. Then, Einstein coaddition of $\mathbf{u}$ and $\mathbf{v}$, $\mathbf{u} \boxplus \mathbf{v} = \mathbf{w}$, obeys the gyroparallelo-
gram law, $\mathbf{w} = \ominus A \oplus D$, just as vector addition in $(\mathbb{R}^3, +)$ obeys the parallelogram law. Einstein
coaddition, $\boxplus$, thus gives rise to the gyroparallelogram addition law of Einsteinian velocities,
which is commutative and fully analogous to the parallelogram addition law of Newtonian
velocities.

addition law in Fig. 8. Fortunately, a cosmic phenomenon that can provide
the ultimate resolution of the problem does exist. It is the stellar aberration,
illustrated classically and relativistically for particle aberration in Figs. 9
and 10.

A cosmic experiment in our cosmic laboratory, the Universe, that can
validate the Einstein gyroparallelogram addition law, Fig. 8, and its asso-
ciated gyrotriangle addition law of Einsteinian velocities shown in Fig. 10,
is the *stellar aberration* [48]. Stellar aberration is particle aberration where
the particle is a photon emitted from a star. Particle aberration, in turn,
is the change in the apparent direction of a moving particle caused by the
relative motion between two observers. The case when the two observers
are E (at rest relative to the Earth) and S (at rest relative to the Sun) is
shown graphically in Fig. 9 (classical interpretation) and Fig. 10 (relativistic
interpretation). Obviously, in order to detect stellar aberration there is no
need to place an observer at rest relative to the Sun since this effect varies
during the year. It is this variation that can be observed by observers at
rest relative to the Earth.

The classical interpretation of particle aberration is obvious in terms of
the triangle law of Newtonian velocity addition (which is the common vector
addition in Euclidean geometry), as demonstrated graphically in Fig. 9. The
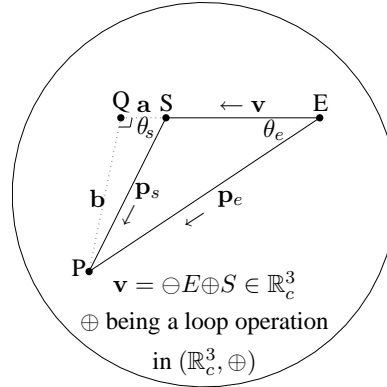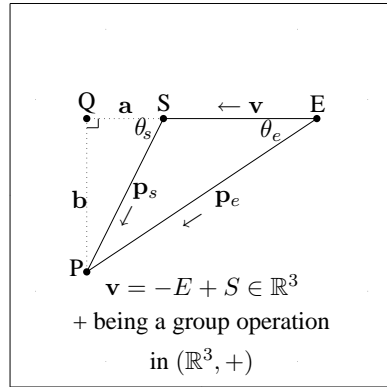
Figure 9. Particle Aberration: Classical interpretation in terms of the triangle law of addition of Newtonian velocities in the standard model of 3-dimensional Euclidean geometry $(\mathbb{R}^3, +)$. Two dimensions are shown for graphical clarity. Here $+$ is the common vector addition in $\mathbb{R}^3$.

A particle P moves with Newtonian velocity $\mathbf{p}_e$ ($\mathbf{p}_s$) relative to the Earth E (the Sun S), making an angle θ_e (θ_s) with the Newtonian velocity $\mathbf{v}$ of the Sun S relative to the Earth E. In order to calculate the Newtonian (classical) particle aberration $\theta_s - \theta_e$, the Euclidean triangle ESP is augmented into the Euclidean right-angled triangle EQP, allowing elementary trigonometry to be employed.

Points are given by their orthogonal Cartesian coordinates (x, y, z), $x^2 + y^2 + z^2 < \infty$. The coordinates are not shown.

The Euclidity of $(\mathbb{R}^3, +)$ is determined by the Euclidean metric in which the distance between two points A, B is $\| - A + B \|$.

