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Loops related to geometric structures

Helmut Karzel

Abstract

There are many connections between loops and geometries:

- one can derive loops from several geometries and then use these loops for a "coordinatization" of the geometries,
- one can start from loops with certain properties and associate to them geometric structures or
- one can use geometric structures – for instance "chain structures" or "graphs" – in order to represent loops.

Some of these relations I like to discuss here.

1. Introduction and historical remarks

In many geometries we observe the following situation. There is a set P of geometric objects (like points, lines, planes, circles etc.) and a distinct set Γ of permutations of P (like collineations, motions, automorphisms etc.) such that for any two objects $a, b \in P$ there is exactly one permutation in Γ – denoted by $[a \to b]$ – mapping a onto b. Thus the pair (P, Γ) is a *regular permutation set.* Such a situation we obtain for instance if we take for P the set of all points of an Euclidean, or more generally an absolute geometry, and for Γ all reflections in points. More precisely, many geometries $(P, \mathfrak{L}, \equiv)$ (P denotes the set of points, $\mathfrak{L}$ the set of lines and $\equiv$ the congruence relation) in particular absolute and some unitary geometries have the properties:

1. For all $a \in P$ there exists exactly one involutory motion $\tilde{a}$ with $Fix\, \tilde{a} = \{a\}$.
2. Any two points $a, b \in P$ have exactly one midpoint $m \in P$ hence $\widetilde{m}(a) = b$.

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3. For all a,b $\in P$ it holds $\widetilde{\tilde{a}(b)} = \tilde{a} \circ \tilde{b} \circ \tilde{a}$.

Now if (P, Γ) is a regular permutation set and if we fix an arbitrary element $o \in P$, then the set P becomes with respect to the binary operation,

$$a + b := [o \to a] \circ [o \to o]^{-1}(b)$$

a loop $(P, +)$. This construction we call *loop derivation of* (P, Γ) *in the element* o. On the other side, for a given loop $(P, +)$ we obtain a regular permutation set. For $a \in P$ let $a^+(x) := a + x$, hence a^+ is a permutation of P. Let $P^+ := \{p^+ \mid p \in P\}$, $\nu : P \to P; x \mapsto (x^+)^{-1}(o)$ and $a^\circ := a^+ \circ \nu$. Then the pair (P, P°) with $P^\circ := \{p^\circ \mid p \in P\}$ is a regular permutation set – called the *permutation derivation of* $(P, +)$ – having the property that p° interchanges the elements o and p. The loop derivation of (P, P°) in the element o reproduces the loop $(P, +)$.

With these derivations we can translate properties of one structure in properties of the other.

Any arbitrary permutation set (E, Γ) (i.e., we claim only that Γ is a subset of the symmetric group $Sym\, E$ of the set E) can be represented as a chain structure $(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K})$ (cf. section 7, 8, 9) and so also any loop $(E, +)$ via the permutation set (E, E^+) and we have inter alia:

Let (E, Γ) *be a permutation set and* $(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K})$ *the corresponding chain structure, then* (E, Γ) *is regular* (*sharply* 2*-transitive; sharply* 3*-transitive*) *if and only if* $(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K})$ *is a web* (2*-structure; hyperbola structure*).

Of particular interest are invariant reflection structures (P, Γ) and their corresponding K-loops (= Bruck loops) (cf. section 6). Among these structures there are the ordinary point reflection spaces $(P, \widetilde{P})$ characterized by the "three reflection properties" (R1) and (R2) which allow us to define lines such that P together with the set $\mathfrak{L}$ of all lines forms an incidence space $(P, \mathfrak{L})$. Examples are the set P of points and the set $\widetilde{P}$ of all point reflections of a hyperbolic space. If we fix a point $o \in P$ in an ordinary point reflection spaces $(P, \widetilde{P})$ and consider the loop derivation $(P, +)$ in o, then each line $L \in \mathfrak{L}$ passing through o is a commutative subgroup of the loop $(P, +)$. Taking the loop $(P, +)$ and the set $\mathfrak{F} := \{F \in \mathfrak{L} \mid o \in F\}$ of all lines containing o we obtain a "coordinatization" $(P, +, \mathfrak{F})$ of the point reflection space $(P, \widetilde{P})$ like in analytic geometry where $(P, +)$ is a vector space and $\mathfrak{F}$ the set of one dimensional vector subspaces. The points of the corresponding point reflection space or affine space, respectively, are the elements of P and the lines are in both cases the cosets $a+F$ with $a \in P$ and $F \in \mathfrak{F}$ (cf. Theorems 10.1, 11.5, 11.6).

Now we give some historical remarks on incidence groups and the generalisation to geometric spaces with a loop structure. A tripel $(P, \mathfrak{L}, \cdot)$ consisting of a group $(P, \cdot)$ and an incidence space $(P, \mathfrak{L})$ such that for each $a \in P$ the map

$$a^{\cdot} : P \to P; \quad x \mapsto a \cdot x$$

is a collineation of the incidence space $(P, \mathfrak{L})$ is called *incidence group*. Of interest there are the following subclasses. An incidence group $(P, \cdot, \mathfrak{L})$ is called:

fibered	if any line $L \in \mathfrak{L}$ containing the neutral element e of the group $(P, \cdot)$ is a subgroup of $(P, \cdot)$,
2-*sided*	if for all $a \in P$ also the map $^{\cdot}a : P \to P; \quad x \mapsto x \cdot a$ is a collineation of the incidence spacs $(P, \mathfrak{L})$,
kinematik space	if $(P, \mathfrak{L}, \cdot)$ is fibered and 2-sided.

If $(P, \mathfrak{L}, \cdot)$ is an incidence group then the set $\mathfrak{F} := \{L \in \mathfrak{L} \mid e \in L\}$ is a *bundle in* e, i.e., $\bigcup \mathfrak{F} = P$ and for all $A, B \in \mathfrak{F}$ with $A \neq B$ it holds $A \cap B = \{e\}$, and we have $\mathfrak{L} = \{a \cdot F \mid a \in P,\ F \in \mathfrak{F}\}$. If $(P, \mathfrak{L}, \cdot)$ is fibered then $\mathfrak{F}$ is a *fibration* (*partition*) of the group $(P, \cdot)$, i.e., $\mathfrak{F}$ is a bundle and a set of proper subgroups of the group $(P, \cdot)$. If $(P, \mathfrak{L}, \cdot)$ is even a kinematik space then $\mathfrak{F}$ is a *kinematik fibration*, i.e., $\mathfrak{F}$ has the additional property that for all $X \in \mathfrak{F}$ and for all $a \in P$ it holds $a \cdot X \cdot a^{-1} \in \mathfrak{F}$. On the other hand, if $\mathfrak{F}$ is a bundle of a group $(P, \cdot)$ in the neutral element e of $(P, \cdot)$ and if we set $\mathfrak{L} := \{a \cdot F \mid a \in P,\ F \in \mathfrak{F}\}$ then $(P, \mathfrak{L}, \cdot)$ is an incidence group if and only if the following condition is satisfied:

$$(f) \qquad \forall a \in P\ \forall X \in \mathfrak{F} \quad e \in a \cdot X \ \Rightarrow\ a \cdot X \in \mathfrak{F}.$$

Clearly if $\mathfrak{F}$ is a fibration of the group $(P, \cdot)$ then $\mathfrak{F}$ satisfies the condition (f) and so there is a one to one correpondence between fibered incidence groups (kinematik spaces) and fibrations (kinematik fibrations) of groups.

The notion of incidence group was generalized by weakening the assumptions concerning the algebraic structure. The group $(P, \cdot)$ was replaced by a loop or even a groupoid by H. Wähling, G. Kist, M. Marchi, E. Zizioli and the author (cf. [8], [26], [18], [22], [11], [27]). In [11] the concepts "fibration" and "kinematic fibration" were used also for loops. In 1987 Elena Zizioli found out that for a general loop these notions are not enough to produce a fibered incidence loop. She showed that the conditions $(f) = (F4)$ and $(F5)$ (cf. section 11) are necessary and sufficient. Such fibrations (satisfying $(F4)$ and $(F5)$) are called *incidence fibrations* (cf. [27], [16], [18] Sec. 8).

E. Kolb and A. Kreuzer [19] defined in a loop $(P, +)$ with the help of the defect function $\delta_{a,b}$ (cf. section 5) the binary relation "$a \sim b \Leftrightarrow \delta_{a,b} = id$". Under the assumption that $\sim$ is an equivalence relation, they showed that the equivalence classes form an incidence fibration.

2. Notations and known results

Permutation sets. In this paper P will always denote a non empty set, $Sym\,P$ the group of all permutations of the set P, $J := \{\sigma \in Sym\,P \mid \sigma^2 = id\}$ and $J^* := J \setminus \{id\}$. A pair (P, Γ) with $\Gamma \subseteq Sym\,P$ is called *permutation set* and we call a permutation set

Bol set	if for each $\gamma \in \Gamma,\ \gamma \circ \Gamma \circ \gamma = \Gamma$,
symmetric	if for each $\gamma \in \Gamma,\ \gamma \circ \Gamma^{-1} \circ \gamma = \Gamma$,
invariant	if for each $\gamma \in \Gamma,\ \gamma \circ \Gamma \circ \gamma^{-1} = \Gamma$,
involution set	if $\Gamma \subseteq J$.

For a permutation set (P, Γ) we define for $a, b \in P$:

$$[a \to b] := \{\gamma \in \Gamma \mid \gamma(a) = b\}.$$

Then we call a point $p \in P$ *semiregular* (*transitive*), if for each $x \in P$ we have $|[p \to x]| \leqslant 1$ ($|[p \to x]| \geqslant 1$), and we call $p \in P$ *regular* if $|[p \to x]| = 1$.

By P_s (P_t) we denote the set of all semiregular (transitive) points and by P_r or $(P, \Gamma)_r$ the set of all regular points of (P, Γ). The pair (P, Γ) is called *regular permutation set* if $P = P_r$.

2.1. *Let (P, Γ) be a permutation set. Then:*

(1) *(P, Γ) is a Bol set if and only if (P, Γ) is symmetric and $\Gamma = \Gamma^{-1}$.*

(2) *If (P, Γ) is symmetric and $\sigma \in Sym\,P$ then $(P, \sigma \circ \Gamma)$ is symmetric.*

(3) *If (P, Γ) is symmetric and $\sigma \in \Gamma$ then $(P, \sigma^{-1} \circ \Gamma)$ is a Bol set with $id \in \sigma^{-1} \circ \Gamma$.*

(4) *If (P, Γ) is a Bol set and $\sigma \in Sym\,P$ with $\sigma \circ \Gamma \circ \sigma = \Gamma$, in particular if $\sigma \in \Gamma$, then $(P, \sigma \circ \Gamma)$ is a Bol set.*

(5) *If (P, Γ) is an involution set then the notions "symmetric", "Bol set" and "invariant" coincide.*

Proof. (1) If (P,Γ) is a Bol set and $\gamma\in\Gamma$ then $\gamma\circ\Gamma\circ\gamma=\Gamma$ implies $\gamma^{-1}\circ\Gamma\circ\gamma^{-1}=\Gamma$ hence $\gamma^{-1}\circ\gamma\circ\gamma^{-1}=\gamma^{-1}\in\Gamma$, i.e., $\Gamma^{-1}=\Gamma$ and so $\gamma\circ\Gamma^{-1}\circ\gamma=\gamma\circ\Gamma\circ\gamma=\Gamma$.

(2) Let $\gamma\in\Gamma$ then $(\sigma\circ\gamma)\circ(\sigma\circ\Gamma)^{-1}\circ(\sigma\circ\gamma)=\sigma\circ\gamma\circ\Gamma^{-1}\circ\sigma^{-1}\circ\sigma\circ\gamma=\sigma\circ(\gamma\circ\Gamma^{-1}\circ\gamma)=\sigma\circ\Gamma$ hence $(P,\sigma\circ\Gamma)$ is symmetric.

(3) By (2) $\sigma^{-1}\circ\Gamma$ is symmetric and $\sigma\circ\Gamma^{-1}\circ\sigma=\Gamma$ implies $\sigma^{-1}\circ\Gamma=\Gamma^{-1}\circ\sigma=(\sigma^{-1}\circ\Gamma)^{-1}$, i.e., by (1) $(P,\sigma^{-1}\circ\Gamma)$ is a Bol set.

(4) follows in the same way as (2). □

Binary operation. If P is provided with a binary operation " $+$ ", we define for $a\in P$:

$$a^+ : P\to P;\quad x\mapsto a+x,$$

$$^+a : P\to P;\quad x\mapsto x+a,$$

$$P^+ := \{a^+ \mid a\in P\}\quad\text{and}\quad {}^+P := \{{}^+a \mid a\in P\}.$$

An element $o\in P$ is called *left (right) zero element* if $o^+=id$ $({}^+o=id)$, and *zero element* if $o^+={}^+o=id$. $(P,+)$ is called *left (right) quasigroup* if $P^+\subseteq Sym\,P$ $({}^+P\subseteq Sym\,P)$ and *quasigroup* if $P^+\cup{}^+P\subseteq Sym\,P$, and *left (right) loop* or *loop*, respectively, if moreover $(P,+)$ has a zero element.