Figure 10. Particle Aberration: Relativistic interpretation in terms of the gyrotriangle law of addition of Einsteinian velocities in the Beltrami-Klein ball model of 3-dimensional hyperbolic geometry $(\mathbb{R}^3_c, \oplus)$. Here $\oplus$ is Einstein addition in the c-ball $\mathbb{R}^3_c \subset \mathbb{R}^3$.

A particle P moves with Einsteinian velocity $\mathbf{p}_e$ ($\mathbf{p}_s$) relative to the Earth E (the Sun S), making an angle θ_e (θ_s) with the Einsteinian velocity $\mathbf{v}$ of the Sun S relative to the Earth E. In order to calculate the relativistic particle aberration $\theta_s - \theta_e$, the gyrotriangle ESP is augmented into the right-gyroangled gyrotriangle EQP, allowing elementary gyrotrigonometry to be employed [62].

Points are given by their orthogonal Cartesian coordinates (x, y, z), $x^2 + y^2 + z^2 < c^2$. The coordinates are not shown.

The hyperbolicity of $(\mathbb{R}^3_c, \oplus)$ is determined by the hyperbolic gyrometric in which the distance between two points A, B is given by $\| \ominus A \oplus B \|$.

relativistic interpretation of particle aberration is, however, less obvious.

Relativistic particle aberration is illustrated in Fig. 10 in terms of analogies that it shares with its classical interpretation in Fig. 9. These analogies are just analogies that gyrocommutative gyrogroups share with commutative groups and gyrovector spaces share with vector spaces. Remarkably, the resulting expressions that describe the relativistic stellar aberration phenomenon, obtained by our gyrovector space approach, agree with expressions that are obtained in the literature by employing the relativistic Lorentz transformation group. Our gyrovector space approach is thus capable of recovering known results in astrophysics, to which it gives new geometric

interpretations that are analogous to known, classical interpretations.

8. Dark matter of the universe

What is the universe made of? We do not know. If standard gravitational theory is correct, then most of the matter in the universe is in an unidentified form that does not emit enough light to have been detected by current instrumentation. Astronomers and physicists are collaborating on analyzing the characteristics of this dark matter and in exploring possible physics or astronomical candidates for the unseen material.

S. Weinberg and J. Bahcall [4, p. v]

Fortunately, our gyrovector space approach is capable of discovering a novel result in astrophysics as well, proposing a viable mechanism for the formation of the dark matter of the Universe.

We have seen in Sec. 8 that the cosmic effect of stellar aberration supports our gyrovector gyrospace approach guided by analogies that it shares with the common vector space approach. Another cosmic effect that may support a relativistic physical novel result obtained by our gyrovector space approach to Einstein's special theory of relativity is related to the elusive relativistic center of mass. The difficulties in attempts to obtain a satisfactory relativistic center of mass definition were discussed by Born and Fuchs in 1940 [7], but they did not propose a satisfactory definition. Paradoxically, "In relativity, in contrast to Newtonian mechanics, the centre of mass of a system is not uniquely determined", as Rindler stated with a supporting example [44, p. 89]. Indeed, in 1948 M.H.L. Pryce [41] reached the conclusion that "there appears to be no wholly satisfactory definition of the [relativistic] mass-centre." Subsequently, Pryce's conclusion was confirmed by many authors who proposed various definitions for the relativistic center of mass; see for instance [3, 17, 32] and references therein, where various approaches to the concept of the relativistic center of mass are studied. Consequently, Goldstein stated that "a meaningful center-of-mass (sometimes called center-of-energy) can be defined in special relativity only in terms of the angular-momentum tensor, and only for a particular frame of reference." [18, p. 320].

Fortunately, the spacetime geometric insight that our novel grouplike loop approach offers enables the elusive "manifestly covariant" relativistic

center of mass of a particle system with *proper time* to be identified. It turns out to be analogous to the classical center of mass to the mass of which a specified fictitious mass must be added so as to render it "manifestly covariant" with respect to the motions of hyperbolic geometry. Specifically, let $S = S(m_k, \mathbf{v}_k, \Sigma_\mathbf{0}, N)$, be an isolated system of N noninteracting material particles the k-th particle of which has mass $m_k > 0$ and velocity $\mathbf{v}_k \in \mathbb{R}_c^3$ relative to a rest frame $\Sigma_\mathbf{0}$, $k = 1, \ldots, N$. Then, classically, the system S of N particles can be viewed as a fictitious single particle located at the center of mass of S, with mass $m_0 = \sum_{k=1}^{N} m_k$ that equals the total mass of the constituent particles of S. Relativistically, however, symmetries are determined by gyrogroup, rather than group, symmetries. As in the classical counterpart, the system S can be viewed in Einstein's special theory of relativity as a fictitious single particle located at the relativistic center of mass of S (specified in [62]), with mass m_0 that we present in (48) below.

In order to obey necessary relativistic symmetries, the mass m_0 of the relativistic center of mass of S must exceed, in general, the total mass of the constituent particles of S according to the equation

$$m_0 = \sqrt{\left(\sum_{k=1}^{N} m_k\right)^2 + 2\sum_{\substack{j,k=1 \\ j<k}}^{N} m_j m_k (\gamma_{\ominus \mathbf{v}_j \oplus \mathbf{v}_k} - 1)} \geq \sum_{k=1}^{N} m_k \qquad (48)$$

as explained in [62].

The additional, fictitious mass $m_0 - \sum_{k=1}^{N} m_k$ in (48) of the system S results from relative velocities, $\ominus \mathbf{v}_j \oplus \mathbf{v}_k$, $j, k = 1, \ldots, N$, between particles of the system S. The fictitious mass of a rigid particle system, therefore, vanishes. The fictitious mass of nonrigid galaxies does not vanish and, hence, could account for the dark matter needed to gravitationally "glue" each nonrigid galaxy together.

Indeed, the cosmic laboratory, our Universe, may support the existence of the predicted fictitious mass in (48) as the mass of the dark matter in the Universe that astrophysicists are forced to postulate but cannot detect [4, 34, 10, 37, 49]. Hence, in order to uncover a viable mechanism that accounts for the formation of dark matter that manifests itself only through gravitational interaction, there is no need to modify the laws of physics, as Milgrom proposed in [34]. Rather, one can find it in our grouplike loop approach that improves our understanding of Einstein's special theory of relativity and its underlying hyperbolic geometry of Bolyai and Lobachevsky [62].

9. The Bloch gyrovector of QIC

Bloch vector is well known in the theory of quantum information and computation (QIC). We will show that, in fact, Bloch vector is not a vector but, rather, a gyrovector [9, 54, 55]. It is easy to predict that in the present twenty-first century it is quantum mechanics that will increasingly influence our lives. Hence, it would be interesting to see what gyrovector spaces have to offer in QIC.

A *qubit* is a two state quantum system completely described by the qubit *density matrix* $\rho_{\mathbf{v}}$,

$$\rho_{\mathbf{v}} = \tfrac{1}{2} \begin{pmatrix} 1 + v_3 & v_1 - iv_2 \\ v_1 + iv_2 & 1 - v_3 \end{pmatrix} \tag{49}$$

parametrized by the vector $\mathbf{v} = (v_1, v_2, v_3) \in \mathbb{B}^3$ in the open unit ball $\mathbb{B}^3 = \mathbb{R}^3_{s=1}$ of the Euclidean 3-space $\mathbb{R}^3$. The vector $\mathbf{v}$ in the ball is known in QIC as the Bloch vector. However, we will see that it would be more appropriate to call it a gyrovector rather than a vector.