If $(P,+)$ is a left loop, hence $P^+\subseteq Sym\,P$, then for all $a,b\in P$ also

$$\delta_{a,b} := ((a+b)^+)^{-1}\circ a^+\circ b^+\in Sym\,P$$

is a permutation fixing the element o. Therefore to each left loop $(P,+)$ there corresponds the subgroup $\Delta := \langle\{\delta_{a,b}\mid a,b\in P\}\rangle$ of $Sym\,P$, generated by all these maps. We have:

2.2. *$(P,+)$ is a quasigroup if and only if (P,P^+) is a regular permutation set.*

2.3. *Let (P,Γ) be a permutation set with $P_r\neq\emptyset$, let $o\in P_r$ be fixed and for $a,b\in P$ we define $a^\bullet := [o\to a]$, $P^\bullet := \{a^\bullet \mid a\in P\}$ and*

$$a\oplus b := [o\to a](b) = a^\bullet(b),$$
$$a+b := [o\to a]\circ[o\to o]^{-1}(b) = a^\bullet\circ(o^\bullet)^{-1}(b).$$

Then

(1) $P^\bullet=\Gamma$.

(2) *$(P,\oplus)$ is a left quasigroup with the property "$\forall a\in P: a\oplus o=a$".*

(3) *$(P,+)$ is a left loop with o as zero element.*

(4) *If (P,Γ) is invariant then (P,Γ) is a regular permutation set, hence*

$P = P_r$.

(5) *If* (P, Γ) *is a regular permutation set then* $(P, \oplus)$ *is a quasigroup with the right zero element* o*, and* $(P, +)$ *is a loop with the zero element* o.

Proof. (4) Let $a, b \in P$ be given, let $c := [o \to a]^{-1}(b)$, $\gamma := [o \to a] \circ [o \to c] \circ [o \to a]^{-1}$ and $d := \gamma(o)$ then (by the invariance) $\gamma \in \Gamma$, hence (by $o \in P_r$) $\gamma = [o \to d]$ and $\gamma(a) = [o \to a] \circ [o \to c](o) = [o \to a](c) = b$. Therefore γ is the unique element in Γ mapping a onto b. $\square$

Definition 1. If (P, Γ) is a permutation set with $P_r \neq \emptyset$ and $p \in P_r$, let $\tilde{p} := [p \to p]$, $\widetilde{P}_r := \{\tilde{p} \mid p \in P_r\}$. Then for each $p \in P_r$ the binary operation

$$+_p : P \times P \to P; \quad (a, b) \mapsto a + b := [p \to a] \circ \tilde{p}^{-1}(b)$$

is called the *loop derivation of* (P, Γ) *in the point* p. Moreover if (P, Γ) is regular and $o \in P$ we set:

$$\nu = \nu_o : P \to P; \quad x \mapsto \widetilde{o} \circ [o \to x]^{-1}(o),$$

$$\omega = \omega_o := \widetilde{o}^{-1} \circ \nu : P \to P; \quad x \mapsto [o \to x]^{-1}(o),$$

$$P^\circ := \Gamma \circ \omega = \{a^\circ := [o \to a] \circ \omega \mid a \in P\}.$$

We remark that $\nu(x) = [o \to o] \circ [o \to x]^{-1}(o) = (x^+)^{-1}(o)$ and we denote

$$-x := \nu(x) = (x^+)^{-1}(o).$$

For $a, b \in P$ we write $a - b := a + (-b)$.

2.4. *If* $(P, +)$ *is a left loop and* $\mu \in Sym\,P$ *any permutation with* $\mu(o) = o$*, then* $(P, P^+ \circ \mu)$ *is a permutation set with* $o \in (P, P^+ \circ \mu)_r$ *and the loop derivation of* $(P, P^+ \circ \mu)$ *in the point* o *gives us back the original left loop* $(P, +)$.

Definition 2. Let $(P, +)$ be a left loop with $-x = \nu(x)$. If $\nu \in Sym\,P$, let $P^\circ := P^+ \circ \nu = \{x^\circ := x^+ \circ \nu \mid x \in P\}$. Then (P, P°) is called *permutation derivation of the left loop* $(P, +)$. If $(P, +)$ is a loop and $p \in P$, let $2'p$ be the solution of the equation $x - p = p$. Then $\widetilde{p} := (2'p)^\circ$ (recall that $\widetilde{p}$ is the unique permutation of P° fixing p) and $\widetilde{\widetilde{p}} := p^+ \circ \nu \circ (p^+)^{-1}$.

2.5. *If $(P,+)$ is a left loop then:*

(1) $\nu \in Sym\,P \Leftrightarrow o \in (P,(P^+)^{-1})_r$.

(2) *If $(P,+)$ is obtained by the loop derivation of a permutation set (P,Γ) in a point $o \in (P,\Gamma)_r$ then* $\nu \in Sym\,P \Leftrightarrow o \in (P,\tilde{o}\circ\Gamma^{-1})_r$.

(3) *If $(P,+)$ is a loop then $\nu \in Sym\,P$ hence we can form the permutation derivation (P,P°) and the loop derivation of (P,P°) in o reproduces the original loop $(P,+)$.*

Definition 3. A loop $(P,+)$ is called:

(*)-*loop*	if (*) $\forall a,b \in P : a-(a-b)=b$;
Bol loop	if for all $a,b\in P$ we have $a^+\circ b^+\circ a^+\in P^+$, i.e., $a+(b+(a+x))=(a+(b+a))+x$ and (P,P^+) is a Bol set;
Bruck loop or *K-loop*	if $(P,+)$ is a Bol loop and if $\nu\in Aut(P,+)$, i.e., $-(a+b)=(-a)+(-b)$.

2.6. *Let (P,Γ) be a Bol set with $P_r\neq\emptyset$ and $(P,+)$ the loop derivation in any point $o\in P_r$ then $(P,+)$ is a Bol loop. If $(P,+)$ is any Bol loop then the permutation derivation (P,P°) is a Bol set.*

Proof. Let $o\in P_r$, $(P,+)$ the loop derivation of (P,Γ) in o and let $a,b\in P$ then (cf. 2.3) $a^+=a^\bullet\circ(o^\bullet)^{-1}$, $b^+=b^\bullet\circ(o^\bullet)^{-1}$ and $a^+\circ b^+\circ a^+=a^\bullet\circ(o^\bullet)^{-1}\circ b^\bullet\circ(o^\bullet)^{-1}\circ a^\bullet\circ(o^\bullet)^{-1}$. Since (P,Γ) is a Bol set, $o^\bullet\circ\Gamma\circ o^\bullet=\Gamma$ hence $\Gamma=(o^\bullet)^{-1}\circ\Gamma\circ(o^\bullet)^{-1}$ and so $(o^\bullet)^{-1}\circ b^\bullet\circ(o^\bullet)^{-1}\in(o^\bullet)^{-1}\circ\Gamma\circ(o^\bullet)^{-1}=\Gamma$, i.e., by 2.3(1) there is a $c\in P$ with $c^\bullet=(o^\bullet)^{-1}\circ b^\bullet\circ(o^\bullet)^{-1}$ and so $a^+\circ b^+\circ a^+=a^\bullet\circ C^\bullet\circ a^\bullet\circ(o^\bullet)^{-1}$. Again since (P,Γ) is a Bol set there is a $d\in P$ with $a^\bullet\circ C^\bullet\circ a^\bullet=d^\bullet$ thus $a^+\circ b^+\circ a^+=d^\bullet\circ(o^\bullet)^{-1}\in P^+$. Therefore (P,P^+) is a Bol set. Moreover by 2.1(1), $\Gamma=\Gamma^{-1}$ and so there is an $a'\in P$ with $a'^\bullet=o^\bullet\circ(a^\bullet)^{-1}\circ o^\bullet$. Hence $(a^+)^{-1}=(a^\bullet\circ(o^\bullet)^{-1})^{-1}=o^\bullet\circ(a^\bullet)^{-1}\circ o^\bullet\circ(o^\bullet)^{-1}=a'^\bullet\circ(o^\bullet)^{-1}=a'^+\in P^+$. By [17] (3.10.3), $(P,+)$ is a Bol loop. □

2.7. *Let $(P,+)$ be a left loop with $\nu\in Sym\,P$ and $P^\circ:=P^+\circ\nu$ then $o\in(P,P^\circ)_r$ and:*

(1) $a\in(P,P^\circ)_r\Leftrightarrow\forall x\in P\ \exists_1 x'\in P$ *such that* $x=x'-a$.

(2) *If $a\in(P,P^\circ)_r$ and if $+_a$ is the loop derivation of (P,P°) in the point a then for all $p,q\in P$ it holds* $p+_a q=p'+\left(a'^+\right)^{-1}(q)$.

(3) *If $(P,+)$ is a Bol loop then $(P,P^\circ)_r=P$ and for all $a\in P$ it holds $p+_a q=p'+(-a'+q)$ and $x'=a+((-a+x)+a)$, in particular, $a'=a+a=:2a$ and moreover, $(P,+_a)$ is a Bol loop.*

Proof. (1) is a consequence of $p^\circ(a) = p^+ \circ \nu(a) = p + (-a) = p - a$.

(2) If $[a \to p]_\circ$ denotes the permutation of P° mapping a onto p then $[a \to p]_\circ = p'^+ \circ \nu$ in particular, $\tilde{a} = [a \to a]_\circ = a'^+$ and so by Definition 1, $p +_a q = [a \to p]_\circ \circ \tilde{a}^{-1}(q) = p'^+ \circ \nu \circ (\nu)^{-1} \circ (a'^+)^{-1}(q) = p' + (a'^+)^{-1}(q)$.

(3) For each loop we have $(P, P^\circ)_r = P$ and in a Bol loop, $(-a)^+ = (a^+)^{-1}$, $(2a)^+ = (a + (o + a))^+ = a^+ \circ a^+$ and so $(a + ((-a + x) + a)) - a = a^+ \circ (-a + x)^+ \circ a^+(-a) = a^+ \circ (-a + x)^+(o) = a^+(-a + x) = x$ implying $x' = a + ((-a + x) + a)$. Consequently, $p +_a q = p' + (a'^+)^{-1}(q) = p' + (-a' + q) = (a + ((-a + p) + a)) + (-2a + q)$ and therefore $p^{+_a} = a^+ \circ (-a + p)^+ \circ (a^+)^{-1}$ implying $p^{+_a} \circ q^{+_a} \circ p^{+_a} = a^+ \circ (-a + p)^+ \circ (-a + q)^+ \circ (-a + p)^+ \circ (a^+)^{-1} \in a^+ \circ P^+ \circ (a^+)^{-1}$. Thus if $r := a + ((-a + p) + ((-a + q) + (-a + p)))$ then $p^{+_a} \circ q^{+_a} \circ p^{+_a} = r^{+_a}$ showing that $(P, +_a)$ is a Bol loop. $\square$

2.8. *Let (P, Γ) be a regular permutation set, let $o \in P$ be fixed and $(P, +)$ the loop derivation in o then:*

(1) *(P, P°) (cf. Definition 1) is a regular permutation set and for each $a \in P$, a° interchanges the points o and a.*

(2) $P^\circ = \Gamma \Leftrightarrow \forall x \in P: \ [o \to x] = [x \to o]$.

(3) *If (P, Γ) is invariant then $\widetilde{o} \circ \nu = \nu \circ \widetilde{o}$ and so $\omega = \widetilde{o}^{-1} \circ \nu = \nu \circ \widetilde{o}^{-1}$ and moreover:*

P° *is invariant* $\Leftrightarrow \forall \alpha \in \Gamma : \alpha \circ \omega \circ \Gamma = \Gamma \circ \omega \circ \alpha \Leftrightarrow \Gamma \cup \{\nu\} \subseteq N(P^\circ)$.

(4) *If P° is invariant then $P^\circ \subseteq J$.*

Proof. (1) By 2.3(5) and 2.5(3), $\nu \in Sym\, P$ and so $\omega = \widetilde{o}^{-1} \circ \nu \in Sym\, P$ hence $P^\circ = \Gamma \circ \omega$ is a regular permutation set. Finally $\omega(o) = [o \to o]^{-1}(o) = o$, $\omega(a) = [o \to a]^{-1}(o)$ and so $a^\circ(o) = [o \to a] \circ \omega(o) = [o \to a](o) = a$ and $a^\circ(a) = [o \to a] \circ \omega(a) = [o \to a] \circ [o \to a]^{-1}(o) = o$.

(2) By Definition 1, $P^\circ = \Gamma \Leftrightarrow \omega = id \Leftrightarrow \nu = \widetilde{o} \Leftrightarrow \forall x \in P : \nu(x) = (x^+)^{-1}(o) = \widetilde{o} \circ [o \to x]^{-1}(o) = \widetilde{o}(x) \Leftrightarrow \forall x \in P: \ [o \to x](x) = o \Leftrightarrow \forall x \in P: \ [o \to x] = [x \to o]$ (since (P, Γ) is a regular permutation set).

(3) Let $x \in P$ then $\nu(x) = \widetilde{o} \circ [o \to x]^{-1}(o) = \widetilde{o} \circ [o \to x]^{-1} \circ \widetilde{o}^{-1}(o) = [o \to \widetilde{o}(x)]^{-1}(o)$ (since Γ is invariant) hence $\widetilde{o} \circ \nu(x) = \widetilde{o} \circ [o \to \widetilde{o}(x)]^{-1}(o) = \nu(\widetilde{o}(x))$ and so $\widetilde{o} \circ \nu = \nu \circ \widetilde{o}$.