The density matrix product of the four density matrices in the following equation, which are parametrized by two distinct Bloch vectors $\mathbf{u}$ and $\mathbf{v}$, can be written as a single density matrix parametrized by the Bloch vector $\mathbf{w}$, multiplied by the trace of the matrix product,

$$\rho_{\mathbf{u}}\rho_{\mathbf{v}}\rho_{\mathbf{v}}\rho_{\mathbf{u}} = tr[\rho_{\mathbf{u}}\rho_{\mathbf{v}}\rho_{\mathbf{v}}\rho_{\mathbf{u}}]\rho_{\mathbf{w}} \tag{50}$$

$\mathbf{u}, \mathbf{v} \in \mathbb{B}^3$. Here $tr[m]$ is the trace of a square matrix m, and

$$\mathbf{w} = \mathbf{u} \oplus_{\mathrm{M}} (2 \otimes \mathbf{v} \oplus_{\mathrm{M}} \mathbf{u}) = 2 \otimes (\mathbf{u} \oplus_{\mathrm{M}} \mathbf{v}) \tag{51}$$

Identity (51) is one of several identities available in [9, 54, 55] that demonstrate the compatibility of density matrix manipulations and gyrovector space manipulations.

Two Bloch vectors $\mathbf{u}$ and $\mathbf{v}$ generate the two density matrices $\rho_{\mathbf{u}}$ and $\rho_{\mathbf{v}}$ that, in turn, generate the *Bures fidelity* $\mathcal{F}(\rho_{\mathbf{u}}, \rho_{\mathbf{v}})$ that we may also write as $\mathcal{F}(\mathbf{u}, \mathbf{v})$. The Bures fidelity $\mathcal{F}(\mathbf{u}, \mathbf{v})$ is a most important distance measure between quantum states $\rho_{\mathbf{u}}$ and $\rho_{\mathbf{v}}$ of the qubit in QIC, given by the equations

$$\mathcal{F}(\mathbf{u}, \mathbf{v}) = \left[\mathrm{tr}\sqrt{\sqrt{\rho_{\mathbf{u}}}\rho_{\mathbf{v}}\sqrt{\rho_{\mathbf{u}}}} \right]^2 = \tfrac{1}{2} \frac{1 + \gamma_{\mathbf{u} \oplus_{\mathrm{E}} \mathbf{v}}}{\gamma_{\mathbf{u}}\gamma_{\mathbf{v}}} \tag{52}$$

The first equation in (52) is well known [38, 67], and the second equation in (52) is a gyrovector space equation verified in [56, Eq. 9.69]. Identity (51)

and the second identity in (52) indicate that in density matrix manipulations in QIC, Bloch vectors appear to behave like gyrovectors in Möbius gyrovector spaces $(\mathbb{R}^3_{s=1}, \oplus_{\mathrm{M}}, \otimes)$ and in Einstein gyrovector spaces $(\mathbb{R}^3_{s=1}, \oplus_{\mathrm{E}}, \otimes)$.

Indeed, since the Bures fidelity has particularly wide currency today in QIC geometry, Nielsen and Chuang had to admit for their chagrin [38, p. 410] that

> "Unfortunately, no similarly [alluding to Euclidean geometric interpretation] clear geometric interpretation is known for the fidelity between two states of a qubit".

It is therefore interesting to realize that while Bures fidelity has no Euclidean geometric interpretation, as Nielsen and Chuang admit, it does have a hyperbolic geometric interpretation, which is algebraically regulated by our grouplike loops and their associated gyrovector spaces.

References

[1] **L. V. Ahlfors**: *Conformal invariants: topics in geometric function theory*, McGraw-Hill Book Co., New York, 1973.

[2] **L. V. Ahlfors**: *Möbius transformations in several dimensions*, Univ. of Minnesota School of Mathematics, Minneapolis, Minn., 1981.

[3] **D. Alba, L. Lusanna and M. Pauri**: *Centers of mass and rotational kinematics for the relativistic N-body problem in the rest-frame instant form*, J. Math. Phys. **43(4)** (2002), $1677 - 1727$.