Let $P^\circ = \Gamma \circ \omega = \Gamma \circ \widetilde{o}^{-1} \circ \nu$ be invariant and let $\alpha \in \Gamma$ then $\alpha \circ \omega \circ P^\circ = \alpha \circ \omega \circ \Gamma \circ \omega = P^\circ \circ \alpha \circ \omega = \Gamma \circ \omega \circ \alpha \circ \omega$ hence $\alpha \circ \omega \circ \Gamma = \Gamma \circ \omega \circ \alpha$. For $\alpha := \widetilde{o}$ and using the commutativity of $\widetilde{o}$ and ν we obtain $\nu \circ \Gamma = \Gamma \circ \nu$ hence $\omega \circ \Gamma = \widetilde{o}^{-1} \circ \nu \circ \Gamma = \widetilde{o}^{-1} \circ \Gamma \circ \nu = \Gamma \circ \widetilde{o}^{-1} \circ \nu = \Gamma \circ \omega = P^\circ$. Together, $\alpha \circ P^\circ = \alpha \circ \omega \circ \Gamma = \Gamma \circ \omega \circ \alpha = P^\circ \circ \alpha$.

Now let $\Gamma \cup \{\nu\} \subseteq N(P^\circ)$. Then $\nu \circ P^\circ = \nu \circ \Gamma \circ \omega = \nu \circ \Gamma \circ \widetilde{o}^{-1} \circ \nu = P^\circ \circ \nu = \Gamma \circ \omega \circ \nu$ hence (using the commutativity), $\nu \circ \Gamma \circ \widetilde{o}^{-1} = \Gamma \circ \widetilde{o}^{-1} \circ \nu = \Gamma \circ \nu \circ \widetilde{o}^{-1}$ and so $\nu \circ \Gamma = \Gamma \circ \nu$. This implies $\omega \circ \Gamma = \widetilde{o}^{-1} \circ \nu \circ \Gamma = \widetilde{o}^{-1} \circ \Gamma \circ \nu = \Gamma \circ \widetilde{o}^{-1} \circ \nu = \Gamma \circ \omega = P^\circ$ and so $\omega \circ P^\circ = \omega \circ \Gamma \circ \omega = P^\circ \circ \omega$. Therefore if $\alpha \circ \omega \in P^\circ$ then by $\alpha \in N(P^\circ)$, $\alpha \circ \omega \circ P^\circ = \alpha \circ P^\circ \circ \omega = P^\circ \circ \alpha \circ \omega$ showing that P° is invariant.

(4) If $a, b \in P$ we denote the map of P° mapping a onto b by $[a \to b]'$. Now let $\varphi \in P^\circ$, $a \in P$ and $b := \varphi(a)$ hence $\varphi = [a \to b]'$. Since P° is invariant we have $(a^\circ)^{-1} \circ [a \to b]' \circ a^\circ = [o \to (a^\circ)^{-1}(b)]'$. By (1) this is equal $[(a^\circ)^{-1}(b) \to o]' = (a^\circ)^{-1} \circ [b \to a]' \circ a^\circ$. Together we obtain, $\varphi = [b \to a]'$ hence $\varphi(b) = a$, i.e., $\varphi \in J$. □

3. Isomorphisms

Let (P, Γ) and (P', Γ') be permutation sets and let $\psi : P \to P'$ be a bijection. Then ψ is called *isomorphism* between (P, Γ) and (P', Γ') and (P, Γ), (P', Γ') are called *isomorphic*, if $\Gamma' = \psi \circ \Gamma \circ \psi^{-1}$. An isomorphism φ is called *automorphism* of (P, Γ) if $(P, \Gamma) = (P', \Gamma')$, hence $\Gamma = \varphi \circ \Gamma \circ \varphi^{-1}$. Thus the automorphism group $Aut(P, \Gamma)$ is exactly the normalizer of Γ in $Sym\, P$. We call (P, Γ) *homogeneous* if $Aut(P, \Gamma)$ acts transitively on P and *self homogeneous* if for all $a, b \in P$ it holds $[a \to b] \cap Aut(P, \Gamma) \neq \emptyset$. Clearly if (P, Γ) is homogeneous and $(P, \Gamma)_r \neq \emptyset$ then (P, Γ) is a regular permutation set, and if (P, Γ) is invariant with $(P, \Gamma)_r \neq \emptyset$, then (P, Γ) is homogeneous (cf. 2.3(4)).

3.1. *Let $\psi : P \to P'$ be an isomorphism from (P, Γ) onto (P', Γ'), let $(P, \Gamma)_r \neq \emptyset$ and $o \in (P, \Gamma)_r$ then $o' := \psi(o) \in \psi((P, \Gamma)_r) = (P', \Gamma')_r$ and we have:*

(1) $\forall a, b \in P \quad \psi \circ [a \to b] \circ \psi^{-1} = [\psi(a) \to \psi(b)]$.

(2) $\forall x \in P \quad \psi \circ [o \to x] \circ \tilde{o}^{-1} = [o' \to \psi(x)]' \circ \widetilde{o'}^{-1} \circ \psi$.

(3) *If $(P, +)$, $(P', +')$ are the loop derivations of (P, Γ) and (P', Γ') in o and o', respectively, then ψ is an isomorphism from $(P, +)$ onto $(P', +')$ and also from the permutation derivation (P, P°) onto the permutation derivation $(P', p'^{\circ'})$ (We have the formula: If $a \in P$ then $\psi \circ a^\circ \circ \psi^{-1} = (\psi(a))^{\circ'}$).*

(4) *If (P, Γ) is invariant then for all $a, b \in P$ and for each $\gamma \in \Gamma$:*

$$\gamma \circ [a \to b] \circ \gamma^{-1} = [\gamma(a) \to \gamma(b)].$$

Proof. Since ψ is an isomorphism we have for all $a, b \in P$: $\psi \circ [a \to b] \circ \psi^{-1} = [\psi(a) \to \psi(b)]$ and so by $o' = \psi(o)$, $\psi(a+b) = \psi([o \to a] \circ [o \to o]^{-1}(b)) = \psi[o \to a] \circ \psi^{-1} \circ \psi \circ [o \to o]^{-1} \circ \psi(b)[\psi(o) \to \psi(a)] \circ [\psi(o) \to \psi(o)]^{-1}(\psi(b)) = \psi(a) +' \psi(b)$. $\square$

3.2. *Let $o \in (P, \Gamma)_r$, $o' \in (P', \Gamma')_r$, let $(P, +)$ and $(P', +')$, resp., be the loop derivations of (P, Γ) in o, and (P', Γ') in o', resp., and let φ be an isomorphism from $(P, +)$ onto $(P', +')$. Then:*

(1) *φ is also an isomorphism from (P, Γ) onto (P', Γ') if and only if $\varphi \circ \tilde{o} = \widetilde{o'} \circ \varphi$.*

(2) *If $\nu \in Sym\, P$, then $\nu' \in Sym\, P'$ and φ is an isomorphism from the permutation derivation $(P, P^\circ = P^+ \circ \nu)$ of $(P, +)$ onto the permutation derivation (P', P'°) of $(P', +')$.*

Proof. (1) For each $a \in P$ we have $a^+ = [o \to a] \circ \tilde{o}^{-1}$ and $(\varphi(a))^{+'} = [o' \to \varphi(a)]' \circ \widetilde{o'}^{-1}$, and since φ is an isomorphism, $\varphi \circ a^+ = (\varphi(a))^{+'} \circ \varphi$. Together we obtain, $\varphi \circ [o \to a] \circ \varphi^{-1} \circ \varphi \circ \tilde{o}^{-1} = [o' \to \varphi(a)]' \circ \widetilde{o'}^{-1} \circ \varphi$. This implies for $a = o$, $[o' \to \varphi(o)]' = \widetilde{o'}$ and so $o' = \varphi(o)$, i.e., φ is only an isomorphism from (P, Γ) onto (P', Γ') if $\varphi \circ \tilde{o} = \widetilde{o'} \circ \varphi$ and then $\varphi \circ [o \to a] \circ \varphi^{-1} = [o' \to \varphi(a)]'$ showing $\Gamma' = \varphi \circ \Gamma \circ \varphi^{-1}$ since $\Gamma = \{[o \to a] \mid a \in P\}$ and $\Gamma' = \{[o \to \varphi(a)]' \mid a \in P\}$.

(2) From $o' = \varphi(o) = \varphi(x + \nu(x)) = \varphi(x) +' \varphi(\nu(x))$ we obtain $\nu'(\varphi(x)) = \varphi(\nu(x))$ and finally, since $\tilde{o} = o^\circ = o^+ \circ \nu = \nu$ and $\widetilde{o'} = \nu'$ the equation $\varphi \circ \tilde{o} = \widetilde{o'} \circ \varphi$. Hence by 3.2(1), φ is an isomorphism from (P, P°) onto (P', P'°). $\square$

From 3.1 and 3.2 one obtains:

3.3. *Let φ be an isomorphism between the permutation sets (P, Γ) and (P', Γ'), let $o \in (P, \Gamma)_r$ (then $\varphi(o) \in (P', \Gamma')_r$) and let $(P, +)$ resp. $(P', +')$ be the loop derivation in o resp. $\varphi(o)$. If $\nu \in Sym\, P$ (then also $\nu' \in Sym\, P'$), let $P^\circ := P^+ \circ \nu$ and $P'^\circ := P'^+ \circ \nu'$ then φ is an isomorphism between $(P, +)$ and $(P', +')$ and between the permutation sets (P, P°) and (P', P'°).*

3.4. *Let $(P, \Gamma)_r \neq \emptyset$, $a \in (P, \Gamma)_r$, $\psi \in Aut(P, \Gamma)$ and $b := \psi(a)$ then:*

(1) $\forall x \in P \quad \psi \circ [a \to x] \circ \tilde{a}^{-1} = [b \to \psi(x)] \circ \tilde{b}^{-1} \circ \psi$,

(2) *ψ is an isomorphism between the left loops $(P, +_a)$ and $(P, +_b)$ obtained by the loop derivations of (P, Γ) in the points a and b.*

3.5. *Let $(P,+)$ be a left loop with $\nu \in Sym\,P$, (P,P°) with $P^\circ = P^+ \circ \nu$ the permutation derivation of $(P,+)$ and let $\varphi \in Sym\,P$ and $f := \varphi\circ\nu\circ\varphi^{-1}(o)$. Then for $c \in P$:*

(1) $\varphi \in Aut(P,P^\circ) \Leftrightarrow \forall a \in P \;\; \varphi \circ a^+ \circ \varphi^{-1} \circ f^+ = (\varphi(a + \varphi^{-1}(f)))^+$.

(2) *If* $\varphi(o) = o$ *then:* "$\varphi \in Aut(P,P^\circ) \Leftrightarrow \varphi \in Aut(P,+)$".

(3) $\nu \in Aut(P,P^\circ) \Leftrightarrow \nu \in Aut(P,+)$.

(4) $c^+ \in Aut(P,P^\circ) \Leftrightarrow \forall a \in P \;\; c^+ \circ a^+ \circ (c^+)^{-1} \circ (c-(-c))^+ = (c + (a - (-c)))^+$.

(5) $c^\circ \in Aut(P,P^\circ) \Leftrightarrow \forall a \in P \;\; c^+ \circ \nu \circ a^+ \circ \nu^{-1} \circ (c^+)^{-1} \circ (c-(-c))^+ = (c - (a - c))^+$.

Proof. By definition, $\varphi \in Aut(P,P^\circ)$ if and only if $\varphi \circ a^+ \circ \nu \circ \varphi^{-1} \in P^+ \circ \nu$ for each $a \in P$. For $a = o$ we obtain that there has to be an $f \in P$ with $\varphi \circ \nu \circ \varphi^{-1} = f^+ \circ \nu$ and so $\varphi \circ \nu \circ \varphi^{-1}(o) = f^+ \circ \nu(o) = f^+(o) = f$. Thus $\varphi \circ a^+ \circ \nu \circ \varphi^{-1} = \varphi \circ a^+ \circ \varphi^{-1} \circ \varphi \circ \nu \circ \varphi^{-1} = \varphi \circ a^+ \circ \varphi^{-1} \circ f^+ \circ \nu \in P^+ \circ \nu$, i.e., $\varphi \circ a^+ \circ \varphi^{-1} \circ f^+ \in P^+$. Since $\varphi \circ a^+ \circ \varphi^{-1} \circ f^+(o) = \varphi(a + \varphi^{-1}(f))$ we have proved (1). If $\varphi(o) = o$ then $f = o$ and condition (1) assumes the form

$$\varphi \in Aut(P,P^\circ) \Leftrightarrow \forall a \in P \;\; \varphi \circ a^+ \circ \varphi^{-1} = (\varphi(a))^+.$$

But this tells us that φ is an automorphism of the left loop $(P,+)$. Since $\nu(o) = (o^+)^{-1}(o) = id(o) = o$, (3) is a consequence of (2).

Since $f := c^+ \circ \nu \circ (c^+)^{-1}(o) = c - (-c) = c^\circ(-c) = c^\circ \circ \nu \circ (c^\circ)^{-1}(o)$ and so $c^+(a + (c^+)^{-1}(f)) = c + (a - (-c))$ and $c^\circ(a + (c^\circ)^{-1}(f)) = c^\circ(a - c) = c - (a - c)$, (4) and (5) are consequences of (1). $\square$

From 3.2 we obtain:

3.6. *Let $(P,+)$ be a left loop with $\nu \in Sym\,P$, then:*

(1) $P^+ \subseteq Aut(P,P^\circ) \Leftrightarrow \forall a,b \in P \;\; a^+ \circ b^+ = (a + (b - (-a)))^+ \circ (-a)^+$.