[4] **J. Bahcall, T. Piran and S. Weinberg (eds.)**: *Dark Matter in the Universe*, sec. ed. Kluwer Academic Publishers Group, Dordrecht, 2004.

[5] **J. F. Barrett**, *Special relativity and hyperbolic geometry*, Univ. Sunderland, Sunderland, UK, 1998. Physical Interpretations of Relativity Theory. Proceedings, London, UK, 11–14 September 1998.

[6] **G. S. Birman and A. A. Ungar**: *The hyperbolic derivative in the poincaré ball model of hyperbolic geometry*, J. Math. Anal. Appl. **254** (2001), $321 - 333$.

[7] **M. Born and K. Fuchs**: *The mass centre in relativity*, Nature **145** (1940), 587.

[8] **J. L. Chen and A. A. Ungar**: *From the group* $\mathrm{sl}(2, \mathbf{C})$ *to gyrogroups and gyrovector spaces and hyperbolic geometry*, Found. Phys. **31(11)** (2001), $1611 - 1639$.

[9] **J. L. Chen and A. A. Ungar**: *The Bloch gyrovector*, Found. Phys. **32(4)** (2002), $531 - 565$.

[10] **D. B. Cline**: *The search for dark matter*, Sci. Amer., March 2003, $50 - 59$.

[11] **L. Corry**: *The influence of David Hilbert and Hermann Minkowski on Einstein's views over the interrelation between physics and mathematics*, Endeavor **22(3)** (1998), $95 - 97$.

[12] **A. Einstein**: *Zur Elektrodynamik Bewegter Körper (on the electrodynamics of moving bodies)*, Ann. Physik (Leipzig) **17** (1905), $891 - 921$.

[13] **T. Feder**: *Strong near subgroups and left gyrogroups*, J. Algebra **259** (2003), $177 - 190$.

[14] **S. D. Fisher**: *Complex variables*, Dover Publications Inc., Mineola, NY, 1999, Corrected reprint of the second (1990) edition.

[15] **T. Foguel and A. A. Ungar**: *Involutory decomposition of groups into twisted subgroups and subgroups*, J. Group Theory **3** (2000), $27 - 46$.

[16] **T. Foguel and A. A. Ungar**: *Gyrogroups and the decomposition of groups into twisted subgroups and subgroups*, Pac. J. Math. **197** (2001), $1 - 11$.

[17] **R. P. Gaĭda, V. I. Tretyak and Yu. G. Yaremko**: *Center-of-mass variables in relativistic Lagrangian dynamics of a particle system*, Teoret. Mat. Fiz. **101** (1994), $402 - 416$.

[18] **H. Goldstein**: *Classical mechanics*, Addison-Wesley Publishing Co., Reading, Mass., second edition, 1980.

[19] **R. E. Greene and S. G. Krantz**: *Function theory of one complex variable*, John Wiley & Sons Inc., New York, 1997.

[20] **H. Haruki and T. M. Rassias**: *A new invariant characteristic property of Möbius transformations from the standpoint of conformal mapping*, J. Math. Anal. Appl. **181** (1994), $320 - 327$.

[21] **H. Haruki and T. M. Rassias**: *A new characteristic of Möbius transformations by use of Apollonius points of triangles*, J. Math. Anal. Appl. **197** (1996), $14 - 22$.

[22] **H. Haruki and T. M. Rassias**: *A new characteristic of Möbius transformations by use of Apollonius quadrilaterals*, Proc. Amer. Math. Soc. **126** (1998), $2857 - 2861$.

[23] **H. Haruki and T. M. Rassias**: *A new characterization of Möbius transformations by use of Apollonius hexagons*, Proc. Amer. Math. Soc. **128** (2000), $2105 - 2109$.

[24] **M. Hausner**: *A vector space approach to geometry*, Dover Publications Inc., Mineola, NY, 1998. Reprint of the 1965 original.