(2) *If $P^+ \subseteq Aut(P,P^\circ)$ then $(P,+)$ is a loop and for the structure group $\Delta := \langle\{\delta_{a,b} \mid a,b \in P\}\rangle$ of the loop generated by the permutations $\delta_{a,b} := ((a+b)^+)^{-1} \circ a^+ \circ b^+$ we have $\Delta \le Aut(P,+)$ and therefore $Aut(P,P^\circ) = P^+ \rtimes_Q Aut(P,+)$ is equal the quasidirect product of the loop $(P,+)$ with the automorphism group of the loop.*

(3) $P^\circ \subseteq Aut(P,P^\circ) \Leftrightarrow \forall a,b \in P \;\; a^+ \circ (-b)^+ = (a - (b - a))^+ \circ (-a)^+$.

(4) $P^\circ \subseteq Aut(P,P^\circ) \Leftrightarrow P^+ \cup \{\nu\} \subseteq Aut(P,P^\circ)$.

Proof. (1) We have: "$P^+ \subseteq Aut(P,P^\circ) \Leftrightarrow$ the functional equation of 3.5(4) is valid for all $c, a \in P$". For $a = -c$ we obtain $(c^+)^{-1} \circ (c-(-c))^+ = ((-c)^+)^{-1}$ and so 3.5(4) takes on the form $(c+(a-(-c)))^+ \circ (-c)^+ = c^+ \circ a^+$.

(2) By 2.4, since $\nu \in Sym\, P$, (P, P°) is a permutation set with $o \in (P, P^\circ)_r$ and so by $P^+(o) = P$ and $P^+ \subseteq Aut(P, P^\circ)$, (P, P°) is a regular permutation set. With (P, P°) also $(P, P^+ = P^\circ \circ \nu^{-1})$ is regular and so by 2.2, $(P, +)$ is a loop. By $P^+ \subseteq Aut(P, P^\circ)$ we have $\Delta \leq Aut(P, P^\circ)$ and since $\delta_{a,b}(o) = ((a+b)^+)^{-1} \circ a^+ \circ b^+(o) = ((a+b)^+)^{-1}(a+b) = o$ each element $\delta \in \Delta$ fixes o and so by 3.5(2), $\Delta \leq Aut(P, +)$.

(3) Again, "$P^\circ \subseteq Aut(P, P^\circ) \Leftrightarrow$ the functional equation of 3.5(5) is valid $\forall c, a \in P$". For $c = o$ we obtain $\nu \circ a^+ \circ \nu^{-1} = (-a)^+$ and so 3.5(5) takes on the form $c^+ \circ (-a)^+ \circ (c^+)^{-1} \circ (c - (-c))^+ = (c - (a - c))^+$. Now by $a = c$, we obtain $(-c)^+ \circ (c^+)^{-1} \circ (c - (-c))^+ = id$, i.e., $(c - (-c))^+ = c^+ \circ ((-c)^+)^{-1}$ and finally $c^+ \circ (-a)^+ = (c - (a - c))^+ \circ (-c)^+$.

(4) Clearly if $P^\circ \subseteq Aut(P, P^\circ)$, then $\nu = o^\circ \in Aut(P, P^\circ)$ and $P^+ = P^\circ \circ \nu^{-1} \subseteq Aut(P, P^\circ)$. If $P^+ \cup \{\nu\} \in Aut(P, P^\circ)$ then $P^\circ = P^+ \circ \nu \subseteq Aut(P, P^\circ)$. □

3.7. *For a loop $(P, +)$ the following conditions are equivalent:*

(1) (P, P°) *is selfhomogeneous.*

(2) $\forall a, b \in P \quad a^+ \circ (-b)^+ \circ ((-a)^+)^{-1} = (a - (b - a))^+$.

(3) (P, P°) *is an invariant regular involution set.*

(4) $(P, +)$ *is a K-loop (= Bruck loop).*

Proof. Let $a, b \in P$ and $c \in P$ the solution of $x - a = b$ then $[a \to b]_\circ = c^\circ$ and so by 3.6(3) the conditions (1) and (2) are equivalent. From the equation (2) we obtain $a^+ \circ (-b)^+ = (a - (b - a))^+ \circ ((-a)^+)$ hence $a + (-b + x) = (a - (b - a)) + (-a + x)$ and so for $x := -(-a)$, $a + (-b - (-a)) = a - (b - a)$, i.e., $-b - (-a) = -(b - a)$, showing that ν is an automorphism of $(P, +)$. Now observing $\nu \in Aut(P, +)$ we obtain for $x := - - (b - a)$: $a + (-b - -(b - a)) = (a - (b - a)) + (-a - -(b - a)) = o$ hence $a = b - (b - a) = b^+ \circ \nu \circ b^+ \circ \nu(a) = b^\circ \circ b^\circ(a)$, i.e., $b^\circ \in J$ in particular, $o^\circ = o^+ \circ \nu = \nu \in J$. Consequently $P^\circ \subseteq J$. Finally $a^\circ = a^+ \circ \nu = (a^\circ)^{-1} = \nu \circ (a^+)^{-1}$ hence $(a^+)^{-1}(x) = \nu \circ a^+ \circ \nu(x) = -(a - x) = -a - -x = -a + x = (-a)^+(x)$ and $\nu \circ a^+ \circ \nu = (-a)^+$. Therefore the equation (2) assumes the form $a^+ \circ b^+ \circ a^+ = (a + (b + a))^+$ saying that $(P, +)$ is a Bol loop hence together with $\nu \in Aut(P, +)$, $(P, +)$ is a Bruck loop and moreover, $a^\circ \circ b^\circ \circ a^\circ = a^+ \circ \nu \circ b^+ \circ \nu \circ a^+ \circ \nu = a^+ \circ (-b)^+ \circ a^+ \circ \nu = (a + (-b + a))^+ \circ \nu \in P^+ \circ \nu = P^\circ$. Consequently (P, P°) is an invariant regular involution set.

By [10], (3) and (4) are equivalent. Now let $(P, +)$ be a Bruck loop. Since $(P, +)$ is also a Bol loop, we have $a^+ \circ b^+ \circ a^+ = (a + (b + a))^+$ and obtain by substituting $a := -b$, $(-b)^+ = (b^+)^{-1}$ and so $(- - b)^+ =$

$((-b)^+)^{-1} = ((b^+)^{-1})^{-1} = b^+$ hence $- - b = b$, i.e., $\nu^2 = id$. Then (since $\nu \in Aut(P,+)$) $a^+ \circ (-b)^+ \circ a^+ = (a+(-b+a))^+ = (a+(\nu(b)+\nu(\nu(a))^+ = (a+\nu(b+\nu(a))^+ = (a-(b-a))^+$ and this is equation (2). □

3.8. *For a left loop $(P,+)$ with $\nu \in Sym\,P$ the following conditions are equivalent:*

(1) $P^+ \subseteq Aut(P,P^\circ)$ *and* $P^+ = (P^+)^{-1}$,

(2) $(P,+)$ *is a Bol loop.*

Proof. (1) ⇒ (2) Let $a, b \in P$ then there is a $c \in P$ with $(a^+)^{-1} = c^+$ hence $c = c^+(o) = (a^+)^{-1}(o)$ implying $a + c = a^+(c) = o$, i.e., $c = -a$ hence $(a^+)^{-1} = (-a)^+$ and so $-(-a) = a$. By 3.6(1), $a^+ \circ b^+ = (a + (b - (-a)))^+ \circ (-a)^+$ and by observing the previous facts we obtain $a^+ \circ b^+ = (a + (b + a))^+ \circ (a^+)^{-1}$ or $a^+ \circ b^+ \circ a^+ = (a + (b + a))^+$ telling us that $(P,+)$ is a Bol loop.

(2) ⇒ (1) Since in a Bol loop, for each $a \in P$, $(a^+)^{-1} = (-a)^+$ and $-(-a) = a$ the characterizing functional equation $a^+ \circ b^+ \circ a^+ = (a+(b+a))^+$ of the Bol loop can be written in the form of the equation of 3.6(1) and therefore the statements of (1) are verified. □

Remark 1. By 3.8, if $(P,+)$ is a Bol loop then the permutation derivation (P,P°) of $(P,+)$ is a homogeneous Bol set (cf. 2.6) and so by 3.4(2), if $(P,+_a)$ is the loop derivation of (P,P°) in an arbitrary point $a \in P$, then $(P,+_a)$ and $(P,+)$ are isomorphic. This supplements 2.7(3) and more precisely we have: The map $(-a)^+$ is an isomorphism from the Bol loop $(P,+_a)$ onto the Bol loop $(P,+)$.

4. Involution sets

By [14] we have:

4.1. *Let $(P,+)$ be a left loop then the following statements are equivalent:*

(1) $\nu \in Sym\,P$ *and* $P^\circ \subseteq J$, *i.e.,* (P,P°) *is an involution set with* $o \in (P,P^\circ)_r$.

(2) $(P,+)$ *satisfies the condition* (*) $\forall\, a, b \in P \;\; a - (a - b) = b$.

4.2. *Let (P,Γ) be a permutation set with $P_r := (P,\Gamma)_r \neq \emptyset$, let $o \in P_r$ be fixed and let $+ := +_o$ be the loop derivation of (P,Γ) in o. Then:*

(1) $\Gamma = \Gamma^{-1} \Leftrightarrow \widetilde{P_r} \subseteq J$ *and* $\tilde{o} \circ (P^+)^{-1} \circ \tilde{o} = P^+$.

(2) *If there is a* $\nu' \in J$ *with* $\nu' \circ (P^+)^{-1} \circ \nu' = P^+$ *then* $\Gamma = \Gamma^{-1}$.

(3) *If* (P,Γ) *is an involution set then* $(P,+)$ *satisfies the condition* (*).
(4) *If* (P,Γ) *is an invariant involution set then* $(P,+)$ *is a K-loop.*

Proof. (1) If $a \in P_r$ then $\tilde{a} := [a \to a]$ is the unique element of Γ fixing a and also $\tilde{a}^{-1}(a) = a$. Therefore if $\Gamma = \Gamma^{-1}$ then $\tilde{a} = \tilde{a}^{-1}$, i.e., $\tilde{a} \in J$, in particular $\tilde{o} = o^\circ \in J$. By $P^+ = \{x^+ = [o \to x] \circ (o^\circ)^{-1} = x^\circ \circ \tilde{o} \mid x \in P\} = P^\circ \circ \tilde{o} = \Gamma \circ \tilde{o}$ hence $\Gamma = P^+ \circ \tilde{o}$ we have:

$$\Gamma = P^+ \circ \tilde{o} = \Gamma^{-1} = \tilde{o} \circ (P^+)^{-1} \Leftrightarrow \tilde{o} \circ (P^+)^{-1} \circ \tilde{o} = P^+.$$

(2) Let $\nu' \in J$ with $\nu' \circ (P^+)^{-1} \circ \nu' = P^+$ hence for each $a \in P$ $\exists b \in P$ with $\nu' \circ (a^\circ \circ (o^\circ)^{-1})^{-1} \circ \nu' = \nu' \circ o^\circ \circ (a^\circ)^{-1} \circ \nu' = b^\circ \circ o^\circ$. For $a = o$ we obtain $id = \nu' \circ \nu' = b^\circ \circ o^\circ$ hence $b^\circ = (o^\circ)^{-1}$. Consequently $\Gamma^{-1} = \tilde{o} \circ (P^+)^{-1} = \tilde{o} \circ \nu' \circ P^+ \circ \nu' = \Gamma = P^+ \circ \tilde{o}^{-1}$ if and only if $\nu' = \tilde{o}^{-1}$. □

5. Defect functions

Let (P,Γ) be a permutation set with $P_r \neq \emptyset$ then the map

$$\delta : P_r \times P \times \Gamma \to Sym\, P,$$
$$(a,b,\gamma) \mapsto \delta_{a;b,\gamma} = [a \to a] \circ [\gamma(b) \to a] \circ \gamma \circ [a \to b]$$

is called the *defect function of the permutation set* (P,Γ) *in the point* a.

We have $\delta_{a;b,\gamma}(a) = a$. If $P_r = P$ we set

$$\delta : P^3 \longrightarrow Sym\, P; \quad (a,b,c) \mapsto \delta_{a;b,c} := [a \to a] \circ [c \to a] \circ [b \to c] \circ [a \to b].$$

Three points $a,b,c \in P$ are called *defect free* if $\delta_{a;b,c} = id$.

If $(P,+)$ is a left loop then the map

$$\delta : P \times P \to Sym\, P; \quad (a,b) \mapsto \delta_{a,b} = ((a+b)^+)^{-1} \circ a^+ \circ b^+$$

is called the *defect function of the left loop* $(P,+)$ and $a,b \in P$ are called *defect free* if $\delta_{a,b} = id$, i.e., if $(a+b)^+ = a^+ \circ b^+$. Here o is the fixed point of $\delta_{a,b}$. We recall that the definition implies:

5.1. *If* $(P,+)$ *is a K-loop then* $\Delta := \langle\{\delta_{a,b} \mid a,b \in P\}\rangle \leq$ *Aut* $(P,+)$.

6. Reflection structures and point reflection spaces

Let P be a non empty set. If there is a fixed point $o \in P$ and a map

$$^\circ : P \to J; \quad x \mapsto x^\circ \quad \text{with} \quad x^\circ(o) = x \quad \text{for each} \quad x \in P$$

then the tripel $(P, {}^\circ, o)$ is called *reflection structure* (cf.[10]). If there is a map

$$\sim : P \to J;\quad x \mapsto \widetilde{x} \ \text{ with } \ Fix\ \widetilde{x} = \{x\} \ \text{ for each } \ x \in P$$

satisfying the property

(M) for all $a, b \in P$ $\exists_1 m \in P$ with $\widetilde{m}(a) = b$

then the pair $(P, \sim)$ is called *point reflection structure* (cf. [6], [5]). A reflection structure (point reflection structure) is *invariant* if for all $a, b \in P$ there exists $c \in P$ with $a^\circ \circ b^\circ \circ a^\circ = c^\circ$, $(\widetilde{a} \circ \widetilde{b} \circ \widetilde{a} = \widetilde{c})$. An invariant reflection structure $(P, {}^\circ, o)$ is a point reflection structure if for each $a \in P$, $|Fix\ a^\circ| = 1$.

By the definitions and 2.5 follows:

6.1. *Let $(P, {}^\circ, o)$ be a reflection structure and $\Gamma := P^\circ := \{x^\circ \mid x \in P\}$. If (P, Γ) is regular, for each $p \in P$ we denote by $\widetilde{p} \in \Gamma$ the permutation with $\widetilde{p}(p) = p$. Let $\widetilde{P} := \{\widetilde{p} \mid p \in P\}$. Then:*

(1) *(P, P°) is an involution set, o a regular point hence $o \in (P, P^\circ)_r$ and the loop derivation of (P, P°) in the point o gives us a left loop $(P, +)$ satisfying the condition (*).*

(2) *$(P, {}^\circ, o)$ is an invariant reflection structure $\Leftrightarrow$ (P, P°) is an invariant involution set with $(P, P^\circ)_r \neq \emptyset \Leftrightarrow (P, +)$ is a K-loop.*

(3) *$(P, {}^\circ, o)$ is an invariant point reflection structure $\Leftrightarrow$ (P, P°) is an invariant involution set with $(P, P^\circ)_r \neq \emptyset$ and any two points $a, b \in P$ have exactly one midpoint $m \in P$, i.e., if $c^\circ \in P^\circ$ with $m \in Fix\, c^\circ$ then $c^\circ(a) = b \Leftrightarrow \widetilde{P} = P^\circ = \Gamma \Leftrightarrow (P, +)$ is a K-loop uniquely 2-divisible.*

(4) *If the reflection structure $(P, {}^\circ, o)$ (the point reflection structure $(P, \sim)$) is invariant then for all $a, b \in P$ we have $a^\circ \circ b^\circ \circ a^\circ = (a^\circ \circ b^\circ(a))^\circ$, $(\widetilde{a} \circ \widetilde{b} \circ \widetilde{a} = \widetilde{a(b)})$.*

Proof. All we have to show is that $\nu \in Sym\,P$. Let $x \in P$ then $x^+ = [o \to x] \circ [o \to o]^{-1} = x^\circ \circ (o^\circ)^{-1} = x^\circ \circ o^\circ$ hence $(x^+)^{-1} = o^\circ \circ x^\circ$ and so $\nu(x) = -x = (x^+)^{-1}(o) = o^\circ \circ x^\circ(o) = o^\circ(x)$. Therefore $\nu = o^\circ \in Sym\,P$. □

6.2. *If (P, Γ) is an involution set with $P_r = (P, \Gamma)_r \neq \emptyset$, $o \in P_r$ fixed and $x^\circ := [o \to x]$ for each $x \in P$ then $(P, {}^\circ, o)$ is a reflection structure.*

6.3. *If $(P, +)$ is a left loop satisfying (*) and $x^\circ := x^+ \circ \nu$ for each $x \in P$ then $(P, {}^\circ, o)$ is a reflection structure.*

Definition 4. Let $(P,\sim)$ be a point reflection structure and let $\rho := \{(a,b,c) \in P^3 \mid \tilde{a} \circ \tilde{b} \circ \tilde{c} \in J\}$ then $(P,\sim)$ is called *point reflection space* if:

$(R1)$ $\forall a,b,c \in \rho \ : \ \tilde{a} \circ \tilde{b} \circ \tilde{c} \in \widetilde{P}$,

$(R2)$ ρ is a *ternary equivalence relation*, i.e.,

$$(a,b,c) \in \rho \ \Rightarrow \ (b,c,a),(b,a,c) \in \rho \ \text{ and}$$
$$a \neq b \ \wedge (a,b,c),(a,b,d) \in \rho \ \Rightarrow \ (b,c,d) \in \rho.$$

Remark 2. If $(P,\mathfrak{L},\equiv,\alpha)$ is an absolute geometry (cf. [10]) and if $\sim$ is the map which associates to each point $p \in P$ the reflection in p then $(P,\sim)$ is a point reflection space.

From now on let $(P,\sim)$ be a point reflection space, let $\widetilde{P} := \{\widetilde{p} \mid p \in P\}$, and let $\mathbf{G} := \langle \widetilde{\mathbf{P}} \rangle$ be the group generated by the point reflections $\widetilde{p}$. Let the point $o \in P$ be fixed. From (R1) follows that $(P,\widetilde{P})$ is an invariant regular involution set. Therefore by 4.2(4) the loop derivation $(P,+)$ of $(P,\widetilde{P})$ in o is a K-loop. We call $(P,\sim)$ *singular* if $\rho = P^3$ and *ordinary* otherwise. Any two distinct points $a,b \in P$ determine an equivalence class

$$\overline{a,b} := \{x \in P \mid (a,b,x) \in \rho\}.$$

We have:

6.4. *A point reflection space $(P,\sim)$ is singular if one of the following equivalent conditions is satisfied:*

(1) *The set P of all points forms the only equivalence class of ρ,*

(2) $\widetilde{P} \circ \widetilde{P} \circ \widetilde{P} = \widetilde{P}$,

(3) $\widetilde{P} \circ \widetilde{P}$ *is a commutative subgroup of index* 2 *in* $\mathbf{G}$,

(4) *The K-loop $(P,+)$ is a commutative group (isomorphic with $\widetilde{P} \circ \widetilde{P}$).*

For the rest of this section let $(P,\sim)$ be an ordinary point reflection space. Then the set P together with the set $\mathfrak{L} := \{\overline{a,b} \mid a,b \in P, a \neq b\}$ of all equivalence classes of ρ – called *lines* – forms an incidence space $(P,\mathfrak{L})$ where the set $\mathfrak{L}$ contains more than one line. A subset $T \subseteq P$ is called *subspace of* $(P,\sim)$ if for all $a,b \in T$ with $a \neq b$ we have $\overline{a,b} \subseteq T$. Let $\mathfrak{T}$ be the set of all subspaces of $(P,\sim)$.

Remark 3. If $(P,\mathfrak{L},\equiv,\alpha)$ is an ordinary absolute geometry then the set of lines $\mathfrak{L}$ coincides with the set of equivalence classes of the relation ρ.

6.5. *An ordinary point reflection space $(P,\sim)$ has the following properties:*

(1) $(P,\mathfrak{L})$ *in an incidence space.*

(2) *If* $L \in \mathfrak{L}$, *then* $\forall a, b, c \in L\ \exists\ d \in L$ *with* $\widetilde{a} \circ \widetilde{b} \circ \widetilde{c} = \widetilde{d}$.
(3) *If* $a, b \in P$ *with* $a \neq b$, *then* $\overline{a, b} = \{x \in P \mid \widetilde{a} \circ \widetilde{b} \circ \widetilde{c} \in J\}$.
(4) *For each* $T \in \mathfrak{T}$, *and for each* $t \in T$ *it holds* $\widetilde{t}(T) = T$.
(5) *Let* $\sim_T : T \to Sym\,T;\ \ t \mapsto \widetilde{t}_{|T}$. *Then* $(T, \sim_T)$ *is a point reflection space and* $(T, \sim_T)$ *is singular if and only if* T *is a point or a line.*
(6) $\langle \widetilde{P} \rangle = \mathbf{G} \leq \mathbf{Aut}(\mathbf{P}, \sim) = \mathbf{Aut}(\mathbf{P}, \rho) = \mathbf{Aut}(\mathbf{P}, \mathfrak{L}) = \mathbf{Aut}(\mathbf{P}, \mathfrak{T})$ *and the automorphism group* $Aut(P, \sim)$ *acts transitively on the point set* P.

Since an absolute space $(P, \mathfrak{L}, \equiv, \alpha)$ is also an ordered space $(P, \mathfrak{L}, \alpha)$ and an ordered space is an exchange space (cf. [10] Theorem 1.5) there are ordinary point reflection spaces $(P, \sim)$ such that the corresponding incidence space $(P, \mathfrak{L})$ is an exchange space. For these spaces we can state:

6.6. *Let* $(P, \sim)$ *be an ordinary point reflection space such that the corresponding incidence space* $(P, \mathfrak{L})$ *is an exchange space. Then:*
(1) $(P, \mathfrak{L})$ *has a base* B, *two bases have the same cardinality and we define* $dim(P, \sim) := |B| - 1$ *as dimension of* $(P, \sim)$.
(2) *If* $dim(P, \sim) \geqslant 3$ *then* $(P, \sim)$ *is desarguesian.*
(3) *If* $dim(P, \sim) \geqslant 3$ *let* $\mathfrak{E} := \{T \in \mathfrak{T} \mid dim\,T = 2\}$ *be the set of all planes, for each* $p \in P$ *let* $\mathfrak{L}(p) := \{L \in \mathfrak{L} \mid p \in L\}$ *and* $\mathfrak{E}(p) := \{E \in \mathfrak{E} \mid p \in E\}$, *then* $(\mathfrak{L}(p), \mathfrak{E}(p), \subset)$ *is a projective space.*

7. Nets, chain structures, webs and their properties

Let $(P, \mathfrak{G}_1, \mathfrak{G}_2)$ be a 2-*net*, i.e., P is a non empty set and $\mathfrak{G}_1, \mathfrak{G}_2$ are subsets of the powerset of P called *generators* such that:

$(I1)$ $\forall p \in P,\ \forall i \in \{1, 2\}\ \ \exists_1\ [p]_i \in \mathfrak{G}_i$ with $p \in [p]_i$,
$(I2)$ $\forall X \in \mathfrak{G}_1, \forall Y \in \mathfrak{G}_2\ |X \cap Y| = 1$.

By $(I1)$, $(I2)$, if $A, B \in \mathfrak{G}_i$ then A and B have the same cardinality and there is a binary operation (cf. e.g. [4]):

$$\Box : P \times P \to P;\quad (x, y) \mapsto xy := [x]_1 \cap [y]_2$$

which has the properties:

7.1. *Let* $a, b, c, d \in P$ *and let* $\{a, b\}^{\Box} := \{ab, ba\}$ *and* $\{a, b; c, d\}^{\Box} := \{ab, ba; cd, dc\}$. *Then:*
(1) $(ab)(cd) = ad$,

(2) $"ab = b \Leftrightarrow [a]_1 = [b]_1", \ "ab = a \Leftrightarrow [a]_2 = [b]_2"$ *and* $aa = a$,
(3) $(\{a, b\}^{\square})^{\square} = \{ab, ba\}^{\square} = \{a, b\}$,
(4) $\{a, b\}^{\square} = \{a, b\} \Leftrightarrow ab = a$ *or* $ab = b \Leftrightarrow |\{a, b\} \cup \{a, b\}^{\square}| < 4$,
(5) $(\{a, b; c, d\}^{\square})^{\square} = \{a, b; c, d\}$.

The set $\{a, b\}$ is called *parallel* (*joinable*) if $\{a, b\}^{\square} = \{a, b\}$, $(\{a, b\}^{\square} \neq \{a, b\})$ and a subset $A \subseteq P$ is called *joinable* if for all $\{a, b\} \in \binom{A}{2}$ we have $\{a, b\}^{\square} \neq \{a, b\}$. Let

$$\mathfrak{C} := \{C \in 2^P \mid \forall X \in \mathfrak{G}_1 \cup \mathfrak{G}_2 \ \ |C \cap X| = 1\}$$

be the set of all *chains* of the 2-net $(P, \mathfrak{G}_1, \mathfrak{G}_2)$. If $\mathfrak{K} \subseteq \mathfrak{C}$ then $(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K})$ is called *chain structure* and $(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{C})$ *maximal chain structure*. We have:

$$\mathfrak{C} \neq \emptyset \Leftrightarrow \forall A \in \mathfrak{G}_1 \ \forall B \in \mathfrak{G}_2 \ \ |A| = |B|.$$

If $(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K})$ satisfies the condition

(I_i) $\forall \{a_1, ..., a_i\} \in \binom{P}{i}$ which are joinable $\exists_1 K \in \mathfrak{K}$ with $\{a_1, ..., a_i\} \subseteq K$

for $i = 1, 2, 3$ then $(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K})$ is called *web*, 2-*structure*, *hyperbola structure*, respectively. If $(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K})$ is a web, for each $p \in P$ we denote the chain $K \in \mathfrak{K}$ which is uniquely determined by $p \in K$ with $[p]_3$, hence $[p]_3 \in \mathfrak{K}$ and $p \in [p]_3$.