[25] **A. N. Issa**: *Gyrogroups and homogeneous loops,*Rep. Math. Phys. **44** (1999), 345 − 358.

[26] **A. N. Issa**: *Left distributive quasigroups and gyrogroups*, J. Math. Sci. Univ. Tokyo **8** (2001), 1 − 16.

[27] **H. Kiechle**: *Theory of K-loops*, Springer-Verlag, Berlin, 2002.

[28] **M. K. Kinyon and O. Jones**: *Loops and semidirect products*, Comm. Algebra **28** (2000), 4137 − 4164.

[29] **M. K. Kinyon and A. A. Ungar**: *The gyro-structure of the complex unit disk*, Math. Mag. **73** (2000), 273 − 284.

[30] **E. Kuznetsov**: *Gyrogroups and left gyrogroups as transversals of a special kind*, Algebra Discrete Math. **3** (2003), 54 − 81.

[31] **J. Lawson and Y. Lim**: *Symmetric sets with midpoints and algebraically equivalent theories*, Results Math. **46** (2004), 37 − 56.

[32] **L. R. Lehner and O. M. Moreschi**: *On the definition of the center of mass for a system of relativistic particles*, J. Math. Phys. **36** (1995), 3377 − 3394.

[33] **J. E. Marsden**: *Elementary classical analysis*, W. H. Freeman and Co., San Francisco, 1974. With the assistance of M. Buchner, A. Erickson, A. Hausknecht, D. Heifetz, J. Macrae and W. Wilson, and with contributions by P. Chernoff, I. Fáry and R. Gulliver.

[34] **M. Milgrom**: *Is 95% of the universe really missing? an alternative to dark matter*, Sci. Amer. August 2002, 42 − 50.

[35] **D. Mumford, C. Series and D. Wright**: *Indra's pearls: The vision of Felix Klein*, Cambridge University Press, New York, 2002.

[36] **T. Needham**: *Visual complex analysis*, The Clarendon Press Oxford University Press, New York, 1997.

[37] **I. Nicolson**: *Dark side of the universe: dark matter, dark energy, and the fate of the cosmos*, John Hopkins University Press, Baltimore, MD, 2007.

[38] **M. A. Nielsen and I. L. Chuang**: *Quantum computation and quantum information*, Cambridge University Press, Cambridge, 2000.

[39] **H. O. Pflugfelder**: *Quasigroups and loops: introduction*, vol. 7 of *Sigma Series in Pure Mathematics*, Heldermann Verlag, Berlin, 1990.

[40] **H. Orlik Pflugfelder**: *Historical notes on loop theory*, Comment. Math. Univ. Carolin. **41** (2000), 359 − 370.

[41] **M. H. L. Pryce**: *The mass-centre in the restricted theory of relativity and its connexion with the quantum theory of elementary particles*, Proc. Roy. Soc. London. Ser. A. **195** (1948), 62 − 81.

[42] **L. Pyenson**: *Relativity in late Wilhelmian Germany: the appeal to a preestablished harmony between mathematics and physics*, Arch. Hist. Exact Sci. **27** (1982), $137 - 155$.

[43] **J. G. Ratcliffe**: *Foundations of hyperbolic manifolds*, vol. 149 of *Graduate Texts in Mathematics*, Springer-Verlag, New York, 1994.

[44] **W. Rindler**: *Introduction to special relativity*, The Clarendon Press Oxford University Press, New York, 1982.

[45] **K. Rózga**: *On central extensions of gyrocommutative gyrogroups*, Pacific J. Math. **193** (2000), $201 - 218$.

[46] **L. V. Sabinin**: *On A. Ungar's gyrogroups*, Uspekhi Mat. Nauk **50(305)** (1995), $251 - 252$.

[47] **L. V. Sabinin, L. L.Sabinina and L. V.Sbitneva**: *On the notion of gyrogroup*, Aequationes Math. **56** (1998), $11 - 17$.