For $A, B \in \mathfrak{C}$ and $p \in P$ let

$$pA := [p]_1 \cap A, \quad Ap := [p]_2 \cap A \quad \text{and} \quad \widetilde{AB} : P \to P; \quad x \mapsto (Bx)(xA).$$

Moreover we consider the 1- and 2-*perspectivities*:

$$[A \xrightarrow{1} B] : A \to B; \quad x \mapsto xB, \qquad [A \xrightarrow{2} B] : A \to B; \quad x \mapsto Bx.$$

We note:

7.2. *Let $A, B, C \in \mathfrak{C}$ then:*
(1) $\widetilde{AB} \in Sym\, P$ *with* $\widetilde{AB} \in Aut(P, \mathfrak{G}_1 \cup \mathfrak{G}_2)$ *and* $\widetilde{AB}(\mathfrak{G}_1) = \mathfrak{G}_2$,
$\widetilde{BA} = (\widetilde{AB})^{-1}$, $Fix\, \widetilde{AB} = A \cap B$,
(2) $\widetilde{AB}(C) \in \mathfrak{C}$, $\widetilde{AB}(A) = B$, $\widetilde{AB}(B) = A$,
(3) $\widetilde{A} := \widetilde{AA}$ *is an involution with* $Fix\, \widetilde{A} = A$, *called* reflection in A,
(4) $\widetilde{AB} = \widetilde{CD} \Leftrightarrow (A, B) = (C, D)$,
(5) $\widetilde{A, B}_{|A} = [A \xrightarrow{2} B]$, $\widetilde{A, B}_{|B} = [B \xrightarrow{1} A]$,
(6) $\widetilde{A, B} \circ \widetilde{C} \circ \widetilde{B, A} = \widetilde{\widetilde{A, B}(C)}$, *in particular* $\widetilde{A} \circ \widetilde{C} \circ \widetilde{A} = \widetilde{\widetilde{A}(C)}$.

By 7.2(2) there is the following ternary operation:

$$\tau : \mathfrak{C} \times \mathfrak{C} \times \mathfrak{C} \to \mathfrak{C}; \quad (A, B, C) \mapsto \tau(A, B, C) := \widetilde{AC}(B).$$

Two chains $A, B \in \mathfrak{C}$ are called *orthogonal*, denoted by $A \perp B$, if $A \neq B$ and $\widetilde{A}(B) = B$. Then $A \perp B$ implies $B \perp A$. For a subset $\mathfrak{K} \subseteq \mathfrak{C}$ we denote by $\mathfrak{K}^{\perp} := \{C \in \mathfrak{C} \mid \forall K \in \mathfrak{K}\ \widetilde{C}(K) = K\}$ the *orthogonal complement* of the chain set $\mathfrak{K}$.

Definition 5. Let $(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K})$ be a chain structure, $T \in \mathfrak{C}$ and $X \in \mathfrak{G}_1 \cup \mathfrak{G}_2$. $(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K})$ is called *covering* if $\bigcup \mathfrak{K} = P$. T is called *transversal* of $(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K})$ if for each $K \in \mathfrak{K}$

$$T \cap K \neq \emptyset \ \text{ and for each } \ t \in T\ \exists_1 K \in \mathfrak{K} \ \text{ such that } \ t \in K.$$

T is called *orthogonal transversal* of $(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K})$ if moreover for each $K \in \mathfrak{K}$ it holds $T \perp K$. X is called *transversal* (*quasi-transversal*) of $(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K})$ if the map

$$\mathfrak{K} \to X; \quad K \mapsto K \cap X$$

is a bijection (injection).

7.3. *Let $E \in \mathfrak{C}$ be fixed and for $A, B \in \mathfrak{C}$ let $A \cdot B := \widetilde{AB}(E)$. Then $(\mathfrak{C}, \cdot)$ is a group isomorphic to $\mathrm{Sym}\, E$ with the neutral element E and we have the representations:*

$$\tau(A, B, C) = \widetilde{AC}(B) = A \cdot B^{-1} \cdot C \quad \textit{and} \quad \widetilde{A}(B) = A \cdot B^{-1} \cdot A.$$

Definition 6. A subset $\mathfrak{S} \subseteq \mathfrak{C}$ is called *symmetric*, if for all $A, B \in \mathfrak{S}$ it holds $\tau(A, B, A) = \widetilde{A}(B) \in \mathfrak{S}$, and *double symmetric*, if for all $A, B, C \in \mathfrak{S}$ we have $\tau(A, B, C) = \widetilde{A, C}(B) \in \mathfrak{S}$.

Clearly each double symmetric subset $\mathfrak{S}$ is also symmetric, and the set of all symmetric and double symmetric subsets, respectively, is closed with respect to intersections. This allows us to define the two closure operations: if $\mathfrak{A}$ is an arbitrary subset of $\mathfrak{C}$ and if $\mathfrak{C}_S$ and $\mathfrak{C}_{SS}$, respectively, denotes the set of all symmetric and double symmetric subsets of $\mathfrak{C}$, let

$$\mathfrak{A}^{\sim} := \bigcap \{\mathfrak{S} \subseteq \mathfrak{C}_S \mid \mathfrak{A} \subseteq \mathfrak{S}\} \ \text{ and } \ \mathfrak{A}^{\sim\sim} := \bigcap \{\mathfrak{S} \subseteq \mathfrak{C}_{SS} \mid \mathfrak{A} \subseteq \mathfrak{S}\},$$

respectively, be the smallest symmetric and double symmetric subset of $\mathfrak{C}$ containing $\mathfrak{A}$.

Let $\widetilde{\mathfrak{A}} := \{\widetilde{A} \mid A \in \mathfrak{A}\}$ and $\widetilde{\widetilde{\mathfrak{A}}} := \{\widetilde{A, B} \mid A, B \in \mathfrak{A}\}$. Then:

7.4. *Let* $\mathfrak{A} \subseteq \mathfrak{C}$ *and* $\mathcal{M}(\mathfrak{A}) := \{\widetilde{AB}, \widetilde{AB} \circ \widetilde{C} \mid A, B, C \in \mathfrak{A}\}$. *Then:*

(1) $\mathfrak{A} = \mathfrak{A}^{\sim} \Leftrightarrow \widetilde{\mathfrak{A}}$ *is normal in* $\widetilde{\mathfrak{A}}$, (*i.e.,* $\forall\ \alpha, \beta \in \widetilde{\mathfrak{A}}\ \ \alpha \circ \beta \circ \alpha \in \widetilde{\mathfrak{A}}$),

(2) $\mathfrak{A} = \mathfrak{A}^{\sim\sim} \Leftrightarrow \widetilde{\mathfrak{A}}$ *is normal in* $\widetilde{\mathfrak{A}^{\sim\sim}}$,
(*i.e.,* $\forall\ \alpha \in \widetilde{\mathfrak{A}}\ \forall\ \beta \in \widetilde{\mathfrak{A}^{\sim\sim}}\ \ \beta \circ \alpha \circ \beta^{-1} \in \widetilde{\mathfrak{A}}$),

(3) *if an element* $E \in \mathfrak{A}$ *is fixed and the multiplication defined according to* 7.3, *then* $\mathfrak{A} = \mathfrak{A}^{\sim\sim}$, *i.e.,* $\mathfrak{A}$ *is double symmetric* $\Leftrightarrow \mathfrak{A} \leq \mathfrak{C}$,

(4) $\mathcal{M}(\mathfrak{A}) \leq Sym\, P \Leftrightarrow \mathfrak{A} \leq \mathfrak{C}$.

Remark 4. For a chain structure $(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K})$ the *General Rectangle Axiom*

$(R)\quad \forall A, B, C \in \mathfrak{K}\ :\ \{[[a]_1 \cap B]_2 \cap [[a]_2 \cap C]_1 \mid a \in A\} \in \mathfrak{K},$

formulated in [13] p.89, claims exactly that the set $\mathfrak{K}$ of chains is double symmetric, i.e., for all $A, B, C \in \mathfrak{K}$ it holds $A \cdot B^{-1} \cdot C \in \mathfrak{K}$. Therefore if a chain structure $(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K})$ satisfies the General Rectangle Axiom (R) then $\mathfrak{K}$ is symmetric. For a web $(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K})$ the axiom (R) is called *Reidemeister Condition.*

Remark 5. Another axiom formulated by W.Benz [3] for hyperbola structures $(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K})$ and called *Symmetry Axiom* is the following:

$(S)\quad \forall K, L \in \mathfrak{K}\ :\ |\widetilde{L}(K) \cap K| \geqslant 2\ \Rightarrow\ \widetilde{L}(K) = K.$

By [2] and [9] we have the result: *For a hyperbola structure* $(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K})$ *the Symmetry Axiom* (S) *implies the Rectangle Axiom* (R) *and so for a hyperbola structure* $(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K})$ *satisfying the Symmetry Axiom* (S) *the set of chains* $\mathfrak{K}$ *is symmetric.* But there are hyperbola structures $(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K})$ where $\mathfrak{K}$ is even double symmetric and the Symmetry Axiom (S) is violated (cf. [13] p. 90 ff).

7.5. *Let* $A, B, C, \dots \in \mathfrak{C}$, $E \in \mathfrak{C}$ *fixed and let* " $\cdot$ " *be defined according to* 7.3. *Then:*

(1) $Fix\,\widetilde{AB} = A \cap B, \quad Fix\,(\widetilde{AB} \circ \widetilde{C}) = (B \cap C)\square(A \cap C)$,

(2) $\widetilde{AB} \circ \widetilde{CD} \circ \widetilde{FG} = \widetilde{(AD^{-1}F)(GC^{-1}B)}$,

(3) $\widetilde{AB} \circ \widetilde{CD} = \widetilde{(AD^{-1}U)(UC^{-1}B)} \circ \widetilde{U}$,

(4) $\widetilde{A} \circ \widetilde{B} \circ \widetilde{A} = \widetilde{AB^{-1}A} = \widetilde{\widetilde{A}(B)}$,

(5) $\widetilde{A} \circ \widetilde{BC} \circ \widetilde{A} = \widetilde{(AC^{-1}A)(AB^{-1}A)} = \widetilde{\widetilde{A}(C), \widetilde{A}(B)}$,

(6) $\widetilde{AB} \circ \widetilde{E} \circ \widetilde{CD} \circ \widetilde{E} = \widetilde{(AC)(DB)} \circ \widetilde{E}$.

8. Symmetric chain structures as permutation sets

Let $(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{C})$ be a maximal chain structure and let $\mathfrak{S} \subseteq \mathfrak{C}$ be a symmetric subset of chains, hence for all $S, T \in \mathfrak{S}$, $\widetilde{S}(T) \in \mathfrak{S}$. Therefore for each $S \in \mathfrak{S}$ the map

$$\widetilde{S} : \mathfrak{S} \to \mathfrak{S}; \quad X \mapsto \widetilde{S}(X)$$

is an involution of $Sym\, \mathfrak{S}$ and so $\underbrace{(\mathfrak{S}, \widetilde{\mathfrak{S}})}$ with $\widetilde{\mathfrak{S}} := \{\widetilde{S} \mid S \in \mathfrak{S}\}$ is an involution set. By 7.2(6), $\widetilde{S} \circ \widetilde{T} \circ \widetilde{S} = \widetilde{\widetilde{S}(T)}$, hence $(\mathfrak{S}, \widetilde{\mathfrak{S}})$ is an invariant involution set. From 2.3 and 4.2 we obtain:

8.1. *Let $(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{S})$ be a symmetric chain structure such that $(\mathfrak{S}, \widetilde{\mathfrak{S}})_r \neq \emptyset$ and let $E \in (\mathfrak{S}, \widetilde{\mathfrak{S}})_r$ then:*

(1) *$(\mathfrak{S}, \widetilde{\mathfrak{S}})$ is a regular invariant involution set,*

(2) *$\forall A, B \in \mathfrak{S}\ \exists_1 C \in \mathfrak{S}\ \ \widetilde{C}(A) = B$ (i.e., $[A \to B] = \widetilde{C}$),*

(3) *the loop derivation*

$$+ : \mathfrak{S} \times \mathfrak{S} \to \mathfrak{S}; \quad (A, B) \mapsto A + B := [E \to A] \circ \widetilde{E}(B)$$

of $(\mathfrak{S}, \widetilde{\mathfrak{S}})$ in E produces a K-loop $(\mathfrak{S}, +)$.

8.2. *Let $(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{S})$ be a web with $\mathfrak{S}^{\perp} \neq \emptyset$, let $T \in \mathfrak{S}^{\perp}$ and for each $S \in \mathfrak{S}$ let $\widetilde{S}_{|T}$ be the restriction of $\widetilde{S}$ onto T then:*

(1) *$(T, \widetilde{\mathfrak{S}}_{|T})$ with $\widetilde{\mathfrak{S}}_{|T} := \{\widetilde{S}_{|T}) \mid S \in \mathfrak{S}\}$ is a regular involution set and for each $S \in \mathfrak{S}$ we have $S = \{x \Box \widetilde{S}_{|T}(x) \mid x \in T\}$.*

(2) *The following statements are equivalent:*

 (*i*) *$\widetilde{\mathfrak{S}}$ is symmetric,*

 (*ii*) *$(\mathfrak{S}, \widetilde{\mathfrak{S}})$ is a regular invariant involution set.*

Proof. (1) Let $a, b \in T$ and $C := [ab]_3$ be the chain $C \in \mathfrak{S}$ of our web containing the point ab then $C \perp T$ and this implies $\widetilde{C}(T) = T$ and $\widetilde{T}(ab) = ba \in \widetilde{T}(C) = C$ hence $\widetilde{C}(a) = b$ and so $[a \leftrightarrow b] := \widetilde{C}_{|T}$ showing that $(T, \widetilde{\mathfrak{S}}_{|T})$ is a regular involution set.

(2) is a consequence of (1) and 8.1. $\square$

9. Immersions of permutation sets in chain structures

We consider firstly the permutation set $(E, Sym\, E)$ where E is a not empty set. To $(E, Sym\, E)$ there corresponds the following maximal chain structure $(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{C})$ (cf. e.g. [1]) with:

$$P := E \times E, \quad \mathfrak{G}_1 := \{E \times x \mid x \in E\}, \quad \mathfrak{G}_2 := \{x \times E \mid x \in E\}$$

and $\mathfrak{C}$ the set of all chains of the net $(P, \mathfrak{G}_1, \mathfrak{G}_2)$. If we identify E with the subset $\{(x,x) \mid x \in E\}$ of P then $E \in \mathfrak{C}$ and in that way we see that $\mathfrak{C}$ is not empty. Let

$$\kappa_E : Sym\, E \to \mathfrak{C}; \quad \sigma \mapsto \kappa_E(\sigma) := \{(x, \sigma(x)) \mid x \in E\}.$$

Then for each $\sigma \in Sym\, E$, $\kappa_E(\sigma)$ is a chain of $\mathfrak{C}$, the *graph* of σ, and the map κ_E is a bijection between $Sym\, E$ and $\mathfrak{C}$. The inverse map of κ_E is given by

$$\lambda_E : \mathfrak{C} \to Sym\, E; \quad C \mapsto \widetilde{CE} \circ \widetilde{CE}_{|E}.$$

Moreover if $\mathfrak{C}$ is turned into a group $(\mathfrak{C}, \cdot)$ according to 7.3 then κ_E is an isomorphism from the symmetric group $(Sym\, E, \circ)$ onto the group $(\mathfrak{C}, \cdot)$. Now let (E, Γ) be an arbitrary permutation set. Then $\mathfrak{K} := \kappa_E(\Gamma)$ is a subset of $\mathfrak{C}$ and $(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K})$ a chain structure called the *envelope* of (E, Γ). We write $\mathfrak{Ev}(E, \Gamma) := (P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K})$. Between a permutation set (E, Γ) and her envelope $\mathfrak{Ev}(E, \Gamma) := (P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K})$ there are the following connections:

9.1. *Let (E, Γ) be a permutation set and $(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K}) := \mathfrak{Ev}(E, \Gamma)$ her envelope then:*

(1) $\Gamma \leq Sym\, E \Leftrightarrow \mathfrak{K} \leq \mathfrak{C}$,

(2) *(E, Γ) is a regular permutation set $\Leftrightarrow$ $(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K})$ is a web,*

(3) *(E, Γ) is a regular permutation group $\Leftrightarrow$ $(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K})$ is a web with $\mathfrak{K} \leq \mathfrak{C}$, i.e., a web satisfying the Reidemeister Condition; in this case we call $(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K})$ a* webgroup*,*

(4) *(E, Γ) is a sharply 2-transitive permutation set $\Leftrightarrow$ $(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K})$ is a 2-structure,*

(5) *(E, Γ) is a sharply 2-transitive permutation group $\Leftrightarrow$ $(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K})$ is a 2-structure with $\mathfrak{K} \leq \mathfrak{C}$, i.e., a 2-structure satisfying the rectangle axiom (R) (cf. [7]); in this case we call $(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K})$ a* 2-group*,*

(6) *(E, Γ) is a sharply 3-transitive permutation set $\Leftrightarrow$ $(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K})$ is a hyperbola structure,*

(7) *(E, Γ) is a sharply 3-transitive permutation group $\Leftrightarrow$ $(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K})$ is a hyperbola structure with $\mathfrak{K} \leq \mathfrak{C}$, i.e., a hyperbola structure satisfying the rectangle axiom; in this case we call $(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K})$ a* hyperbola group.

From now on we consider only permutation sets (E, Γ) with $(E, \Gamma)_r \neq \emptyset$.

9.2. *Let* $(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K})$ *be a chain structure with a transversal* $X \in \mathfrak{G}_1$ *of* $(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K})$, *let* $o \in X$ *be fixed and let* $E \in \mathfrak{K}$ *with* $o \in E$. *Then for each* $a \in E$, $o\Box a$ *is an element of* X *and so there is exactly one* $A \in \mathfrak{K}$ *with* $o\Box a \in A$. *Therefore if we set* $a^+ := \lambda_E(A)$ *and* $E^+ := \{a^+ \mid a \in E\}$ *then* (E, E^+) *becomes a permutation set with* $o \in (E, E^+)_r$, *and* $(E, +)$ *with* $a + b := a^+(b)$ *becomes a left loop. Moreover we have* $\kappa_E(a^+) = A$ *hence* $\mathfrak{K} = \kappa_E(E^+)$ *and the following three propositions are equivalent:*

(*i*) $(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K})$ *is a web,*

(*ii*) $\forall X \in \mathfrak{G}_1$ X *is a transversal of* $(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K})$,

(*iii*) $\forall X \in \mathfrak{G}_2$ X *is a transversal of* $(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K})$.

9.3. *Let* $(E, +)$ *be a left loop,* $X := o\Box E$, $\mathfrak{K} := \kappa_E(E^+)$ *and for* $a, b \in E$, $A := \kappa_E(a^+)$, $B := \kappa_E(b^+)$ *then* $\kappa_E(a^+ \circ b^+ \circ a^+) = A \cdot B \cdot A = \widetilde{A}(B^{-1})$ *and we have:*

(1) $X \in \mathfrak{G}_1$ *is a transversal of* $(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K})$,

(2) $a^+ \circ b^+ \circ a^+ \in E^+ \Leftrightarrow A \cdot B \cdot A = \widetilde{A}(B^{-1}) \in \mathfrak{K}$,

(3) $(E, +)$ *is a Bol loop* $\Leftrightarrow \mathfrak{K}$ *is symmetric.*

From 9.3 we obtain the theorems:

Theorem 9.4. *Let* $(E, +)$ *be a Bol loop and let* $\mathfrak{K} := \kappa_E(E^+)$ *then* $\mathfrak{K}$ *is symmetric hence (by section 8)* $(\mathfrak{K}, \widetilde{\mathfrak{K}})$ *is an invariant involution set and the following statements are equivalent:*

(*i*) $(\mathfrak{K}, \widetilde{\mathfrak{K}})$ *is regular,*

(*ii*) *for all* $a, b \in E$ *the equation* $b = x + (-a + x)$ *has exactly one solution* $x \in E$,

(*iii*) $(E, +)$ *is uniquely 2-divisible, i.e.,* $\forall a \in E\ \exists_1 x \in E$ *with* $x + x = a$,

(*iv*) $\forall A \in \mathfrak{K}\ \exists_1 A' \in \mathfrak{K}\ \ A' \cdot A' = A$ (*this implies: if* $A + B := A' \cdot B \cdot A'$ *then* $(\mathfrak{K}, +)$ *is a K-loop*),

(*v*) $\mathfrak{K}^\perp \neq \emptyset$.

Proof. Let $A, B \in \mathfrak{K}$ and $a^+ := \lambda_E(A)$, $b^+ := \lambda_E(B)$. If there is a $C \in \mathfrak{K}$ such that $B = \widetilde{C}(A) = C \cdot A^{-1} \cdot C$ and if $c^+ := \lambda_E(C)$ then $b^+ = c^+ \circ (-a)^+ \circ c^+$ hence $b = c + (-a + c)$. If $c \in E$ is a solution of the equation $b = x + (-a + x)$ and $C := \kappa_E(C^+)$ then $\widetilde{C}(A) = B$. This shows the equivalence of (*i*) and (*ii*) and if we set $a := o$ in (*ii*) we see that (*ii*) implies (*iii*). Finally we assume (*iii*). For each $a \in E$ let $a' \in E$ such that $a' + a' = a$ then if $A := \kappa_E(a^+)$ and $A' := \kappa_E(a'^+)$ we have $\widetilde{A'}(E) = A' \cdot E \cdot A' = A' \cdot A' = \kappa_E(a'^+) \cdot \kappa_E(a'^+) = \kappa_E(a'^+ \circ a'^+) = \kappa_E(a^+) = A$. This shows (iv) and $E \in (\mathfrak{K}, \widetilde{\mathfrak{K}})_r$ and so by 2.3(4), $(\mathfrak{K}, \widetilde{\mathfrak{K}})$ is a regular invariant involution set, i.e., (iv) implies (i).

The equivalence of (i) and (v) is a consequence of 8.1. □

We set $A + B := \widetilde{A'} \circ \widetilde{E}(B) = \widetilde{A'}(B^{-1}) = A' \cdot B \cdot A' = \kappa_E(a'^+) \cdot \kappa_E(b^+) \cdot \kappa_E(a'^+) = \kappa_E(a'^+ \circ b^+ \circ a'^+) = \kappa_E(a' + (b + a'))^+ \in \kappa_E(E^+) = \mathfrak{K}$ and so by 8.1(3), $(\mathfrak{K}, +)$ is a K-loop and moreover we have the result of P. T. Nagy and K. Strambach [23].

Theorem 9.5. *Let $(E, +)$ be a Bol loop uniquely 2-divisible and for $a \in E$ let $a' \in E$ such that $a' + a' = a$, then $(E, \oplus)$ with $a \oplus b := a' + (b + a')$ is a K-loop.*

10. Loops derived from point reflection spaces

In this section let $(P, \sim)$ be a point reflection space, let a point $o \in P$ be fixed and let $(P, +)$ be the loop derivation of $(P, \sim)$ in o. If $(P, \sim)$ is singular then by 6.4 the loop $(P, +)$ is a commutative group. Therefore we assume that $(P, \sim)$ is ordinary. Then by 9.4, $(P, +)$ is a proper K-loop uniquely 2-divisible. For $p \in P$ let $p' \in P$ such that $p' + p' = p$. We recall that the operation "+" is given by $a + b := \widetilde{a'} \circ \widetilde{o}(b)$ and that the pair $(P, \mathfrak{L})$, where $\mathfrak{L}$ denotes the set of equivalence classes of the relation ρ, is an incidence space (cf. section 6).

We show:

Theorem 10.1. *Let $\mathfrak{F} := \mathfrak{L}(o) := \{L \in \mathfrak{L} \mid o \in L\}$ be the set of all equivalence classes containing o and let $a, b \in P$. Then:*

(1) *if $a^+ \circ b^+ = b^+ \circ a^+$ then $a^+ \circ b^+ \in P^+$, more precisely, $a^+ \circ b^+ = (a + b)^+$,*
(2) *For each $F \in \mathfrak{F}$, F is a commutative subgroup of the loop $(P, +)$, and if $a \in F \setminus \{o\}$ then $F = \{x \in P \mid a^+ \circ x^+ = x^+ \circ a^+\}$,*
(3) *$\mathfrak{L} = \{a + F \mid a \in P,\ F \in \mathfrak{F}\}$,*
(4) *the collineation group $Aut(P, \mathfrak{L})$ contains P^+,*
(5) *the set $\mathfrak{F}$ is a fibration of the K-loop $(P, +)$ consisting of commutative subgroups of the loop $(P, +)$, i.e., for all $A, B \in \mathfrak{F}$ and for each $a \in P$:*
   (F.1) $|A| \geqslant 2$,
   (F.2) $\bigcup \mathfrak{F} = P$,
   (F.3) if $A \neq B$ then $A \cap B = \{o\}$.

Proof. (1) By $p^+ = \widetilde{p'} \circ \widetilde{o}$, the equation $a^+ \circ b^+ = \widetilde{a'} \circ \widetilde{o} \circ \widetilde{b'} \circ \widetilde{o} = b^+ \circ a^+ = \widetilde{b'} \circ \widetilde{o} \circ \widetilde{a'} \circ \widetilde{o}$ implies $\widetilde{a'} \circ \widetilde{o} \circ \widetilde{b'} = \widetilde{b'} \circ \widetilde{o} \circ \widetilde{a'}$, i.e., $\widetilde{a'} \circ \widetilde{o} \circ \widetilde{b'} \in J$ and so by (R1), there is a $c \in P$ such that $\widetilde{a'} \circ \widetilde{o} \circ \widetilde{b'} = \widetilde{c}$. Therefore $a^+ \circ b^+ = \widetilde{c} \circ \widetilde{o} \in P^+$ and since $a^+ \circ b^+(o) = a^+(b) = a + b$ this implies $a^+ \circ b^+ = (a+b)^+$.

(2) For $p \in P^+$ the equation $p = \widetilde{p'}(o)$ implies by 6.1(4), $\widetilde{p} = \widetilde{\widetilde{p'}(o)} = \widetilde{p'} \circ \widetilde{o} \circ \widetilde{p'}$ and $p' \neq o$ hence $\widetilde{o} \circ \widetilde{p'} \circ \widetilde{p} = \widetilde{p'} \in J$ and $\widetilde{o} \circ \widetilde{p} \circ \widetilde{p'} = \widetilde{o} \circ \widetilde{p'} \circ \widetilde{o} \in J$ and so by 6.5(3), $p' \in \overline{o,p}$ and $p \in \overline{o,p'}$ hence $\overline{o,p} = \overline{o,p'}$. Therefore: $x \in F = \overline{o,a} = \overline{o,a'} \Leftrightarrow x' \in F = \overline{o,a'} \Leftrightarrow \widetilde{x'} \circ \widetilde{o} \circ \widetilde{a'} = \widetilde{a'} \circ \widetilde{o} \circ \widetilde{x'} \Leftrightarrow x^+ \circ a^+ = \widetilde{x'} \circ \widetilde{o} \circ \widetilde{a'} \circ \widetilde{o} = \widetilde{a'} \circ \widetilde{o} \circ \widetilde{x'} \circ \widetilde{o} = a^+ \circ x^+ \;\Rightarrow\; x + a = x^+(a) = x^+ \circ a^+(o) = a^+ \circ x^+(o) = a + x$.