[48] **A. B. Stewart**: *The discovery of stellar aberration*, Sci. Amer. March 1964, $100 - 108$.

[49] **V. Trimble**: *Existence and nature of dark matter in the universe*, Ann. Rev. Astron. Astrophys. **25** (1987), $425 - 472$.

[50] **A. A. Ungar**: *Thomas rotation and the parametrization of the Lorentz transformation group*, Found. Phys. Lett. **1** (1988), $57 - 89$.

[51] **A. A. Ungar**: *The relativistic noncommutative nonassociative group of velocities and the Thomas rotation*, Resultate Math. **16** (1989), $168 - 179$.

[52] **A. A. Ungar**: *Thomas precession and its associated grouplike structure*, Amer. J. Phys. **59** (1991), $824 - 834$.

[53] **A. A. Ungar**: *Beyond the Einstein addition law and its gyroscopic Thomas precession: The theory of gyrogroups and gyrovector spaces*, vol. 117 of *Fundamental Theories of Physics*, Kluwer Academic Publishers Group, Dordrecht, 2001.

[54] **A. A. Ungar**: *The density matrix for mixed state qubits and hyperbolic geometry*, Quantum Inf. Comput. **2** (2002), $513 - 514$.

[55] **A. A. Ungar**: *The hyperbolic geometric structure of the density matrix for mixed state qubits*: Found. Phys. **32** (2002), $1671 - 1699$.

[56] **A. A. Ungar**: *Analytic hyperbolic geometry: Mathematical foundations and applications*, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2005.

[57] **A. A. Ungar**: *Gyrovector spaces and their differential geometry*, Nonlinear Funct. Anal. Appl. **10** (2005), $791 - 834$.

[58] **A. A. Ungar**: *Thomas precession: a kinematic effect of the algebra of Einstein's velocity addition law. Comments on: "Deriving relativistic momentum and energy. II. Three-dimensional case" by S. Sonego and M. Pin*, European J. Phys. **27(3)** (2006), $L17 - L20$.

[59] **A. A. Ungar**: *The relativistic hyperbolic parallelogram law*, in *Geometry, integrability and quantization*, Softex, Sofia, 2006, $249 - 264$.

[60] **A. A. Ungar**: *Einstein's velocity addition law and its hyperbolic geometry*, Comput. Math. Appl. **53** (2007), $1228 - 1250$.

[61] **A. A. Ungar**: *From Möbius to gyrogroups*, Amer. Math. Monthly, 2007 (in print).

[62] **A. A. Ungar**: *Analytic hyperbolic geometry and Einstein's special theory of relativity*, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2008.

[63] **J. Vermeer**: *A geometric interpretation of Ungar's addition and of gyration in the hyperbolic plane*, Topology Appl. **152(3)** (2005), $226 - 242$.

[64] **S. Walter**: *Minkowski, mathematicians, and the mathematical theory of relativity*, in *The expanding worlds of general relativity* Berlin, 1995, $45 - 86$. Birkhäuser Boston, Boston, MA, 1999.

[65] **S. Walter**: *The non-Euclidean style of Minkowskian relativity*: in *The symbolic universe* (J. J. Gray (ed.), M. Keynes, England), $91 - 127$, Oxford Univ. Press, New York, 1999.

[66] **S. Walter**: *Book Review: Beyond the Einstein Addition Law and its Gyroscopic Thomas Precession: The Theory of Gyrogroups and Gyrovector Spaces*, by A. A. Ungar, Found. Phys. **32(2)** (2002), $327 - 330$.

[67] **X. B. Wang, L. C. Kwek and C. H. Oh**: *Bures fidelity for diagonalizable quadratic Hamiltonians in multi-mode systems*, J. Phys. A, **33(27)** (2000), $4925 - 4934$.

Department of Mathematics
North Dakota State University
Fargo, North Dakota 50105
United States of America
E-mail: Abraham.Ungar@ndsu.edu