(3), (4) If $p \in P$ then by 6.5(6) $p^+ = \widetilde{p'} \circ \widetilde{o} \in Aut(P, \mathfrak{L})$, and therefore if $L \in \mathfrak{L}$, $p \in L$ then $F := (p^+)^{-1}(L) \in \mathfrak{L}$ and $o \in F$, i.e., $F \in \mathfrak{F}$ and $a + F = a^+(F) = L$. $\square$

11. Loops with fibrations

In 1987 Elena Zizioli introduced for loops the notion of an *incidence fibration* (cf. [27], [16]) in the sense of the following definition:

Given a loop $(P, +)$ and a set $\mathfrak{F} \subseteq 2^P$, $\mathfrak{F}$ is called a *fibration* of $(P, +)$ if:

$(F1)$ $\forall X \in \mathfrak{F} \;\; |X| \geqslant 2$,

$(F2)$ $\bigcup \mathfrak{F} = P$,

$(F3)$ $\forall A, B \in \mathfrak{F} \; A \neq B \;\; A \cap B = \{o\}$.

If furthermore the following conditions

$(F4)$ $\forall a \in P \; \forall X \in \mathfrak{F} \;\; o \in a + X \Longrightarrow a + X \in \mathfrak{F}$,

$(F5)$ $\forall X \in \mathfrak{F} \; \forall \delta \in \Delta \;\; \delta(X) \in \mathfrak{F}$,

are valid then $\mathfrak{F}$ is called an *incidence fibration*.

Remark 6. If $(P, +, \mathfrak{F})$ is a fibered loop then to each $a \in P^*$ there is exactly one fiber $A \in \mathfrak{F}$ with $a \in A$ which we denote by $[a]$. Then $(F4)$ and $(F5)$ can be expressed in the form:

$(F4)'$ $\forall a \in P^* \;\; a + [-a] = [a]$,

$(F5)'$ $\forall a \in P^* \; \forall \delta \in \Delta \;\; \delta([a]) = [\delta(a)]$.

By [27] we have:

11.1. *If $\mathfrak{F}$ is an incidence fibration of a loop $(P, +)$ let $\mathfrak{L} := \{a + X \mid a \in P,\; X \in \mathfrak{F}\}$. Then $(P, \mathfrak{L}, +)$ is an incidence loop, i.e., $(P, \mathfrak{L})$ is an incidence space, $(P, +)$ is a loop and for each $a \in P$ the map a^+ is a collineation of $(P, \mathfrak{L})$.*

Remark 7. The fibration $\mathfrak{F}$ corresponding to an ordinary point reflection space $(P, \sim)$ according to 9.1 is an incidence fibration of the loop $(P, +)$ since the fiberes are subgroups of $(P, +)$ and the maps a^+ collineations of $(P, \mathfrak{L})$. Moreover if $A \in \mathfrak{F}$ and $a \in A \setminus \{o\}$ then $A = \{x \in P \mid x^+ \circ a^+ = a^+ \circ x^+\}$ is the centralizer of the element a in $(P.+)$.

Now we ask, when do the centralizers of an arbitrary loop form a fibration or an incidence fibration, respectively? To answer this, we consider the following two exchange conditions:

$(Z1)$ For all $a, b \in P^*$ if $b \in [a]$ then $[a] \subseteq [b]$.
$(Z2)$ For all $a, b \in P^*$ if $a^+ \circ b^+ = b^+ \circ a^+$ then $a^+ \circ b^+ \in P^+$.

11.2. *Let $(P, +)$ be a loop, for any $a \in P^*$ let $[a] := \{x \in P \mid a^+ \circ x^+ = x^+ \circ a^+\}$ be the centralizer of a and let $\mathfrak{Z} := \{[a] \mid a \in P^*\}$. Then:*

(1) *$\mathfrak{Z}$ is a fibration of $(P, +)$ if and only if the exchange condition $(Z1)$ is verified,*
(2) *$\mathfrak{Z}$ is an incidence fibration if and only if $(Z1)$ and the condition: "$\forall a \in P^*\ \forall \delta \in \Delta :\quad a + [-a] = [a]$ and $\delta([a]) = [\delta(a)]$" are valid,*
(3) *if $\mathfrak{Z}$ is a fibration then on each fiber $[a]$ the addition "$+$" is commutative,*
(4) *if $(P, +)$ satisfies $(Z1)$ and $(Z2)$ then each fiber $[a]$ is a commutative subsemigroup of $(P, +)$ and $[a]^+ := \{x^+ \mid x \in [a]\}$ is a commutative subsemigroup of $Sym\, P$,*
(5) *if $\Delta \leq Aut(P, +)$ (i.e., $(P, +)$ is an A_l-loop, cf.* [17] *p. 35) then $\mathfrak{Z}$ satisfies $(F5)$.*

Proof. (3), (4) Let $x, y \in [a] \setminus \{o\}$. Then $x + y = x^+ \circ y^+(o) = y^+ \circ x^+(o) = y + x$. If $(Z2)$ is valid then $(x + y)^+(o) = x + y = x^+ \circ y^+(o)$ implies $(x + y)^+ = x^+ \circ y^+$ and so $a^+ \circ (x + y)^+ = a^+ \circ x^+ \circ y^+ = x^+ \circ a^+ \circ y^+ = x^+ \circ y^+ \circ a^+ = (x + y)^+ \circ a^+$, i.e., $x + y \in [a]$ and moreover $(x + y) + z = (x + y)^+(z) = x^+ \circ y^+(z) = x + (y + z)$ showing that $[a]$ and $[a]^+$ are semigroups.

(5) Clearly if $a \in P^*$ and $\alpha \in Aut(P, +)$ then $\alpha([a]) = [\alpha(a)]$. $\square$

If $\mathfrak{Z}$ is an incidence fibration we say that the loop $(P, +)$ has a *c(entralizer)-fibration*. In order to obtain more informations we claim from now on that our loop $(P, +)$ satisfies the *left inverse property*

$$\forall a \in P \quad a^+ \circ (-a)^+ = id.$$

11.3. *Let $(P,+)$ be a loop satisfying the left inverse property and $(Z1)$ then:*
(1) $\nu \in Aut(P,\mathfrak{Z}) \cap J$, *more precisely ν is the identity on $\mathfrak{Z}$,*
(2) $(F4)' \Leftrightarrow \forall a \in P^* \ \ [a]+[a] \subseteq [a]$,
(3) *if for each $a \in P^*$ $[a]+[a] \subseteq [a]$ then $[a]$ is a commutative subgroup of the loop $(P,+)$,*
(4) *$(P,+)$ has a c-fibration if and only if $(Z1)$ and the condition: "$\forall a \in P^* \ \forall \delta \in \Delta \ \ [a]+[a]=[a]$ and $\delta([a])=[\delta(a)]$" are valid.*

Proof. Let $a \in P^*$ then $(-a)^+ = (a^+)^{-1}$ and so $a^+ \circ (-a)^+ = (-a)^+ \circ a^+$, hence by $(Z1)$, $[a]=[-a]$, i.e., $\nu \in Aut(P,\mathfrak{Z}) \cap J$ and $a+[-a] = a+[a] \subseteq [a]+[a] \subseteq [a]$. If $x \in [a]$ then $y := (a^+)^{-1}(x) = (-a)^+(x) = -a+x \in [a]+[a] \subseteq [a]$ hence $x \in a^+([a]) = a+[a]$. Together $a+[-a]=[a]$ and this shows the equivalence in (2). □

By 10.1, the loop $(P,+)$ derived from an ordinary point reflection space is a K-loop satisfying the exchange conditions $(Z1)$ and $(Z2)$. Since a K-loop is an A_l-loop with left inverse property, $(P,+)$ has a c-fibration.

11.4. *Let $(P,+)$ be a loop with left inverse property and where $\mathfrak{Z}$ is a c-fibration, let $(P,+,\mathfrak{L})$ (with $\mathfrak{L} := \{a+[b] \mid a \in P,\ b \in P^*\}$) the corresponding incidence loop (according 11.1) and let $a \in P^*$ then:*
(1) *the restriction of ν onto $[a]$ is an automorphism of the commutative group $([a],+)$,*
(2) *for each $p \in P$ $\widetilde{\widetilde{p}} \in J$ and $\widetilde{\widetilde{p}}$ fixes the bundle $p+\mathfrak{Z}$ linewise,*
(3) *if $\nu \in Aut(P,+)$ then $P^\circ = P^+ \circ \nu \subseteq Aut(P,\mathfrak{L}) \cap J$ and for $p \in P$ $\nu \circ p^+ \circ \nu = (\nu(p))^+ = (-p)^+$, $\widetilde{\widetilde{p}} = p^+ \circ p^+ \circ \nu = p^+ \circ p^\circ$, hence $\widetilde{\widetilde{P}} \subseteq Aut(P,\mathfrak{L}) \cap J$.*

We summarize:

Theorem 11.5. *Let $(P,\sim)$ be an ordinary point reflection space (cf. Definition 4), let $o \in P$ be fixed and let $(P,+)$ the loop derivation of $(P,\sim)$ in o. Then $(P,+)$ is a proper K-loop uniquely 2-divisible, satisfying $(Z1)$ and $(Z2)$ and $\mathfrak{Z}$ is an incidence fibration.*

Theorem 11.6. *Let $(P,+)$ be a proper K-loop uniquely 2-divisible satisfying $(Z1)$ and $(Z2)$ and let*

$$\sim : P \to J; \quad p \mapsto \widetilde{p} := p^+ \circ \nu \circ (-p)^+.$$

Then:
(1) *$\mathfrak{Z}$ is an incidence fibration,*
(2) *$(P,\sim)$ is an ordinary point reflection space.*

Proof. (1) A K-loop is an A_l-loop with left inverse property. Therefore by 11.2 and 11.3, $(Z1)$ and $(Z2)$ enforce that $\mathfrak{Z}$ is an incidence fibration.

(2) Since in an uniquely 2-divisible K-loop, ν is an involutory automorphism of $(P,+)$ with $Fix\,\nu = \{o\}$ and $(-p)^+ = (p^+)^{-1}$, the map $\widetilde{p}$ is an involution fixing exactly the point p and $\widetilde{p} = (p+p)^+ \circ \nu = (p+p)^\circ$ showing $P^\circ = P^+ \circ \nu = \widetilde{P} := \{\widetilde{p} \mid p \in P\}$. By 3.7 and the 2-divisibility, $(P, P^\circ) = (P, \widetilde{P})$ is a selfhomogeneous invariant regular involution set satisfying (M). Hence $(P, \sim)$ is a point reflection structure and $\widetilde{p} = p^\circ \circ \nu \circ p^\circ$. If $(a,b,c) \in P^3$ are given and $a' := b^\circ \circ \widetilde{a} \circ b^\circ(o) = b^\circ \circ \widetilde{a}(b)$, $c' := b^\circ \circ \widetilde{c} \circ b^\circ(o) = b^\circ \circ \widetilde{c}(b)$ then $a'^\circ = b^\circ \circ \widetilde{a} \circ b^\circ$ and $c'^\circ := b^\circ \circ \widetilde{c} \circ b^\circ$ and we have: $(a,b,c) \in \rho \Leftrightarrow \widetilde{a} \circ \widetilde{b} \circ \widetilde{c} \in J \Leftrightarrow a'^\circ \circ \nu \circ c'^\circ \in J \Leftrightarrow a'^+ \circ c'^+ = c'^+ \circ a'^+$ implying by $(Z2)$, $d^\circ \circ \nu = d^+ = a'^+ \circ c'^+ = a'^\circ \circ \nu \circ c'^\circ \circ \nu$ for $d := a' + b'$ hence $d^\circ = b^\circ \circ \widetilde{a} \circ b^\circ \circ \nu \circ b^\circ \circ \widetilde{c} \circ b^\circ \Leftrightarrow \tilde{a} \circ \tilde{b} \circ \tilde{c} = b^\circ \circ d^\circ \circ b^\circ \in P^\circ = \tilde{P}$. Thus $(R1)$ is valid.

In order to show $(R2)$ we use the same notation as in the proof of $(R1)$. By the invariance of P° and $(R1)$, the relation ρ is symmetric. Therefore let $a \neq b$ and $(a,b,c), (a,b,d) \in \rho$ hence $a'^+ \circ c'^+ = c'^+ \circ a'^+$ and $a'^+ \circ d'^+ = d'^+ \circ a'^+$, i.e., $c', d' \in [a']$ and so $d'^+ \circ c'^+ = c'^+ \circ d'^+$ implying again by $(Z2)$ for $e := c' + d'$, $\tilde{c} \circ \tilde{b} \circ \tilde{d} = b^\circ \circ e^\circ \circ b^\circ \in P^\circ = \tilde{P}$. Consequently $(c,b,d) \in \rho$ and so also $(b,c,d) \in \rho$. Since $(P,+)$ is not commutative, $\rho \neq R^3$ and therefore $(P,\sim)$ is an ordinary point reflection space. $\square$

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Zentrum Mathematik
T.U. München
D-80290 München
Germany

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